Bieberbach's theorem

Part 2
Seminar: Geometry of Lie groups
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1 Introduction

In the last talk we have seen the definition of a *crystallographic group of dimension* n Γ as a discrete group of isometries of the Euclidean space $\mathbb{E}^n$ which is cocompact. Furthermore, we characterized them by means of their subgroups of translation $T(\Gamma)$.

During this talk we will try to extend the classification of these groups, deriving two additional theorems due to Bieberbach. First, we will show that for each dimension there are (up to isomorphism) only finitely many crystallographic groups. Afterwards, we will then derive that even the isomorphisms between groups of the same dimension are rather special in the sense that they are induced by conjugation with a fixed affine transformation of $\mathbb{E}^n$.

The numbering of the lemmas and the theorems corresponds to the one in [1].

2 Finiteness

We will begin with some preliminary results necessary for the finiteness theorem. The main ingredients concern the connection between lattices in $\mathbb{R}^n$ and certain vector subspaces.

Lemma 2. Let $B_r(a) \subset \mathbb{E}^n$ be an open ball around $a \in \mathbb{E}^n$ with radius $r > 0$. Then we have

$$\mathrm{Vol}(B_r(a)) = c_n r^n,$$

where c_n is a constant independent of a and r.

Proof. Translating the ball to the origin and using spherical coordinates, we see that c_n is just given by

$$c_n = \frac{1}{n}\,\mathrm{Vol}(S^{n-1}),$$

where S^{n-1} denotes the $(n-1)$-dimensional sphere. □

Definition. A lattice $L \leq \mathbb{R}^n$ is called *full scale* if all nonzero vectors of L have norm at least 1.

Lemma 3. Let $L \leq \mathbb{R}^n$ be a full scale lattice and for each $r \geq 0$, let $N(r)$ be the number of $v \in L$, such that $|v| \leq r$. Then we have

$$N(r) \leq (2r+1)^n.$$

Proof. All the vectors have at least distance 1 from each other, as L is full scale. Hence, taking a ball with radius $\frac{1}{2}$ around each point with norm not greater than r, we see that these balls do not overlap. Additionally, all the balls are contained in the larger ball $B_{r+\frac{1}{2}}(0)$. Comparing (with the help of Lemma 2) the volumes of the added up small balls and the large ball, we yield

$$N(r)\left(\frac{1}{2}\right)^n \leq \left(r+\frac{1}{2}\right)^n$$

which proves the assumption. □

Lemma 4. Let $(v_1, \ldots, v_n)$ be a basis for $\mathbb{R}^n$. Then for each $x \in \mathbb{R}^n$ there exist $k_1, \ldots, k_n \in \mathbb{Z}$ such that

$$\left| x - \sum_{i=1}^{n} k_i v_i \right| \leq \frac{1}{2} \sum_{i=1}^{n} |v_i|$$

holds.

Proof. Since the v_i form a basis there exist $t_i \in \mathbb{R}$, such that $x = \sum_i t_i v_i$. We set

$$k_i := \begin{cases} \lfloor t_i \rfloor & \text{if } t_i - \lfloor t_i \rfloor \leq \frac{1}{2}, \\ \lceil t_i \rceil & \text{else.} \end{cases}$$

□

Lemma 5. Let $V \subset \mathbb{R}^n$ be a linear subspace with normalized basis $v_1, \ldots, v_m$, where the basis lies in a full scale lattice $L \leq \mathbb{R}^n$. If a vector $u \in L$ lies not in V, then its $V^\perp$component w has norm

$$|w| > (m+3)^{-n}.$$

Proof. Assume on the contrary that $u \in L \setminus V$ has a $V^\perp$-component w with norm $0 < |w| \leq (m+3)^{-n}$. Set $k = (m+3)^n$. Then $k|w| \leq 1$ and so all lu, $l \in \{0, \ldots, k\}$, have distance at most 1 from V. By Lemma 4, we can add suitable integral linear combinations of $v_1, \ldots, v_n$, such that their V-components have norm at most $\frac{m}{2}$. By definition the $V^\perp$-components will not change by this. Additionally, since $w \neq 0$, these updated vectors will still be distinct. Hence, we have found $k+1$ distinct vectors in L whose norms are less or equal to $r = (\frac{m}{2}) + 1$. Using Lemma 3, we get

