Heidelberg University

Seminar-Lecture 7 SS 16
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Homogeneous Spaces

"Characterization, construction and important examples"

1 What's the deal?

The main aim for studying how groups act on other mathematical objects through their representations ,is to detect internal symmetries of the object. By operating with a particular group and observing what remains unchanged/invariant under the action, we understand internal properties of the structure on which we operating.
Homogeneous spaces are a particular class of manifolds that behave per construction very symmetrically under the action of some groups, and they can be fully reconstructed just by looking at their behaviour under curtain actions. Namely any point on the manifold is correlated pairwise to any other just by the operation of elements of the group. The exact definition follows.

2 Review on group actions

Today we are going to restrict ourselves into Lie group actions on manifolds. So let G be always a Lie group and M a manifold. So we can recap some important definitions that were used in previous lectures,and we will also use today.

Definition 1 ***(Left actions):*** *A left action of a Lie group G on M is a map:*

$$\begin{aligned} \theta : G \times M &\longrightarrow M \\ (g,\ p) &\longmapsto g \cdot p =: \theta_g(p) \end{aligned}$$

s.t. $g_1 \cdot (g_2 \cdot p) = (g_1 \cdot g_2) \cdot p$ *and* $e \cdot p = p$; $g_1, g_2 \in G,\ p \in M,\ e$ *identity in* G.

Definition 2 ***(Summary of basic notions on group actions):***

1. *An action is said to be continuous if the defining map of the action is continuous.*

2. *A manifold with such a G-action (G, M, θ) is called a **G-space**. If M is smooth and the action is smooth then we call M a smooth G-space.*

3. *For all $g \in G$ is $\theta_g : M \longrightarrow M$ a homeomorphism with inverse $\theta_g^{-1} : M \longrightarrow M$. In particular if the action is smooth then θ_g is a diffeomorphism on M.*

4. *The **orbit** of $p \in M$ under the action of G is defined as:*

$$G \cdot p := \{g \cdot p : g \in G\} < M$$

5. *The **isotropy group** of $p \in M$ is defined as:*

$$G_p := \{g \in G : g \cdot p = p\} < G$$

6. *The action is said to be **free** if:*

$$G_p = \{e\};\ \forall p \in M$$

7. *a continuous action is said to be* ***proper*** *if the map:*

$$G \times M \longrightarrow M \times M$$
$$(g, p) \mapsto (g \cdot p, p)$$

is a proper map. Where proper map is defined as a map between topological spaces such that, its preimage of a compact subset is compact itself. A useful characterization of proper actions we have seen last time was: that an action is proper iff following holds: For two sequences $(p_i)_i; \forall p_i \in M$ *and* $(g_i)_i; \forall g_i \in G$ *s.t. both* $(p_i)_i, (g_i \cdot p_i)_i$ *converge in* M*, then a subsequence of* $(g_i)_i$ *converges in* G*.*

8. *And the most important notion for today is the following: The action is said to be* ***transitive*** *if:*

$$\forall p, q \in M \ \exists g \in G : \ p = g \cdot q$$

9. *A map* $F : M \to N$ *for manifolds* M, N *is said to be* ***equivariant*** *if the following diagram commutes:*

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[d, "\theta_g"'] & N \arrow[d, "\phi_g"] \\ M \arrow[r, "F"'] & N \end{tikzcd}$$

where ϕ, θ *denote the actions of* G *on* N, M *respectively.*

3 Homogeneous spaces and their construction

Definition 3 ***(Homogeneous-space):*** *A smooth manifold* M *endowed with a* <u>*transitive*</u>*,* <u>*smooth*</u> *action by a Lie group is called a Homogeneous G-space or just* ***Homogeneous-space****.*

Example 1 ***(Basic Examples):*** *Before we start exploring their properties lets take a look at some examples to gain intuition on how they work.*

1. The Euclidean group:

   $\mathcal{E}(n) := \mathbb{R}^n \rtimes_\phi O(n); \ (b, A)(b', A') := (b + \phi_A(b'), AA'), \ \phi_A(x) := Ax$
   Acting on $\mathbb{R}^n$ *via:*
   $\mathcal{E}(n) \curvearrowright \mathbb{R}^n : \ (b, A) \cdot x := b + \phi_A(x) = b + Ax$

   These are the rigid-body motions. The action is obviously transitive as we can obtain any vector from any other just by linear transformations in $\mathbb{R}^n$*. So* $\mathbb{R}^n$ *becomes a homogeneous space under the action of* $\mathcal{E}(n)$*.*

2. $\mathbb{S}^{n-1}$ vs $O(n)$:

   We observe the action $O(n) \times \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ as the restriction of the natural action $GL(n) \curvearrowright \mathbb{R}^n$, hence is the restriction smooth because $\mathbb{S}^{n-1}$ is an embedded submanifold of $\mathbb{R}^n$. This action is also transitive, by taking a unit vector and rotating it orthogonally in $\mathbb{R}^n$.

