RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II
Heidelberg, 25.10.2016

Exercise sheet 2 - short

Constant curvature

Not to be handed in, will be discussed on November 2nd

Exercise 1. Show that the group of orientation-preserving isometries of $\mathbb{H}^2$, the upper half-plane model of two-dimensional hyperbolic space, can be identified with $\mathrm{PSL}(2,\mathbb{R})$. Here, the action of an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm 1\}$ is given by

$$z \mapsto \frac{az+b}{cz+d}.$$

Exercise 2. Consider the Klein model $\mathbb{K}^2$ of the hyperbolic plane. Define the cross-ratio of four points $p, q, r, s \in \mathbb{R}^2$ as

$$\beta(p,q,r,s) = \frac{|q-r||s-p|}{|p-r||s-q|},$$

where $|\cdot|$ denotes the Euclidean norm.
For any two distinct points $p, q \in \mathbb{K}^2$, let r and s be the points of intersection between the line through p, q and the boundary $\partial\mathbb{K}^2$. Assume that they appear in the order r, p, q, s. Show that the hyperbolic distance between p and q is then given by

$$d(p,q) = \frac{1}{2}|\log \beta(p,q,r,s)|.$$

Exercise 3. In this exercise, we compute geodesics and the curvature of the model spaces $\mathbb{M}^n_k$ for $k \neq 0$. The curvature will be computed by a different method from the one used in the lecture.

(a) Show that a geodesic through the point x with initial tangent vector $v \in \mathrm{T}_x\mathbb{M}^n_k = \mathrm{span}(x)^\perp$ is given by

$$k>0: \quad \gamma(t) = x\cos(t\sqrt{k}\|v\|) + \frac{v}{\sqrt{k}\|v\|}\sin(t\sqrt{k}\|v\|),$$
$$k<0: \quad \gamma(t) = x\cosh(t\sqrt{-k}\|v\|) + \frac{v}{\sqrt{-k}\|v\|}\sinh(t\sqrt{-k}\|v\|).$$

(b) For orthonormal vectors $u, v \in \mathrm{T}_x\mathbb{M}^n_k$, a 1-parameter family of geodesics is given by

$$\gamma_{w(s)}(t) = \exp_x(tw(s)),$$

where $w(s) = u\cos(s) + v\sin(s)$. Use Jacobi fields to compute the curvature of $\mathbb{M}^n_k$ for both $k>0$ and $k<0$.