$$(m+3)^n + 1 = k + 1 \leq N(r) \leq (2r+1)^n = (m+3)^n,$$

which is a contradiction. Therefore, $|w| > (m+3)^n$ needs to hold. □

Definition. An n-dimensional crystallographic group Γ is called *normalized* if its lattice $L(\Gamma)$ is full scale and contains n linearly independent unit vectors.

Lemma 6. Let Γ be an n-dimensional crystallographic group. Then Γ is isomorphic to a normalized n-dimensional crystallographic group.

Proof. By rescaling Γ, which is an isomorphism, we can assume that the shortest nonzero vector in $L(\Gamma)$ is a unit vector. We now proceed by induction and assume that $L(\Gamma)$ is full scale and contains $m < n$ linearly independent unit vectors $v_1, \ldots, v_m$. We try to construct a Γ' which is isomorphic to Γ and whose lattice is also full scale and contains one additional linearly independent unit vector.

Consider $V = \langle v_1, \ldots, v_m \rangle_{\mathbb{R}}$. The point group Π of Γ acts on $\mathbb{R}^n$ and on $L(\Gamma)$. We have to consider two cases. First, we assume that Π does not leave V invariant. In this case there must exist an $A \in \Pi \subset O(n)$ and an $i \leq m$, such that $Av_i \notin V$. However, $Av_i \in L(\Gamma)$ and hence, setting $v_{m+1} = Av_i$ and $\Gamma' = \Gamma$ we are done.

Now assume that Π leaves Vinvariant. In this case it also leaves $V^\perp$ invariant. For all $t > 0$ we define $\alpha_t \in \mathrm{Aut}(\mathbb{R}^n)$ by

$$\alpha_t(u) = v + tw,$$

where $u = v + w \in \mathbb{R}^n$, $v \in V$ and $w \in V^\perp$. Let $a + A \in \Gamma$. As A leaves V an $V^\perp$ invariant, we have

$$\alpha_t(a + A)\alpha_t^{-1} = \alpha_t(a) + A.$$

Now, $\Gamma_t = \alpha_t \Gamma \alpha_t^{-1}$ is a discrete group of isometries of $\mathbb{E}^n$ and $T(\Gamma_t) = \alpha_t T(\Gamma) \alpha_t^{-1}$ by the previous argument. Thus, $T(\Gamma_t)$ is of finite index in Γ_t and has rank n. Therefore, Γ_t is a crystallographic group of dimension n. Looking at $L(\Gamma_t)$, we see the relation

$$L(\Gamma_t) = \alpha_t(L(\Gamma)).$$

Let $u \in L(\Gamma) \setminus V$ and write $u = v + w$ as above. Then for $0 < t \leq |w|^{-1}(m+3)^{-n}$, we have that $v + tw \in L(\Gamma_t) \setminus V$ and $|tw| \leq (m+3)^{-n}$. By Lemma 5 this means that $L(\Gamma_t)$ is not full scale. So there exists a $t > 0$ such that $L(\Gamma_t)$ is not full scale. However, $L(\Gamma_1) = L(\Gamma)$ is full scale. So choose

$$s := \inf\{t > 0 \mid (\Gamma_t) \text{ is full scale}\}.$$

Then we have $0 < s \leq 1$. For fixed u we know that $\alpha_t(u)$ is a continuous function with regard to t, thus from $|\alpha_t(u)| \geq 1$ for all $t > s$ follows $|\alpha_s(u)| \geq 1$. Hence, $L(\Gamma_s)$ is full scale.

