RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II
Heidelberg, 01.11.2016

Exercise sheet 3 - long

Isometries, triangles and curvature

To be handed in by November 15th, 1pm, will be discussed on November 16th

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Exercise 1. Let M be a complete connected Riemannian manifold, and let $d : M \times M \to \mathbb{R}$ be the distance function induced by the Riemannian metric. We assume that $f : M \to M$ is distance-preserving, i.e. we have $d(f(x), f(y)) = d(x, y)\ \forall x, y \in M$. The goal of this exercise is to show that f has to be a Riemannian isometry – in particular, we do not need to assume differentiability of f, but get it automatically. Show that:

(a) If c is a geodesic, then so is $f \circ c$.

(b) Define the map $f' : \mathrm{T}_pM \to \mathrm{T}_{f(p)}M$ (our candidate for the differential of f) as follows: If c is a geodesic with $c(0) = p$ and $c'(0) = X$, let

$$f'(X) = \frac{\mathrm{d}}{\mathrm{d}t} f(c(t))|_{t=0}.$$

Then we have $\|f'(X)\| = \|X\|$ and $f'(aX) = af'(X)$ for $a \in \mathbb{R}$.

(c) To complete the proof that f' is a linear map, we now show that $\langle f'(X), f'(Y)\rangle = \langle X, Y\rangle$. The idea is to express $\langle X, Y\rangle$ in terms of objects we know to be preserved, the norm and the distance function. First show that

$$2\frac{\langle X, Y\rangle}{\|X\|\|Y\|} = \frac{\|X\|^2 + \|Y^2\|}{\|X\|\|Y\|} - \frac{\|tX - tY\|^2}{\|tX\|\|tY\|}$$

for any $t \in \mathbb{R} \setminus \{0\}$, then prove that

$$\lim_{t\to 0} \frac{\|tX - tY\|^2}{\|tX\|\|tY\|} = \lim_{t\to 0} \frac{d\left(\exp(tX), \exp(tY)\right)^2}{\|tX\|\|tY\|}.$$

Conclude from this that $\langle f'(X), f'(Y)\rangle = \langle X, Y\rangle$ and consequently

$$f'(X + Y) = f'(X) + f'(Y).$$

(d) f' is a linear isometry (for any choice of $p \in M$) and f is a smooth diffeomorphism with differential f'.

Exercise 2.

(a) Fix a geodesic $l \subset \mathbb{H}^2$ and a point $p \in l$. Let $T_a, a \in \mathbb{R}$ be the hyperbolic translation along l for some fixed orientation of l, and let $R_\alpha, \alpha \in \mathbb{R}$ be the (counterclockwise) rotation around p by the angle α. Show that we have the following equivalence for any choice of $a, b, c \in \mathbb{R}_+$ and $\alpha, \beta, \gamma \in (0, \pi)$, with $\alpha' = \pi - \alpha$, $\beta' = \pi - \beta$, $\gamma' = \pi - \gamma$:
A hyperbolic triangle Δ with side lengths a, b, c and corresponding opposite angles α, β, γ exists iff the composition

$$R_{\gamma'} \circ T_b \circ R_{\alpha'} \circ T_c \circ R_{\beta'} \circ T_a$$

is the identity map.

(b) Prove the following hyperbolic sine and cosine identities:

$$\cosh(c) = \cosh(a)\cosh(b) - \sinh(a)\sinh(b)\cos(\gamma)$$

$$\frac{\sinh(a)}{\sin(\alpha)} = \frac{\sinh(b)}{\sin(\beta)} = \frac{\sinh(c)}{\sin(\gamma)}$$

$$\cosh(c) = \frac{\cos(\alpha)\cos(\beta) + \cos(\gamma)}{\sin(\alpha)\sin(\beta)}$$

Hint: Use the identity $T_a R_{\gamma'} T_b = R_{-\beta'} T_{-c} R_{-\alpha'}$ from (a) in the hyperboloid model.

Exercise 3. In this exercise, we give a description of curvature as the second derivative of parallel transport along specific shrinking curves.

(a) Let M be a smooth manifold, $E \to M$ a vector bundle and ∇ a connection on E. Let $\gamma\colon [0,1] \to M$ be a smooth curve and $s \in \Gamma_\gamma(E)$ a section of E along γ. For any fixed T, denote by $P_t\colon E_{\gamma(t)} \to E_{\gamma(T)}$ the parallel transport along (the suitable segment of) γ. Show that

$$\nabla_{\dot\gamma(T)} s = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=T} P_t(s(t)).$$

(b) Now let $M = \mathbb{R}^2$, $p = (0,0) \in \mathbb{R}^2$, and define a family of curves $c_t\colon [0,3t] \to \mathbb{R}^2$ by

$$c_t(\tau) = \begin{cases} (\tau, 0) & \text{if } \tau < t, \\ (2t - \tau, \tau - t) & \text{if } t \le \tau \le 2t, \\ (0, 3t - \tau) & \text{if } \tau > 2t. \end{cases}$$

Prove that for every $\xi \in E_p$ there exists a unique section s such that

i. $s(p) = \xi$,

ii. $\nabla_{\frac{\partial}{\partial x_1}} s(\cdot, 0) = 0$,

iii. $\nabla_{\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1}} s = 0$.

(c) In the setting above, let $P_{c_t}^\nabla\colon E_p \to E_p$ be the (forwards) parallel transport along c_t. Prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} P_{c_t}^\nabla(\xi) = \nabla_{\frac{\partial}{\partial x_2}} s(p) = 0.$$

(d) In the same situation, prove that

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} P_{c_t}^{\nabla}(\xi) = \nabla_{\frac{\partial}{\partial x_2}} \nabla_{\frac{\partial}{\partial x_2}} s(p).$$

(e) Finally, prove that

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} P_{c_t}^{\nabla} = -R\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)(p).$$

Here, $R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ is the curvature tensor and the derivative on the left hand side is a derivative of linear maps $P_{c_t}^{\nabla} : E_p \to E_p$.