Wintersemester 2015/16

University of Heidelberg

Riemannian geometry seminar

The 8 geometries of 3-manifolds

by
Sven Grützmacher

supervised by:
Dr. Gye-Seon Lee

Contents

Introduction

It is a well known result that all closed 2-manifolds can all be realised as a quotient of $X = E^2, S^2$ or $\mathbb{H}^2$ by a freely acting subgroup of $\text{Isom}(X)$. So we can say that every closed 2-manifold has a geometric structure modelled on one of the spaces above.

These metrics are very nice since the space looks the same at every point and in every direction one can look...it has a lot of symmetry.

Naturally we ask ourselves if the same result holds for higher dimensional manifolds and in this talk we want to take a look at the case just one dimension higher. It turns our that there are more geometries than E^3, S^3 and $\mathbb{H}^3$ which also means that this extra dimension gives rise to more less-symmetric geometries and spaces.

To be able to talk about these problems we start in section 1 by defining what we mean by „M has a geometry modelled on X" and review very few things about orbifolds.

After this we consider a special class of fibred spaces called Seifert fibre spaces. Their structure as S^1- bundles together with 2 invariants will be a powerfull tool to classify most of the geometries we face in 3 dimensions.

With that we are ready to take a short look at all 8 geometries. In this talk we will only give detailed insights on 2 of the geometries, i.e. two that have a structure as Seifert fibre spaces.

The last section will discuss Thursons geometrization conjecture. We will state the conjecture, a few facts and end this talk by not only showing Seifert fibre bundles can be modelled on those geometries but also that spaces which are modelled on certain geometries have to admit a structure as a Seifert fibre bundle.

Geometries and orbifolds

We should first ask ourselves what a „geometry" should be. Starting with Thurston's definition we can get:

Definition 1. Let F be a manifold. We say F posseses a geometry modelled on X if F is the quotient of X by a subgroup Γ of $\text{Isom}(X)$ such that the projection is a covering map.

As mentioned before, in two dimensions we can only choose X to be either S^3, E^3 or $\mathbb{H}^3$. Adding one dimension now makes a big difference as it is easy to find $3-$manifolds that cannot posses a structure modelled on one of those 3 spaces as our next example shows:

Example 2. $S^2 \times S^1$ has no geometry modelled on E^3, S^3 or $\mathbb{H}^3$.

Proof. We consider the universal cover $S^2 \times \mathbb{R}$ of $S^2 \times S^1$. First of all this space is not homeomorphic to one of the three spaces above, but it still admits a nice metric given simply by the product-metric.

Now intuitively $S^2 \times \mathbb{R}$ has a different geometry because the space looks different when we look in a different direction. On the one hand we have a „curved" direction along S^2 and on the other hand we have a „flat" direction along $\mathbb{R}$.

To understand this argument mathematically we can take a look at the isometry group and the stabilizer on $S^2 \times \mathbb{R}$. In the case of constant curvature $\text{Isom}(X)$ acts transitively (in every point X looks the same) and the stabilizer S_p is $O(3)$ (X looks the same in every direction from p). If we compute this on $S^2 \times \mathbb{R}$ the isometry group still acts transitively but the stabilizer of a point turns out to be isomorphic to $O(2) \times \mathbb{Z}_2$.

Thus the geometric structure is different. □

This example leads to a similar definition of what a geometry should be:

Definition 3. Let M be a manifold

a) a metric on M is said to be **locally homogeneous** iff given x and y we can find neighborhoods U and V and an isometry $U \longrightarrow V$

b) we say M admits a geometric structure iff M has a complete, locally homogeneous metric

Note that such a metric always gives us a metric on a cover $\tilde{M}$ such that the covering map is an isometry. Since the universal cover is simply connected one can show that such a metric is always homogeneous in this case, i.e. the isometry group acts transitively on the universal cover. Thus we can now try to understand what possible geometries there can be.

But for the sake of simplicity we will exclude the non-orientable cases in this talk. Thus fibred klein bottles are not all that interesting here.

To complete this section we want to take a short look at orbifolds. I will not give the defition here because it can be found in [Sco83] or Harry's talk. But there is one interesting invariant, the Euler-Characteristic. This number is not only used to classify 2-dimensional orbifolds but will also be a nice tool to classify some of the geometries later on.

