RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

MATHEMATISCHES INSTITUT

Vorlesung Differentialgeometrie II
Heidelberg, 18.10.2016

EXERCISE SHEET 1 - SHORT

Revision

Not to be handed in, will be discussed on October 26th

Exercise 1. Let M, N be Riemannian manifolds, N connected, and $h, k \colon N \to M$ local isometries. Assume there is a point $p \in N$ with $h(p) = k(p)$ and $D_p h = D_p k$. Show that $h = k$.

Exercise 2.

(a) Let M be a homogeneous Riemannian manifold, i.e. for every pair of points $x, y \in M$ there is an isometry $f_{xy} \colon M \to M$ with $f_{xy}(x) = y$. Show that M is complete.

(b) Let $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ be the upper half plane. We define a metric g on it by

$$g(v, w) = \frac{1}{y} \langle v, w \rangle$$

for all $v, w \in T_{(x,y)}\mathbb{R}^2_+ \cong \mathbb{R}^2$. Show that $(\mathbb{R}^2_+, g)$ is not complete.
Hint: Look at vertical segments approaching the boundary of the half plane.

Exercise 3. Let $S^2 \subset \mathbb{R}^3$ be the sphere (of radius 1).

(a) Let $R \in \mathrm{SO}(3)$ be a rotation, $\gamma \colon [0, 1] \to S^2$ a smooth curve and $P_\gamma \colon T_{\gamma(0)}S^2 \to T_{\gamma(1)}S^2$ the parallel transport along γ. Show that the parallel transport is equivariant with respect to R, that is

$$P_{R\circ\gamma}(R(v)) = R(P_\gamma(v)) \qquad \forall v \in T_{\gamma(0)}S^2.$$

(b) Show that $t \mapsto (\cos t, \sin t, 0)$ is a geodesic.

(c) Let $x \in S^2$ and $v \in T_x S^2$. Find the geodesic $c \colon \mathbb{R} \to S^2$ with $c(0) = x$ and $\dot{c}(0) = v$.