RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Geometry of Manifolds
Heidelberg, 13.6.2017

Exercise sheet 3

Thick-Thin Decomposition and Limit Set

To be handed in by Friday, June 23rd, 2pm

Exercise 1 (Characterization of glued three-manifolds)**.** Prove that a three-dimensional gluing is a topological three-manifold if and only if the link of every vertex is homeomorphic to S^2.

Exercise 2 (Closed geodesics on hyperbolic manifolds)**.** Let $M = \mathbb{H}^n/\Gamma$ be a complete hyperbolic manifold. Define $[S^1, M]$ to be the set of homotopy classes of continuous maps $S^1 \to M$. Show:

(i) There is a bijection between the conjugacy classes in $\pi_1(M, x_0)$ and $[S^1, M]$, where $x_0 \in M$ is any fixed basepoint.

(ii) The conjugagy class of a hyperbolic element in Γ corresponds one-to-one to a closed geodesic of length equal to its minimum displacement.

(iii) Every closed geodesic has the minimum length in its homotopy class.

Exercise 3 (Centralizers of isometry subgroups)**.** Let $\mathbb{H}^n/\Gamma$ be a finite-volume hyperbolic manifold. Show that the centralizer of Γ in $\mathrm{Isom}(\mathbb{H}^n)$ is trivial.
Hint: What can be said about the fixed points of commuting elements?

Don't judge a sheet by its cover. Take also a look at its back.

Exercise 4 (Hyperbolic structures on a torus)**.** Let $\mathbb{H}^n/\Gamma$ be a complete hyperbolic manifold.

(i) Show that every subgroup of Γ isomorphic to $\mathbb{Z} \times \mathbb{Z}$ consists of parabolic elements fixing the same point at infinity.

(ii) Is it possible to equip the n-torus with a hyperbolic structure?

Exercise 5 (Star-shaped neighborhoods)**.** Let $\mathbb{H}^n/\Gamma$ be a complete hyperbolic manifold. For every isometry $\varphi \in \Gamma$ we define

$$S_\varphi(\varepsilon) := \{x \in \mathbb{H}^n \mid d(\varphi(x), x) \leq \varepsilon\}.$$

Show that $S_\varphi(\varepsilon)$ is star-shaped, centered at a point $p \in \partial\mathbb{H}^n$ or at a line l according to whether φ is parabolic fixing p or hyperbolic fixing l.

Exercise 6 (Limit set)**.** Let Γ be a non-trivial discrete subgroup of isometries of $\mathbb{H}^n$. Recall that for a fixed point $x \in \mathbb{H}^n$ the *limit set* $\Lambda(\Gamma)) \subseteq \mathbb{H}^n$ of Γ is defined as the set of all the accumulation points of the orbit $\Gamma(x)$ in $\partial\mathbb{H}^n$.
Show:

(i) The limit set does not depend on x.

(ii) If $\Gamma' < \Gamma$ is a finite index subgroup then $\Lambda(\Gamma') = \Lambda(\Gamma)$.