RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Geometry of Manifolds
Heidelberg, 23.5.2017

Exercise sheet 2

Hyperbolic, flat and elliptic manifolds

To be handed in by Friday, June 2nd, 2pm

Exercise 1 (Area of hyperbolic polygons). (i) Let Δ be any hyperbolic triangle in $\mathbb{H}^2$ with angles $\alpha, \beta, \gamma \geq 0$, i.e. $\alpha + \beta + \gamma < \pi$. Show that the hyperbolic area of Δ is given by

$$\mathrm{Area}(\Delta) = \pi - (\alpha + \beta + \gamma).$$

Hint: Prove the formula first for triangles having one vertex at ∞.

(ii) Prove that a hyperbolic polygon P with inner angles $\alpha_1, \ldots, \alpha_n$ has area

$$\mathrm{Area}(P) = (n-2)\pi - \sum_{i=1}^{n} \alpha_i$$

Exercise 2 (Fundamental domain of $\mathrm{PSL}(2,\mathbb{Z})$). The subgroup $\Gamma := \mathrm{PSL}(2,\mathbb{Z})$ of $\mathrm{PSL}(2,\mathbb{R})$ acts on the upper half plane $\mathbb{H}^2$ via Möbius transformations.

(i) Show:
Γ is generated by the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

(ii) Is the action properly discontinuous? Is it free?

(iii) Show that a fundamental domain of Γ is given by the polygon

$$D := \left\{ z \in \mathbb{H}^2 \;\middle|\; |\,\mathrm{Re}(z)| \leq \frac{1}{2}, |z| \geq 1 \right\}.$$

Hint: Show that D is the Dirichlet domain $D(2i)$.

(iv) What is the hyperbolic area of D?

Don't forget to turn me around!

Exercise 3 (Triangle groups)**.** For a triple $a, b, c > 0$ of integers satisfying

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$$

let Δ be a hyperbolic triangle with angles $\frac{\pi}{a}, \frac{\pi}{b}, \frac{\pi}{c}$. We denote by ρ_a, ρ_b, ρ_c the hyperbolic reflections along the corresponding sides of Δ. Let $\Gamma := \Gamma(a, b, c)$ be the *triangle group* generated by these reflections. Clearly, Γ is a subgroup of the group of hyperbolic isometries. By reflecting iteratively Δ along its sides we get a tesselation of the hyperbolic plane with isometric triangles.

(i) Show that the action of Γ is properly discontinuous but not free.

(ii) What is a fundamental domain of this action?

Exercise 4 (Flat manifolds in dimension 2)**.** (i) Prove that every closed flat orientable surface is a torus.

(ii) Show that the Klein bottle is a closed flat surface that is not orientable.

(iii) Show that the Klein bottle is covered by the torus. What is the degree of this covering?

Exercise 5 (Elliptic manifolds in even dimension)**.** Let M be an elliptic manifold, i.e.

$$M = S^n / \Gamma$$

where $\Gamma \leq \mathrm{Isom}(S^n)$ acts properly discontinuously and freely on S^n. Show that if n is an even number then

$$M = S^n \text{ or } M = \mathbb{RP}^n.$$

Exercise 6 (Elliptic manifolds in dimension 3)**.** Let $p, q > 1$ be coprime integers and set $\theta := e^{\frac{2\pi i}{p}}$. We identify $\mathbb{R}^4$ with $\mathbb{C}^2$ and see S^3 as

$$S^3 = \left\{ (w, z) \in \mathbb{C}^2 \;\middle|\; |w|^2 + |z|^2 = 1 \right\}.$$

Define the map $f(w, z) := (\theta w, \theta^q z)$ and let $\Gamma := \langle f \rangle$. Show:

(i) f is an isometry of S^3.

(ii) $L(p, q) := S^3 / \Gamma$ is an elliptic manifold.

The elliptic manifold $L(p, q)$ we constructed this way is known as *lens space*.