Let $u_0 \in L(\Gamma_s) \setminus V$ be the shortest element. We claim that u_0 is a unit vector. Assume on the contrary that u_0 were not a unit vector, i. e. $|u_0| > 1$. We write $u_0 = v_0 + w_0$ as before and see that for all $u \in L(\Gamma_s) \setminus V$, we have

$$|v|^2 + |w^2 \geq |v_0|^2 + |w_0|^2$$

Setting $t = |u_0|^{-1}$ and acting with α_t on $L(\Gamma_s)$ (leading to $L(\Gamma_{st})$), we yield

$$\begin{aligned}
|\alpha_t(u)|^2 &= |v + tw|^2 \\
&= |v|^2 + t^2|w|^2 \\
&\geq |v|^2 + t^2(|v_0|^2 + |w_0|^2 - |v|^2) \\
&= |v|^2(1 - t^2) + t^2|u_0|^2 \\
&\geq t^2|u_0|^2 = 1.
\end{aligned}$$

Since the vectors in $L(\Gamma_s) \cap V$ are not changed and they were full scale before, it follows that $L(\Gamma_{st})$ is full scale. However, $st < s$ which is a contradiction to the minimality of s. Hence u_0 is a unit vector. Setting $v_{m+1} = u_0$ and choosing $\Gamma' = \Gamma_s$ finishes the induction step. $\square$

Proposition 1. Let $A \leq B$ be two free abelian groups of finite rank. Then A and B have the same rank if and only if B/A is finite.

Proof. We consider the short exact sequence

$$0 \to A \to B \to B/A \to 0.$$

We know that $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, so tensoring with it leaves the sequence exact and we obtain

$$0 \to A \otimes \mathbb{Q} \to B \otimes \mathbb{Q} \to B/A \otimes \mathbb{Q} \to 0.$$

First, let us assume that A and B have the same rank. After tensoring with $\mathbb{Q}$ this means that the $\mathbb{Q}$-vector spaces $A \otimes \mathbb{Q}$ and $B \otimes \mathbb{Q}$ have the same dimension. Since the sequence stays exact, the first map is injective and thus, using the dimension formula, it is also a bijection. Again as the sequence needs to be exact, this means that $B/A \otimes \mathbb{Q}$ vanishes. This implies that B/A is only torsion and hence finite.

If we assume now that B/A is finite, then again $B/A \otimes \mathbb{Q}$ vanishes and the other two $\mathbb{Q}$-vector spaces are isomorphic. Hence, they have the same dimension, which implies that the ranks of A and B must coincide. $\square$

Theorem (7.5.3, Bieberbach's theorem 2). There are only finitely many isomorphism classes of n-dimensional crystallographic groups for each $n \in \mathbb{N}$.

Proof. By Lemma 6, we can reduce to the case of normalized crystallographic groups. Let Γ be such a group. Then $L(\Gamma)$ contains n linearly independent vectors $w_1, \ldots, w_n$. Set for each i, $\tau_i := w_i + I$ the corresponding elements in $T(\Gamma)$ and let $H = \langle \tau_1, \ldots, \tau_n \rangle_{\mathbb{Z}} \leq T(\Gamma)$. Then H and $T(\Gamma)$ have the same rank and by Proposition 1 H has finite index in $T(\Gamma)$. Now, $T(\Gamma)$ has finite index in Γ. Hence, H has finite index in Γ. We can choose representatives $\tau_i := w_i + A_i$, $i = n+1, \ldots, m$, where by conjugation with elements of H the w_i can be chosen with norm not greater than $\frac{n}{2}$ (c. f. Lemma 4).Thus every element $\Phi \in \Gamma$ can be uniquely written as

$$\Phi = \left(\sum_{i=1}^{n} a_i \tau_i + I \right) \tau_p,$$

where $a_i \in \mathbb{Z}$ and $p \in \{(n+1, \ldots, m)\}$. We call this the *normal form* of Φ. Since it is unique there are for each $i, j = 1, \ldots, m$ unique integers c_{ijk} and $f(i,j) > n$ such that

$$\tau_i \tau_j = \left(\sum_{k=1}^{n} c_{ijk} \tau_k + I \right) \tau_{f(i,j)}. \tag{1}$$