Definition 4. Let be $\mathcal{O}$ a compact orbifold triangulated by a finite simplicial complex such that Γ_x is constant (up to isomorphism) on the images of the interior of the simplices. Let $\Gamma(\sigma)$ be the group on the interior of a simplex σ. Then

$$\chi^{orb}(\mathcal{O}) \coloneqq \sum_{\sigma_i} (-1)^{\dim(\sigma_i)} \frac{1}{|\Gamma(\sigma_i)|}$$

where $\mathcal{O}$ is triangulated by $\{\sigma_i\}$. For simplicity we will simply write $\chi(X)$.

Seifert fibre spaces

Definition 5. a) A **fibred solid torus** is solid torus which is finitely covered by a trivial fibred solid torus (that is $S^1 \times D^2$ with the product foliation by circles).

b) A **fibred solid klein bottle** is solid klein bottle which is finitely covered by a trivial fibred solid torus.

c) A **Seifert fibre space** is a $3-$manifold which admits a decomposition into disjoint circles S_α (called fibers) such that every fiber has a neighborhood isomorphic to a fibred solid torus or klein bottle.

Note that a fibred solid torus can be easily contructed from a trivial fibred solid torus by cutting along a disk $\{x\} \times D^2$ and glueing back with a q/p-th turn. This object will then be p-fold covered by a trivial solid torus and we will call ist $T(p,q)$. To get a fibred solid klein bottle glue via a reflection.

Since a Seifert fibre space M is foliated by circles the idea comes up to shrink every circle to a point. Doing so we then obtain a surface X and a projection map $M \longrightarrow X$. If we first consider a trivial fibred torus X is clearly the disk D^2 and the map $M \longrightarrow X$ is the usual covering map. Taking a closer look at $T(p,q)$ we can use the p-fold cover to induce a homeomorphism on D^2 by a rotation by $2\pi/p$. Thus X can be identified with cone orbifold with cone angle $2\pi/p$. Now $M \longrightarrow X$ fails to be a covering map in the usual sense but we will still see it as one in a generalized way. To sum this up we get:

Definition 6. Let M be a Seifert fibre space and X be the surface one obtains by collapsing every fibre to a single point. Then

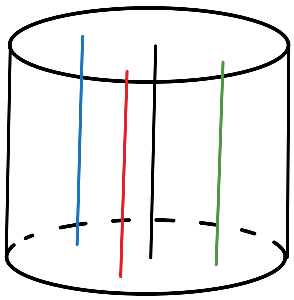
i) X is called the **base space** of M (an orbifold in general)

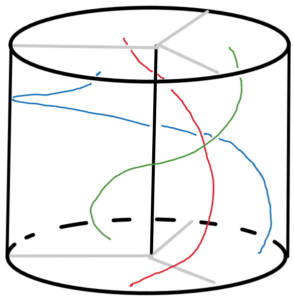
ii) we call X togehter with the generalized covering map $M \longrightarrow X$ a circle bundle, namely a **Seifert bundle**

iii) a fiber whose neighborhood is not a trivial torus is called **critical**

To give a short intuition we want to take a look at examples:

Example 7. a trivial fibred torus and a fibred torus





It has been shown that every Seifert fibre space M is aspherical unless it is covered by S^3 or $S^2 \times \mathbb{R}$, i.e. all homotopy groups except π_1 vanish. Hence we should take a short look at $\pi_1(M)$ which has in fact a rather special structure:

Lemma 8. *Let M be a seifert fibre space with base orbifold X. Then we have an exact sequence*

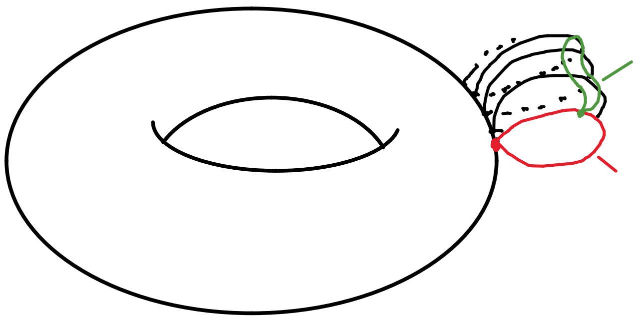
$$1 \longrightarrow K \longrightarrow \pi_1(M) \longrightarrow \pi_1(X) \longrightarrow 1$$

