RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Geometry of Manifolds
*Heidelberg, 3.5.2017*

Exercise sheet 1

**Different models of hyperbolic space**

*To be handed in by Friday, May 12th, 2pm*

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**Exercise 1** (Isometries of hyperbolic space)**.** Consider the hyperboloid model. We want to prove that each isometry of $I^n$ is a product of reflections along hyperbolic hyperspaces.

(i) Let $S$ be a $k$-space in $I^n$ and $p \in S$ a point. Show that there is a unique $(n-k)$-subspace $S'$ intersecting $S$ orthogonally in $p$.

(ii) Show that reflections along hyperspaces in $S^n$ generate $\mathrm{Isom}(S^n) \cong O(n)$.
*Hint: For a given isometry $\phi$ find suitable reflections $\rho_i$ such that $\rho_n \dots \rho_1 \phi$ is the identity.*

(iii) Show that reflections along hyperbolic hyperspaces in $I^n$ generate $\mathrm{Isom}(I^n)$.

**Exercise 2** (Inversion from $H^2$ to $D^2$)**.** Consider the inversion $\phi : \mathbb{C} \to \mathbb{C}$ along the circle centered in $-i$ with radius $\sqrt{2}$. We have already seen that $\phi$ maps the half-plane $H^2$ to the Poincaré disc $D^2$.
Show that $\phi$ is given by

$$\phi(z) = \frac{\bar{z} + i}{i\bar{z} + 1}.$$

**Exercise 3** (Positive triples of points)**.** An ordered triple of distinct points in $\mathbb{R} \cup \{\infty\}$ is called *positive* if they are oriented counterclockwise, like $(0, 1, \infty)$.
Show that the group $\mathrm{PSL}_2(\mathbb{R})$ acts freely and transitively on positive triples of points in $\mathbb{R} \cup \{\infty\}$.
*Remember: A group $G$ acts freely on a set $M$ if $g.m = m$ for some $m \in M$ implies $g = e_G$. And the action is transitive if for all $x, y \in M$ there is $g \in G$ such that $g.x = y$.*

**Exercise 4** (Distance in the hyperbolic model $I^n$)**.** Let $p, q$ be two points in $I^n$. Show that the hyperbolic distance $\mathrm{d}(p, q)$ between $p$ and $q$ is given by

$$\cosh(\mathrm{d}(p, q)) = -\langle p, q \rangle.$$

*Hint: Consider the parametrization of the geodesic segment starting in $p$ and ending in $q$.*

**There's more on the back! :)**

**Exercise 5** (Distance in the half-plane model $H^2$). (i) Let $w, x, y, z \in \bar{H}^2 \subseteq \mathbb{C} \cup \{\infty\}$. The *cross-ratio* of the ordered tuple $(w, x, y, z)$ is defined via

$$\mathrm{CR}(w, x, y, z) := \frac{(w-y)(x-z)}{(x-y)(w-z)}.$$

Show that the cross-ratio is invariant under isometries of $H^2$.
*Hint: What are the generators of* $\mathrm{Isom}(H^2)$*?*

(ii) Show that the hyperbolic distance between $i$ and $\lambda i$ for $\lambda \in \mathbb{R}_{>0}$ is given by

$$d(i, \lambda i) = |\ln \lambda|.$$

(iii) Let $p \neq q$ be two points in $H^2$ and consider the unique geodesic $\gamma$ containing $p$ and $q$, which is a half-circle or line orthogonal to the real axis. Denote by $p^*$ and $q^*$ the two endpoints of $\gamma$ on $\mathbb{R} \cup \{\infty\}$ such that the four points lie on $\gamma$ in the order $p^*, p, q, q^*$. Show that the hyperbolic distance is given by

$$d(p, q) = |\ln \mathrm{CR}(p^*, p, q, q^*)|.$$

*Hint: Use a suitable isometry such that* $p$ *and* $q$ *are mapped to "nice" points.*

**Exercise 6** (Triangles in the Poincare disc model $D^2$). We have already seen that the angle sum in a hyperbolic triangle is always less than $\pi$. Show:

(i) For any triple of positive angles $\alpha, \beta, \gamma$ with $\alpha + \beta + \gamma < \pi$ there is a triangle with inner angles $\alpha, \beta, \gamma$.
*Hint: Start with a triangle in* $D^2$ *having a vertex* $0$ *and a horizontal side.*

(ii) This triangle is unique up to isometries.