3. The Möbious group: *The group $SL(2, \mathbb{R})$ operates transitively on the uper half plane $\mathbb{U} := \{z \in \mathbb{C} : \mathfrak{Im}(z) > 0\}$ under*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}$$

   The induced diffeomorphisms on $\mathbb{U}$ are exactly the Möbius-transformations.

The following theorems 2 and 5 are of great importance, because the first one shows us how to construct homogeneous spaces and the other one classifies them in terms of the first, as quotients of Lie groups by closed subgroups. For these we have to recall the statement of the quotient manifold theorem that we have seen last time:

Theorem 1 ***(Recall of the quotient manifold theorem/QMT:)*** *Suppose G is a Lie goup acting <u>smoothly,freely</u> and <u>properly</u> on a smooth manifold M.Then the orbit space M/G is a topological manifold of dimension equal to $dim(M) - dim(G)$, and has a unique smooth structure with the property that the quotient map $\tilde{\pi} : M \longrightarrow M/G$ is a smooth submersion (that is a differentiable map between differentiable manifolds whose differential is everywhere surjective).*

Theorem 2 ***(Homogeneous-space construction theorem)*** *Let G be a Lie group and let H be a closed subgroup of G.(i) The left coset space G/H is a topological manifold of dimension equal to $dim(G) - dim(H)$, and has a unique smooth structure s.t. the quotient map $\pi : G \longrightarrow G/H$ is a smooth submersion. (ii)The left action of $G \curvearrowright G/H$ given by:*

$$g_1 \cdot (g_2 H) = (g_1 g_2)H$$

turns G/H into a homogeneous-space.
<u>Proof:</u>
At first we observe how H operates from the right on G so that we can use the quotient manifold theorem on $H \curvearrowright G$ and give a natural smooth structure to G/H. This operation is:

$$\begin{aligned} H \times G &\longrightarrow G \\ (h, g_1) &\mapsto g_1 h =: g_2 \end{aligned}$$

- *It follows directly from the closed subgroup theorem,that H is a properly embedded Lie subgroup of G.*

- *$H \curvearrowright G$ is smooth because the operation coincides with the restriction of the multiplication operation of G:*

$$m : G \times G \longrightarrow G \ \textit{restr.} \rightarrow m_{|H} : H \times G \longrightarrow G$$

- *The action is in particular free:*

$$g \in G, h \in H \subset G \implies [gh = g \implies h = e]$$

- *The action is also proper: Let $(g_i)_i, (h_i)_i$ be Sequences in G and H respectively s.t. $(g_i)_i$ and $(g_ih_i)_i$ both converge in G. By the continuity of the inverse map we have that $h_i = g_i^{-1}(g_ih_i)$ also converges in G and because H is closed in G, with the subspace topology ,follows that (h_i) converges in H. This is as we have denoted above equivalent to the fact that $H \curvearrowright G$ is proper.*

So we have a smooth, free and proper action of a Lie group H on the smooth manifold G. Then follows from the QMT that G/H has a unique smooth structure as a quotient manifold, s.t. the quotient map $\pi : G \longrightarrow G/H$ is a smooth submersion.

Now after all these considerations we can go back and try to understand the action $G \curvearrowright_\theta G/H$. Consider the following diagram:

$$\begin{tikzcd} G \times G \arrow[r, "m"] \arrow[d, "Id_G \times \pi"'] & G \arrow[d, "\pi"] \\ G \times G/H \arrow[r, "\theta"'] & G/H \end{tikzcd}$$

where $Id_G \times \pi$ is a smooth submersion as a product of those and

$$\theta : (g_1, g_2H) \mapsto g_1 \cdot (g_2H) := (g_1g_2)H.$$

Under this last map we hope to turn $(G, G/H)$ into a homogeneous space.To end proof by proving that θ is well defined and smooth we have to make use of the following ,very fundamental theorem of differential geometry:

Theorem 3 ***(Passing smoothly to the quotient):*** *Let M, N, P be smooth manifolds and $\pi : M \longrightarrow N$ a surjective smooth submersion. If P is a smooth manifold and $F : M \longrightarrow P$ a smooth map that is constant on the fibers of π (that is : $[\pi(p) = \pi(q) \implies F(p) = F(q)]$), then there exist a unique smooth map $\theta : N \longrightarrow P$ such that $\theta \circ \pi = F$.*