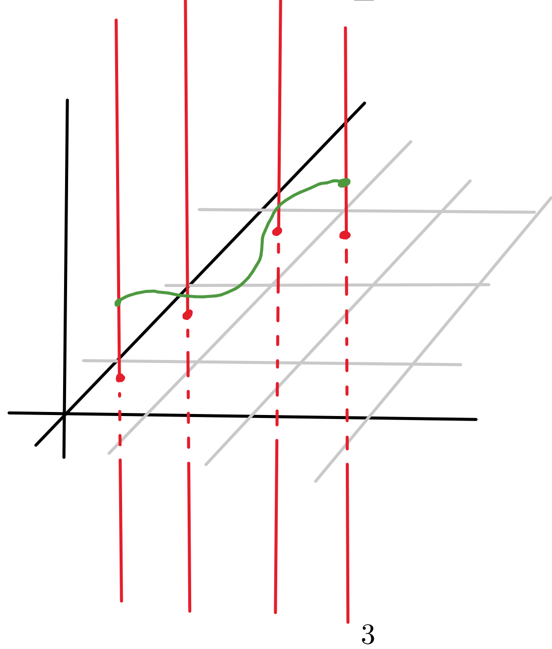
where $K < \pi_1(M)$ is cyclic and generated by a regular fibre. Furthermore K is only finite when M is covered by S^3.

Proof. Since $\pi_1(M)$ acts on $\tilde{M}$ by decktransformation and thus preserves the fibration $(p = p \circ f)$ this gives us an action on $\tilde{X}$ by acting on the resprective points. Since this is a deck transformation on $\tilde{X}$ we get an action of $\pi_1(M)$ on $\pi_1(X)$ resulting in the surjectiv homomorphism on the right.

If we consider the kernel of this action this have to be the covering translations which project to the identity on $\tilde{X}$. Thus the kernel K acts freely and has to be cyclic. The finiteness comes from the fact that S^3 is simply connected. □

Example 9. Let $M = T^2 \times S^1$ and thus $\tilde{M} = \mathbb{R}^3$.





To finish this section and thus our overview of Seifert fibre spaces we take a look at a fiber invariant $e(\eta)$ if η is our Seifert bundle which in a sense measures how far off we are from getting a section in our bundle.

Let M be our bundle over a base space $X \neq S^2$ which will be a regular surface without singularities for now. Our lemma above gives us the exact sequence

$$1 \longrightarrow K \longrightarrow \pi_1(M) \longrightarrow \pi_1(X) \longrightarrow 1$$

where K is infinite cyclic (why not finite?). Let $K = \langle k \rangle$ and note that if X is not closed $\pi_1(X)$ is free and thus this sequence splits.

Now let X be closed and consider the representation (that's why we need $X \neq S^2$)

$$\pi_1 X = \langle \bar{a}_1, \bar{b}_1, \ldots, \bar{a}_g \bar{b}_g \mid \prod_{i=1}^{g} [\bar{a}_i, \bar{b}_i] \rangle$$

Now take an element $a_i \in \pi_1(M)$ that projects to $\bar{a}_i$ and a b_i resprectively. Because the sequence is exact it follows that $\prod_{i=1}^{g} [a_i, b_i]$ lies in the kernel of $\pi_1(M) \longrightarrow \pi_1(X)$ and thus in K. Since K is cyclic we obtain an integer r such that

$$\prod_{i=1}^{g} [a_i, b_i] = k^r$$

To ensure that r coincides with the euler number of the unit tangent bundle of a closed orientable surface we say:

Definition 10. Let η be a Seifert bundle. Then the integer $e(\eta) = -r$ is called the **euler number** of η.

Note that we only defined this for bundles which have a regular surface as base space. In the general case one would also consider the orbit-invariants (p, q) of critical fibers. In that case one would first define an invariant b and then sacrifice some information to get e. For more information on this see [Sco83]. For this talk it will make no difference and I will mention it when needed.