These integers completely determine Γ since one can find the normal form of a product $\Phi\Psi$ by inductively applying the above formula. In addition, if there is another normalized, n-dimensional crystallographic group Γ' with identical integers. Then Γ and Γ' are already isomorphic. This follows from the observation that we can define the isomorphism on the normal form of the elements. Using the multiplication formula shows that it is really a group homomorphism. The bijectivity stems from the fact that the c_{ijk} and $f(i,j)$ coincide.

Hence our problem has reduced to showing, that there are only finitely many possible choices for the c_{ijk} and $f(i,j)$. Now consider Equation 1. If we multiply both sides by $\tau_{f(i,j)}$ we are left with a translation on the right hand side. Thus the left hand side will also be a translation and all in all the translational part will consist of three vectors possibly multiplied by an element of $O(n)$ and then added up. Since we made sure that all vectors w_i have norm at most $\frac{n}{2}$, we yield

$$\left|\sum_{k=1}^{n} c_{ijk} w_k\right| \leq \frac{3n}{2}.$$

Denoting by v_k the component of w_k orthogonal to all other w_i, we yield $|c_{ijk} v_k| \leq \frac{3n}{2}$. By Lemma 5, we have $|v_k| > (n+2)^{-n}$. Hence, for each c_{ijk} we have

$$|c_{ijk}| \leq \frac{3n}{2}(n+2)^n.$$

This is the first part.

Turning towards $f(i,j)$, it suffices to show that m is bounded. We first consider

$$m - n = [\Gamma : H] = [\Gamma : T(\Gamma)] \cdot [T(\Gamma) : H].$$

We know that the translations among $\tau_{n+1}, \dots, \tau_m$ form a complete set of coset representatives of H in $T(\Gamma)$. By construction they have norm at most $\frac{n}{2}$ and there are by Lemma 3 at most $(n+1)^n$ of them. So we see

$$[T(\Gamma) : H] \leq (n+1)^n.$$

Next we observe that $[\Gamma : T(\Gamma)] = |\Pi|$. Let A be in Π. Then A is uniquely determined by its action on a basis of $\mathbb{R}^n$. $(w_1, \dots, w_n)$ is a basis. We remember that all w_i have norm 1. Applying Lemma 3 we get that Aw_i is one of at most 3^n unit vectors in (Γ) (we know that the point group acts on $L(\Gamma)$). So together we get

$$[\Gamma : T(\Gamma)] \leq (3^n)^n.$$

So all in all we have

$$m \leq n + (3^n)^n (n+1)^n,$$

which finishes the proof. □

Remark. The exact numbers are:

n	1	2	3	4	5	6
	2	17	219	4 783	222 018	28 927 915

Table 1: The number of isomorphism classes for the first few n.

3 Isomorphisms & the splitting group

In order to show that all isomorphisms between crystallographic groups are induced by affine projections, we need to introduce a special extension of Γ, called Γ^*, which we will define shortly. The advantage is that Γ^* has an additional nice property, namely it makes a certain sequence split. We will then be able to extend all isomorphisms between crystallographic groups to these new groups and solve the problem there.

Definition. Let Γ be an n-dimensional crystallographic group and let m be the order of its point group Π. Denote by

$$T(\Gamma)^{\frac{1}{m}} := \left\{ \frac{w}{m} + I \mid w + I \in T(\Gamma) \right\}.$$

Then we call

$$\Gamma^* := \langle T(\Gamma)^{\frac{1}{m}}, \Gamma \rangle \leq I(\mathbb{E}^n)$$

the *splitting group* of Γ.

