RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Geometry of Manifolds
Heidelberg, 5.7.2017

Exercise sheet 4

Mostow Rigidity

To be handed in by Friday, July 14th, 2pm

Exercise 1 (Schottky groups)**.** Let $\gamma_1, \dots, \gamma_n \in \mathrm{Isom}(D^n)$ be hyperbolic isometries of the Poincaré disc model and $D_1, \dots, D_n$ be open subsets of D^n satisfying

- $D_i \cap g(D_i) = \emptyset$ for all $g \neq 1$ in $\langle \gamma_i \rangle$,
- $D := \bigcap_{i=1}^n D_i$ is nonempty,
- $D_i \cup D_j = D^n$ for all $i \neq j$.

Show:

(i) The group $\Gamma := \langle \gamma_1, \dots, \gamma_n \rangle$ is the free product of the groups $\langle \gamma_i \rangle$, $i = 1, \dots, n$,

(ii) $D \cap g(D) = \emptyset$ for all $g \neq 1$ in Γ,

(iii) The group Γ is discrete.

Exercise 2 (Trigonometry in hyperbolic space)**.** (i) Prove that for any two points w, z in the upper half plane the hyperbolic distance is given by the formula

$$\cosh d(w,z) = 1 + \frac{|w-z|^2}{2 \,\mathrm{Im}\, w \,\mathrm{Im}\, z}.$$

(ii) Let α, β and $\frac{\pi}{2}$ be the angles of a hyperbolic right triangle and let a, b and c be the lengths of the opposite sides. Show that

$$\sinh a = \sinh c \sin \alpha.$$

(iii) Let $\alpha, \beta, 0$ be the angles of an infinite hyperbolic triangle with just one ideal vertex and let c be the length of the finite side. Show that

$$\sinh c = \frac{\cos \alpha + \cos \beta}{\sin \alpha \sin \beta}.$$

Hint: Use Möbius transformations in order to bring everything into a suitable position.

Insert here your favorite quote about turning the sheet over.

Exercise 3 (Quasi-isometries)**.** (i) Show that if there exists a quasi-isometry $F : X \to Y$ then there exists a quasi-isometry $G : Y \to X$ and a constant $k \geq 0$ such that $d(G \circ F(x), x) \leq k$ and $d(F \circ G(y), y) \leq k$ for all $x \in X$ and all $y \in Y$.

(ii) Prove that the composition of two quasi-isometries is a quasi-isometry.

Exercise 4 (Pseudo-isometries)**.** Let $f : M \to N$ be a smooth map between Riemannian n-manifolds. The *maximum dilatation* of f is given by

$$\sup_{x \in M} \sup_{v \in T_x M, v \neq 0} \frac{|df_x(v)|}{|v|}.$$

Prove that if f has maximum dilatation C then f is C-Lipschitz.

Exercise 5 (Lobachevski function)**.** Prove that the Lobachevski function given by

$$\Lambda(\Theta) = -\frac{1}{2} \int_0^\Theta \log |2 \sin t| dt$$

is well-defined and continuous on $\mathbb{R}$.

Hint: Consider the complex function $\phi(w) := \frac{-\log(1-w)}{w}$ which is analytic (why?) and define three paths in $\mathbb{C}$ as follows. Let α be the straight path from 0 to $e^{2i\varepsilon}$ for some $0 < \varepsilon < \Theta$, γ the straight path from 0 to $e^{2i\Theta}$ and β the path on the unit circle from $e^{2i\varepsilon}$ to $e^{2i\Theta}$. Now integrate ϕ over all three paths and see what happens.

Exercise 6 (Mostow rigidity)**.** Show that in general the natural map

$$\mathrm{Isom}(M) \to \mathrm{Out}(\pi_1(M))$$

fails to be an isomorphism when the dimension of M is 2.

Hint: What can be said about the order of the two groups?