In order to show later results concerning the 8 geometries we need to observe the following:

Theorem 1. *Let η be a Seifert bundle over a closed orbifold X with total space M. Let $\tilde{M}$ be a finite covering of degree d so that $\tilde{M}$ is the total space of a Seifert bundle $\tilde{\eta}$ over an orbifold $\tilde{X}$.*

Let l be the degree of the covering $\tilde{X} \longrightarrow X$ and m be the degree with which fibers of $\tilde{\eta}$ cover fibers of η, so that $lm = d$.

Then we get $e(\tilde{\eta}) = e(\eta)\frac{l}{m}$.

This finally leads to a very useful theorem to distinguish Seifert bundles later on.

Theorem 2. *If a Seifert bundle η has compact total space M, then M possessses a finite covering which is a trivial circle bundle over a surface , with the induced Seifert bundle structure iff $e(\eta) = 0$.*

The 8 geometries

Thurston's theorem involves 8 different geometries of which we will only examine two more closely. First we have the three well known geometries $E^3, S^3, \mathbb{H}^3$ like in dimension

2. But, as said in the introduction we can now find more candidates, i.e. the product spaces $\mathbb{H}^2 \times \mathbb{R}$ and $S^2 \times \mathbb{R}$, Nil and Sol and the universal cover of $Sl_2(\mathbb{R})$ denoted by $\widetilde{Sl_2}$.

In this talk we set focus on $\mathbb{H}^2 \times \mathbb{R}$ and $\widetilde{SL_2(\mathbb{R})}$ as examples. We will see, that they (among 6 of the 8 above) admit a structure as a Seifert fibre space and can be classified by $e(\eta)$ and $\chi(X)$ where X is the base space. Our two candidates share the property $\chi(X) < 0$ which led to this decision.

Overview

Before we start I want to give a short overview of the six other geometries.

The geometry of E^3
This is the well known euclidian 3-space, i.e. the geometry of constant curvature zero. The isometries are given by maps $x \mapsto Ax + b$ where $A \in O(3)$. If we put this together we get a exact sequence

$$1 \longrightarrow \mathbb{R}^3 \longrightarrow \mathrm{Isom}(E^3) \longrightarrow O(3) \longrightarrow 1$$

because the kernel of $\mathrm{Isom}(E^3) \longrightarrow O(3)$ is exactly the set of all translations. In contrast to the two dimensional case the isometries are not only rotations and translations but also srew motions.If you follow [Sco83] the conclusion will be that there are exactly 10 closed manifolds with a geometry modelled on E^3. In regard of the last chapter these are exactly the Seifert fibre spaces where both χ and e vanish.

The geometry of $\mathbb{H}^3$
The hyperbolic three space is the geometry with negative curvature -1. It is well known especially because of the work of Milnor and Thurston which is the reason why I won't say very much here.
There are a few constructions of $\mathbb{H}^3$ like taking the upper half-space $\mathbb{R}^3_+ = \{(x, y, z) \mid z > 0\}$ with the metric $ds^2 = \frac{1}{z^2}(dx^2 + dy^2 + dz^2)$ or the sphere with radius i in the minkowski-space.
The group of orientation preserving isometries of $\mathbb{H}^3$ are exactly the Möbius transformations on $\mathbb{C} \cup \{\infty\}$. For further information the works of Thurston and Milnor will offer detailed results.

The geometry of S^3
This is the geometry with constant positive curvature and can simply be seen as the set of all ordered pairs $(z, \omega) \in \mathbb{C}^2$ such that $|z|^2 + |\omega|^2 = 1$. Like all other spheres S^3 can be embedded in $\mathbb{R}^4$. Like in one dimension less the geodesics are given bei $S^3 \cap P$ where P is a two-plane through the origin. So geodesics are also also circles. The isometry group is simply given by $O(4)$.
As with E^3 the geometries modelled on S^3 are all connected to the theory of Seifert fibre spaces. If one considers the induced fibre structure we get $\chi > 0$ and $e \neq 0$.