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "F"] & \\ N \arrow[r, dashed, "\exists!\theta"'] & P \end{tikzcd}$$

So to make use of this theorem we have to prove that the combination:

$$\pi \circ m : G \times G \to G \to G/H$$

is constant on the fibers of $Id_G \times \pi$: To see that let $g_1, g_2 \in G$ and so we have:

$$\begin{aligned}
(Id_G \times \pi)(g_1, g_2) =& (g_1, g_2 H) \\
(Id_G \times \pi)^{-1}(g_1, \bar{g_2}) =& \{(g_1, g) | gH = g_2 H\} \\
=& : (G \times G)_{(g_1, g_2)} \\
\pi \circ m_{|(G\times G)_{(g_1,g_2)}} :& (G \times G)_{(g_1, g_2)} \to G/H \\
& (g_1, g) \mapsto \pi(g_1 g) = g_1 g H = g_1 g_2 H \ \forall g
\end{aligned}$$

$\implies \pi \circ m$ constant on the fibers of $Id_G \times \pi$.
The uniqueness of θ in Theorem 3 gives us the required smooth well defined structure we wanted and per definition gives θ a group action.This action is finally also transitive: Let $(g_1, g_2) \in G \times G$. Then $g_2 g_1^{-1} \in G$ and satisfies the following relation:

$$\theta(g_2 g_1^{-1}, g_1 H) = (g_2 g_1^{-1}) \cdot g_1 H = g_2 H$$

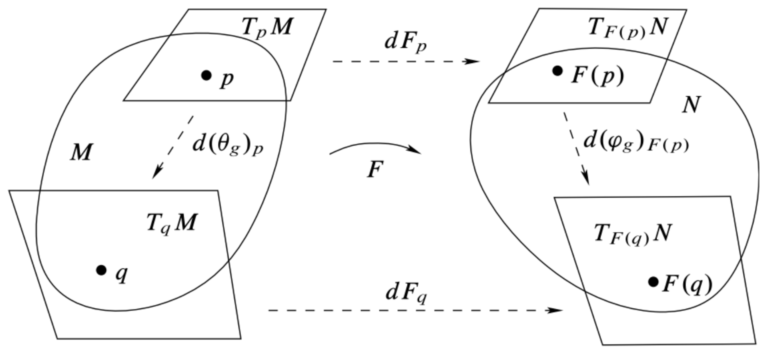
that is,we connect $\bar{g_1}, \bar{g_2}$ *through* $g_2 g_1^{-1}$. So the action is indeed transitive, and this completes the proof of the construction theorem.

Theorem 4 ***(Equivariant map theorem):***
Let M and N be smooth manifolds and G a Lie group. Suppose $F : M \longrightarrow N$ is a smooth map that is equivariant with respect to a transitive smooth G action on M and any smooth G action on N. Then F has constant rank.Thus, if:

1. *F is surjective, then it is a smooth submersion,*
2. *F is injective, then it is a smooth immersion,*
3. *F is bijective, then it is a diffeomorphism.*

That is, in the form of a commutative diagram:

$$\begin{tikzcd}
T_p M \arrow[r, "dF_p"] \arrow[d, "d(\theta_g)_p"'] & T_{F(p)} N \arrow[d, "d(\phi_g)_{F(p)}"] \\
T_q M \arrow[r, "dF_q"'] & T_{F(q)} N
\end{tikzcd}$$



Definition 4 ***(Orbit map):***
On the frame of this theorem we can define, for an action $\theta : G \times M \longrightarrow M$ at each $p \in M$, a map:

$$\theta^{(p)} : G \longrightarrow M$$
$$g \mapsto g \cdot p$$

*which we will call the **orbit map**. This map has some important features:*

- *its image lays completely in the orbit $G \cdot p$*
- *its preimage is the isotropy group of p, that is: $\underline{(\theta^{(p)})^{-1}(p) = G_p}$.*
- *it is smooth: $G \times \{p\} \hookrightarrow G \times M \xrightarrow{\theta} M$*
- *In particular it is equivariant: $\theta^{(p)}(gg') = (gg') \cdot p = g \cdot (g' \cdot p) = g \cdot \theta^{(p)}(g')$*
- *and has constant rank (Equivariant rank theorem + Transitivity of the action).*