Proposition 2. Γ^* is an n-dimensional crystallographic group with

$$T(\Gamma^*) = T(\Gamma)^{\frac{1}{m}} \quad \text{and}$$
$$L(\Gamma^*) = \frac{1}{m} L(\Gamma).$$

Additionally its point group coincides with the point group of Γ.

Proof. Γ^* cannot contain any matrix terms not already present in Γ. Hence, the point groups coincide. Also by construction we see that $T(\Gamma^*)$ coincides with $T(\Gamma)^{\frac{1}{m}}$. This shows also the form of $L(\Gamma^*)$. Also, since Π coincides for the two groups, we see that $T(\Gamma^*)$ has finite index in Γ^*. Next we see that

$$[T(\Gamma^*) : T(\Gamma)] = [L(\Gamma^*) : L(\Gamma)] = [(m^{-1}\mathbb{Z})^n : \mathbb{Z}^n] = m^n$$

holds. By Proposition 1, we yield that $T(\Gamma^*)$ has rank n. All in all we have shown, that Γ^* is a crystallographic group of dimension n. □

Lemma 7. The following exact sequence splits:

$$1 \to T(\Gamma^*) \to \Gamma^* \to \Pi \to 1.$$

Furthermore, there exists a splitting $\sigma\colon \Pi \to \Gamma^*$ such that $\sigma(\Pi)$ has a fixed point $x \in \mathbb{E}^n$ and $(x + I)^{-1}\sigma(\Pi)(x + I) = \Pi$ holds.

Proof. Denote by $p\colon \Gamma^* \to \Pi$ the natural projection. Then for each $A \in \Pi$ we choose a preimage $\Phi_A \in \Gamma$ (observe the missing *). Then for each $A, B \in \Pi$ there must exist a $\tau(A, B) \in T(\Gamma)$, such that

$$\Phi_A \Phi_B = \tau(A, B)\Phi_{AB}$$

holds. Let $\Phi_A = a_A + A$ for each A. Then by computing the above expression and comparing the two sides we yield

$$\tau(A, B) = a_A + Aa_B - a_{AB} + I.$$

With this motivation in mind we define

$$f\colon \Pi \times \Pi \to L(\Gamma),\ (A, B) \mapsto a_A + Aa_B - a_{AB}.$$

Summing over this definition for all $B \in \Pi$, we arrive at the expression

$$\begin{aligned}\sum_{B\in\Pi} f(A, B) &= ma_A + A\sum_{B\in\Pi} a_B - \sum_{B\in\Pi} a_B \\ &= ma_A + (A - I)s,\end{aligned}$$

where $s \coloneqq \sum_{B\in\Pi} a_B$. With this we define

$$\sigma : \Pi \to \Gamma^*, \sigma(A) \coloneqq -\frac{1}{m}\sum_{B\in\Pi} f(A, B) + a_A + A = -\frac{1}{m}(A - I)s + A.$$

Looking at products, we see

$$\begin{aligned}\sigma(AB) &= -\frac{1}{m}(AB - I) + AB \\ &= -\frac{1}{m}(A - I) - \frac{1}{m}(AB - A) + AB \\ &= \sigma(A)\sigma(B).\end{aligned}$$

Therefore, σ is a homomorphism and the above sequence is split. Additionally, we see that $x \coloneqq \frac{s}{m}$ is a fixed point of $\sigma(A)$ for all $A \in \Pi$. Applying the conjugation with the by x induced translation shows, that this cancels σ. $\square$

Theorem (7.5.4, Bieberbach's theorem 3). Let $\xi\colon \Gamma_1 \to \Gamma_2$ be an isomorphism of n-dimensional crystallographic groups. Then there is an affine bijection α of $\mathbb{R}^n$ such that

$$\xi(\Phi) = \alpha\Phi\alpha^{-1}$$

holds for each $\Phi \in \Gamma_1$.