The geometry of $S^2 \times \mathbb{R}$
This is the first of the two product geometries and possibly the easiest one since there are one seven manifolds which admit a geometric structure on $S^2 \times \mathbb{R}$. Also the isometry group can be easily identified with the product $\mathrm{Isom}(S^2) \times \mathrm{Isom}(\mathbb{R})$. Since we take a closer look at the other product geometry I will only give you the list of the seven manifolds:
non-compact:$S^2 \times \mathbb{R}$, two line bundles over P^2
compact:two S^2-bundles over S^1, $P^2 \times S^2$ and $P^3 \# P^3$

The geometry of Nil

Nil is the 3-dimensional Lie group obtained by considering all 3×3 matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

under multiplication. This group is nilpotent and one gets the sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow Nil \longrightarrow \mathbb{R}^2 \longrightarrow 0$$

this lets us see Nil as a line bundle over $\mathbb{R}^2$. If we take a look at the isometry group we find a similar sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathrm{Isom}(Nil) \longrightarrow \mathrm{Isom}(E^2) \longrightarrow 1$$

where all isometries are orientation preserving.

The final interesting fact is that manifolds modelled on Nil have a natural sturcture as a Seifert fibre space. This time the invariant are $\chi = 0$ and $e \neq 0$.

The geometry of Sol

This geometry is the one with the least symmetry. One way to define it is to consider the action of $\mathbb{R}$ on $\mathbb{R}^2$ by $(x, y) \longrightarrow (e^t x, e^{-t} y)$ thus giving us the exact sequence

$$0 \longrightarrow \mathbb{R}^2 \longrightarrow Sol \longrightarrow \mathbb{R} \longrightarrow 0$$

This geometry is one of the two which cannot be modelled as a Seifert fibre space. In fact if the geometry of Sol can be seen as the geometry of a torus bundle over S^1. So in contrast to the one-dimensional fibres of Seifert fibre space we have two-dimensional fibres here.

This should give us a rough overview over six of the eight geometries. We now take a look at the last two geometries in a little more detail

$$\mathbb{H}^2 \times \mathbb{R}$$

This is the second of the two product geometries that can occur as candidates for Thurstons conjecture. In contrast to $S^2 \times \mathbb{R}$ we can find infinitely many manifolds that have a geometry modelled on $\mathbb{H}^2 \times \mathbb{R}$. Simpy take any hyperbollic surface an consider the product with $\mathbb{R}$. Because of the product structure we know that $\mathrm{Isom}(\mathbb{H}^2 \times \mathbb{R})$ is naturally isomorphic to $\mathrm{Isom}(\mathbb{H}^2) \times \mathrm{Isom}(\mathbb{R})$.

Obviously $\mathbb{H}^2 \times \mathbb{R}$ is a fibre space and thus we get the following lemma:

Lemma 11. *Any 3-manifold which admits a geometry modelled on $\mathbb{H}^2 \times \mathbb{R}$ admits a foliation by lines or circles.*

Proof. Because $\mathrm{Isom}(\mathbb{H}^2 \times \mathbb{R}) \cong \mathrm{Isom}(\mathbb{H}^2) \times \mathrm{Isom}(\mathbb{R})$ consider $(\alpha, \beta) \in \mathrm{Isom}(\mathbb{H}^2 \times \mathbb{R})$. Then $(\alpha, \beta)(\{x\} \times \mathbb{R}) = \{\alpha(x)\} \times \beta(\mathbb{R})$ and thus the fibration is invariant under the action of the isometry group.

It follows that if a 3-manifold admits a geometry modelled on $\mathbb{H}^2 \times \mathbb{R}$ it is a quotient under the isometry group. Thus the foliation by lines $\{x\} \times \mathbb{R}$ descends to a foliation by lines or cirlces. $\square$

Now that we know our manifold M admits a fibration the next question is wether this is the structure of a Seifert fibre space or not. To determine this we take a look at the main theorem of this subsection

Theorem 3. *Let G be a discret group of isometries of $\mathbb{H}^2 \times \mathbb{R}$ which acts freely and has quotient M. Then one of the following holds:*

i) the natural foliation of $\mathbb{H}^2 \times \mathbb{R}$ descends to a Seifert bundle structure on M

ii) the structure descends to a structure of a line bundle over some hyperbolic surface

iii) the foliation descends to a foliation by lines in which each line has non-closed image in M. G must be isomorphic to one of $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$ or the Klein bottle group

Proof. We regard both groups $\mathrm{Isom}(\mathbb{H}^2)$ and $\mathrm{Isom}(\mathbb{R})$ as subgroups of $\mathrm{Isom}(\mathbb{H}^2 \times \mathbb{R})$ in a natural way.

Since G is discrete we have that $K = G \cap \mathrm{Isom}(\mathbb{R})$ must be discrete. As G also acts freely it has to be torsion free and thus K is either 1 oder $\mathbb{Z}$. In either case K is a normal subgroup of G and clearly K is the kernel of $G \longrightarrow \mathrm{Isom}(\mathbb{H}^2)$. Thus we get an exact sequence

$$1 \longrightarrow K \longrightarrow G \longrightarrow \Gamma \longrightarrow 1$$

where Γ is the image of $G \longrightarrow \mathrm{Isom}(\mathbb{H}^2)$.

Now let K be $\mathbb{Z}$ which means $K \cong (1, \mathbb{Z}) \subset \mathrm{Isom}(\mathbb{H}^2 \times \mathbb{R})$. If we now consider $(\mathbb{H}^2 \times \mathbb{R})/K$ we see that each line $\{x\} \times \mathbb{R}$ descends to a circle over some point $z \in \mathbb{H}^2$. Thus we have a Seifert fibre bundle and this gives us i).

If K is trivial then $G \cong \mathrm{Isom}(\mathbb{H}^2)$ and if Γ is discrete it must act freely and torsion free. Thus it is easy to see that M is a line bundle over $\mathbb{H}^2/\Gamma$ with fibers $\mathbb{R}$ and so we get ii).

The last case is then that Γ is not discrete. This part will be left for the reader and can be found in [Sco83]. □

The reason why I didn't show the whole proof is simple. As we have seen, the manifolds of case ii) cannot be closed and the same holds for those in case iii). Regarding this talk we are only interested in closed manifolds and to theorem 3 could have been reduced to just stating i). The last step we have to do is to see which invariants these Seifert fibre bundles have.

Proposition 12. *Let M be a closed manifold that admits a geometry modelled on $\mathbb{H}^2 \times \mathbb{R}$ an let η be the Seifert fibre bundle over the orbifold X for this structure. Then $\chi(X) < 0$ and $e(\eta) = 0$.*

Proof. Let Γ be the image of $G \longrightarrow \mathrm{Isom}(\mathbb{H}^2)$ as in the proof above. Then it is clear that $X = \mathbb{H}^2/\Gamma$ and so $\chi(X) < 0$.

As discussed earlier we find a finite orientable cover $\tilde{M}$ that is a circle bundle $\tilde{(\eta)}$ over a closed orientable surface $\tilde{X}$. Because of Theorem 1 if suffices to show that $e(\tilde{\eta}) = 0$.

Let $\bar{a}_i, \bar{b}_i$ be the standard generators of $\pi_1(\tilde{X})$ and a_i, b_i elements of $\tilde{G}$ that project on the generators. Remember that $\pi_1(\tilde{M}) = \tilde{G} < G$. Thus we get

$$\prod_{i=1}^{g} [a_i, b_i] = k^{e(\tilde{\eta})}$$

where $\langle k \rangle = K = \ker(\tilde{G} \longrightarrow \pi_1(\tilde{X}))$.

Because the isometry group splits nicely into its 2 components we have $a_i = (\bar{a}_i, t_i)$ $b_i = (\bar{b}_i, s_i)$ for some isometries s, t of $\mathbb{R}$. Let $(x, y) \in \mathbb{H}^2 \times \mathbb{R}$ then

$$k^{e(\tilde{\eta})}(x, y) = \prod_{i=1}^{g} [a_i, b_i](x, y) = \left(x, \prod_{i=1}^{g} [t_i, s_i] y \right) = (x, y)$$

Because orientation preserving isometries of $\mathbb{R}$ are just its translations. Thus $k^e = 1$ und so it follows that $e(\eta) = 0$. □

$\widetilde{SL_2(\mathbb{R})}$

The last of the 8 geometries we have to consider. The Lie group Sl_2 is the group of all 2×2 matrices with determinant one. If we want to talk about geometries on its universal cover $\tilde{Sl}_2$ we first have to get a metric on this space and take a look at the isomtries.