Proof. Since the $T(\Gamma_i)$ are characterized via their property to be uniquely maximal abelian, ξ restricts to an isomorphism between the two translational subgroups. Therefore, using the five lemma, we obtain an isomorphism $\bar{\xi}\colon \Pi_1 \to \Pi_2$. We try to extend ξ to the splitting groups. In order to do, we choose for each $A \in \Pi_1$ a preimage $\Phi_A \in \Gamma_1$. Then $\{\Phi_A \mid A \in \Pi_1\}$ is a set of coset representatives for $T(\Gamma_1^*)$ in Γ_1^*. Let τ be an arbitrary element in $T(\Gamma_1^*)$. Then we define

$$\xi^* : \Gamma_1^* \to \Gamma_2^*, \xi^*(\tau\Phi_A) \coloneqq [\xi(\tau^m)]^{\frac{1}{m}}\xi(\Phi_A).$$

Restricted to $T(\Gamma_1^*)$, this is easily seen to be a homomorphism and even an isomorphism between $T(\Gamma_1^*)$ and $T(\Gamma_2^*)$. Additionally, it agrees with $\bar{\xi}$. So if we can show that ξ^* is a homomorphism, we can automatically apply the five lemma which guarantees that it is an isomorphism.

The proof that we really found a homomorphism is an arduous computation with the central ingredient being the proof that we can commute the above formula in the sense that

$$\xi^*(\Phi_A \tau) = \xi(\Phi_A)[\xi(\tau^m)]^{\frac{1}{m}}$$

holds. With this in hand we can easily show the homomorphism property. However, we will skip this computation here.

As a last property we see that ξ^* extends ξ. Now, we can restrict our attention to ξ^* and if we can show that it is represented by an affine map, this is automatically also true for ξ. The advantage of working with Γ_i^* instead of Γ_i lies in the splitting property, which we can exploit. Let $\sigma_i \colon \Pi_i \to \Gamma_i^*$ be the splitting homomorphisms from Lemma 7. By conjugating the Γ_i^* with the translations induced by the respective fixed points, we may assume that $\sigma_i(\Pi_i) = \Pi_i$ holds. ξ^* can be adapted accordingly also by conjugation with translations, i. e. conjugation with affine transformations. After that we have that every element in Γ_i^* has the form τA, where τ lies in $T(\Gamma_i^*)$ and A lies in Π_i.

Next, we choose a basis $(v_1, \ldots, v_n)$ of $L(\Gamma_1)$. Using ξ we can push this to a basis $(w_1, \ldots, w_n)$ of $L(\Gamma_2)$. Now we define $\alpha \in \operatorname{Aut}(\mathbb{R}^n)$ by $\alpha(v_j) = w_j$. For $A \in \Pi_1$ and $a \in L(\Gamma_1^*)$ we have

$$\begin{aligned} A(a+I)A^{-1} &= Aa + I \quad \text{and hence} \\ \xi(A)\alpha(a) + I &= \xi^*(A)(\alpha(a)+I)\xi^*(A)^{-1} \\ &= \xi^*(A(a+I)A^{-1}) \\ &= \xi^*(Aa+I) \\ &= \alpha A a + I. \end{aligned}$$

In short, we yield $\xi^*(A)\alpha = \alpha A$ or $\xi^*(A) = \alpha A \alpha^{-1}$. All in all we get

$$\xi^*(\tau A) = \xi^*(\tau)\xi^*(A) = (\alpha\tau\alpha^{-1})(\alpha A \alpha^{-1}) = \alpha(\tau A)\alpha^{-1},$$

which proves the claim. □

Corollary 1. Two n-dimensional crystallographic groups are isomorphic if and only if they are conjugate in the group of affine bijections of $\mathbb{R}^n$.

References

[1] John G. Ratcliffe. *Foundations of hyperbolic manifolds*. Second. Vol. 149. Graduate Texts in Mathematics. Springer, New York, 2006, pp. xii+779. ISBN: 978-0387-33197-3; 0-387-33197-2.