If we try to fully understand everything going on here this would be too much for this talk so I will just briefly explain how to get a metric on $\tilde{Sl}_2$ and give some useful facts.

Let M be a n-dimensional riemannian manifold. Then there is a natural metric on the tangent bundle TM which is 2n-dimensional. It is well known that an isometry $f : M \longrightarrow M$ induces an isometry $TM \longrightarrow TM$. If we applay this knowledge to $M = \mathbb{H}^2$ we get a metric on $T\mathbb{H}^2$ which then induces a metric on the unit tangent bundle $U\mathbb{H}^2$. Now we can identify $U\mathbb{H}^2$ with PSl_2 and thus get a metric there. Because PSl_2 is double covered by Sl_2 which itself is double covered by its universal cover $\tilde{Sl}_2$ we get our metric here. The following diagram visualizes this construction:

$$\begin{tikzcd}
 & & \widetilde{Sl_2} \arrow[d] \\
\mathbb{H}^2 \arrow[r, dashed] & T\mathbb{H}^2 & Sl_2 \arrow[d] \\
 & U\mathbb{H}^2 \arrow[u, phantom, "\cup"] \arrow[r, leftrightarrow] & PSl_2
\end{tikzcd}$$

To include some facts about $\widetilde{Sl_2}$ we can observe that it has the structure of a line bundle over $\mathbb{H}^2$. Furthermore $\mathbb{R}$ acts as an isometry on the universal cover by translating the fibers.

Like in the subsection before we want to prove a theorem that tells us what structures we can have:

Proposition 13. *Let G be a discret group of isometries of $\widetilde{Sl_2}$ acting freely with quotient M. The foliation of $\widetilde{Sl_2}$ by vertical lines descends to a foliation on M and one of the following occurs:*

i) the foliation gives M the structure of a line bundle over a non-closed surface

ii) it is a Seifert fibration

iii) the foliation is by lines whose image in M is not closed. There G must be isomorphic to $\mathbb{Z}, \mathbb{Z} \times \mathbb{Z}$ or the Klein bottle group

Proof. As in the last section we will only prove i) and ii). The start of this proof is the same like the one for $\mathbb{H}^2 \times \mathbb{R}$. We have the exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathrm{Isom}(\widetilde{Sl_2}) \longrightarrow \mathrm{Isom}(\mathbb{H}^2) \longrightarrow 1$$

if now $K = G \cap \mathbb{R}$ an Γ is again the image in $\mathrm{Isom}(\mathbb{H}^2)$ we get

$$0 \longrightarrow K \longrightarrow G \longrightarrow \Gamma \longrightarrow 1$$

As before K can either be infinite cyclic or trivial and case ii) tolds in the first case whereas i) holds in the latter if Γ is discret. $\square$

Like before the only interesting case for us is ii) since the other cases can not occur for closed manifolds.

Thurston's classification

Not lets set our eyes on the main goal of this talk: the geometrization theorem of Thurston. Before I state it I want to say a few words. First of all a *geometry* shall be a pair (X, G) where X is a manifold and G a group acting transitively on X with compact stabilizers. Note that we say two geomtries are equivalent if X is diffeomorphic to X' such that the action of G lands on the action of G'. Since we can always consider the universal cover we restirct ourselves to simply connected spaces where G is maximal, i.e. the case $(\mathbb{R}^2, \mathbb{R}^2)$ is omitted. We also restrict ourselves to those geometries that admit a compact quotient, that is there exists $H < G$ such that X/H is compact.

Theorem 4. *Any maximal, simply connected, 3-dimensional geometry which admits a compact quotient is equivalent to one of the geometries* $(X, \mathrm{Isom}(X))$ *where* X *is one of the following*

$$\begin{array}{cccc} E^3 & H^3 & S^3 & S^2 \times \mathbb{R} \\ \mathbb{H}^2 \times \mathbb{R} & \widetilde{SL_2(\mathbb{R})} & Nil & Sol \end{array}$$

Theorem 5. *If* M *is a closed* $3-$*manifold which admits a geometry modelled on one of the eight above, then the geometry involved is unique.*

Theorem 6. *Let* M *be a closed* $3-$*manifold.*

i) M *posseses a geometrix structure modelled on Sol iff* M *is finitely covered by a torus bundle over* S^1 *with hyperbolic glueing map.*

ii) M *posseses a geometric structure on one of the eight except* $\mathbb{H}^3$ *and Sol iff* M *is a Seifert fibre space. Furthermore if* M *has the structure of a Seifert fibre bundle* η *over an orbifold* X *then the geometry is determined as follows*

	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e = 0$	$S^2 \times \mathbb{R}$	E^3	$\mathbb{H}^2 \times \mathbb{R}$
$e \neq 0$	S^3	Nil	$\widetilde{Sl_2}$

Proof. In this talk we will only proof a little piece of this theorem while the rest can be found in [Sco83].

i) (perhaps)

ii) here we only prove the cases where $\chi < 0$ and $e = 0$, i.e. $\mathbb{H}^2 \times \mathbb{R}$. A big part has already been done in (cite lemma) so that we only need to show that every Seifert fibre space with these invariants admits a geometric structure.

So let η be a Seifert bundle over an orbifold X such that $\chi(X) < 0$ and $e(\eta) = 0$. Thus we get

$$1 \longrightarrow K \longrightarrow \pi_1(M) \longrightarrow \pi_1(X) \longrightarrow 1$$

where $K = \langle k \rangle$ is infinite cyclic. Now again choose generators $\bar{a}_i, \bar{b}_i, \bar{x}_i$ for $\pi_1(X)$ and elements a_i, b_i, x_i in $\pi_1 M$ that project to those generators. Thus we get

$$\pi_1(X) = \left\langle \bar{a}_1, \bar{b}_1, \ldots, \bar{a}_g, \bar{b}_g, \bar{x}_1, \ldots, \bar{x}_q \mid \bar{x}_i^{\alpha_i} = 1, \prod_{i=1}^{g} [\bar{a}_i, \bar{b}_i] \bar{x}_1 \ldots \bar{x}_q = 1 \right\rangle$$

with the same argument as before we get $\prod_{i=1}^{g} [a_i, b_i] x_1 \ldots x_q = k^{-b}$ and $x_i^{\alpha_i} k^{\beta_i} = 1$.

Now because $\chi(X) < 0$ we can see X as a quotient of $\mathbb{H}^2$ by $\pi_1(X)$ and we just need to define the isometries of $\mathbb{H}^2 \times \mathbb{R}$ that we want. So let

$$\begin{aligned} k &= id_{\mathbb{H}^2} \times \tau_1 & a_i &= \bar{a}_i \times id_{\mathbb{R}} \\ x_i &= \bar{x}_i \times \tau_{-\beta_i/\alpha_i} & b_i &= \bar{b}_i \times id_{\mathbb{R}} \end{aligned}$$

where τ_t is a translation by t.

This defines a group action on $\mathbb{H}^2 \times \mathbb{R}$ by some group G which has all the relations from above except perhaps the last one. So let $\omega = \prod_{i=1}^{g} [a_i, b_i] x_1 \dots x_q \in G$. By construction this maps to $id_{\mathbb{H}^2}$ when projected on the first factor an to a translation by $-\sum \beta_i/\alpha_i$ if projected on the second factor.

Now it would be important to distinguish between the bundle invariants b and e. As I mentioned when defining this invariant there are some differences when X is not simply a closed surface. In this case the additional knowledge would give us $0 = e = -b - \sum \beta_i/\alpha_i$ which then lets us see $\omega = k^b$. Thus G has all relations from the presentation above. In fact, G is isomorphic to $\pi_1(M)$.

The last step is to see that G actually acts free on $\mathbb{H}^2 \times \mathbb{R}$ und thus the quotient is a Seifert fibre space with invariants $\chi(X) < 0$ and $e = 0$. Thus it is isomorphic to the given Seifert bundle.

□

References

[Sco83] Peter Scott. *The geometries of 3-manifolds.* Bull. London Math Soc., 1983.