Theorem 5 ***(Homogeneous-space characterization theorem):*** *Let G be a Lie group, let M be a homogeneous G-space, and let p be any point of M. The isotropy group G_p is a closed subgroup of G, and the map*

$$F : G/G_p \longrightarrow M$$
$$gG_p \mapsto g \cdot p$$

is an equivariant diffeomorphism.
Proof:
As we have seen right above, the isotropy groups are given by the inverse of the orbit map,which is continuous and smooth as it is induced by the group action and p closed $\implies$ G_p closed. With this we conclude that the isotropy groups are closed Lie Subgroup of G.
Now we check if F is indeed well defined:

$$
\begin{aligned}
g_1G_p = g_2G_p &\iff \exists h \in G_p : g_2 = g_1h \\
F(\bar{g_2}) &= g_2 \cdot p = g_1h \cdot p = g_1 \cdot F(\bar{h}) \\
&= g_1 \cdot F(\bar{1}) = g_1 \cdot p = F(\bar{g_1})
\end{aligned}
$$

F is an equivariant map:

$$F\left(\overline{g'g}\right) = g'g \cdot p = g'F(\bar{g})$$

and is also smooth.This follows from directly from Theorem 3 if we consider the following diagram:

$$
\begin{tikzcd}
G \arrow[d, "\pi"'] \arrow[dr, "\theta^{(p)}"] & \\
G/G_p \arrow[r, dashed, "!F"'] & M
\end{tikzcd}
$$

Here is π by definition a smooth surjective submersion. $\theta^{(p)}$ is constant on the fibers of π :

$$
\begin{aligned}
\pi^1(g_1G_p) &= \{g|\{\underbrace{gG_p = g_1G_p}_{\exists\xi\in G_p:g=g_1\xi}\} \\
\theta^{(p)}(g) &= g \cdot p = g_1 \cdot \xi p \\
&\qquad = g_1 \cdot p = \theta^{(p)}(g_1)\forall g \in \pi^{-1}(g_1G_p)
\end{aligned}
$$

and therefore there is a unique smooth map F with the wished properties.
At last F is bijective:
It is sujective as: Let $q \in M$ arbitrary, then follows from transitivity:

$$\exists g \in G : g \cdot p = q \implies F(\bar{g}) = q$$

and injective as: $g_1, g_2 \in G$:

$$
\begin{aligned}
F(\bar{g_1}) = F(\bar{g_2}) &\implies g_1 \cdot p = g_2 \cdot p \implies (g_2^{-1}g_1) \cdot p = p \\
&\implies (g_2^{-1}g_1) \in G_p \implies \bar{g_1} = \bar{g_2}
\end{aligned}
$$

$$\implies [F \textit{is equivariant+ smooth+ bijective}] \overset{Th4}{\implies} F \textit{is a Diffeomorphism}$$

This theorem allows us to characterize every Homogeneous G-space M as a quotient of the Lie group G by a closed subgroup, G/H, and namely by an isotropy group of an element in M. Under these new insights we can return to the examles we discussed before and write them as quotients of Lie groups.

Example 2 *"A wiser approach to the former excitation of our intuition":*

1. *On the Example of the Euclidean group we see that the isotropy group of the origin is the subgroup* $O(n) < \mathcal{E}(n)$*, then is* $\mathbb{R}^n$ *diffeomorphic to* $\mathcal{E}(n)/O(n)$*.*

2. *On the natural action* $O(n) \curvearrowright \mathbb{S}^{n-1}$ *lets say we take as basis point* $p = (0, ..., 0, 1)$*-the north pole , then we see that the isotropy group is* $O(n-1)$ *by keeping p steady and rotating the sphere around the frozen axis that goes through the center and p. Thus by using the characterization theorem we see* $\mathbb{S}^{n-1} \cong O(n)/O(n-1)$ *(diffeomorphic).*

3. *On the transitive action of* $SL(2,\mathbb{R})$ *by Möbius transformations on* $\mathbb{U}$*, has the point* $i \in \mathbb{U}$ *the isotropy group:*

$$\begin{aligned} SL(2,\mathbb{R})_i &= \{C_{(a,b,c,d)} \in SL(2,\mathbb{R}) | a = d, c = -b;\ det(C_{(a,b,c,d)}) = 1\} \\ &= SO(2) < SL(2,\mathbb{R}) \end{aligned}$$

   So from theorem 3 we obtain a diffeomorphism: $\mathbb{U} \longrightarrow SL(2,\mathbb{R})/SO(2)$.

   -Further important examples are:

4. Grassmanians: $Gr(r, n) \cong O(n)/(O(r) \times O(n-r))$

5. Projective space: $P^{n-1} \cong PO(n)/PO(n-1)$

6. Hyperbolic space: $\mathcal{H}^n \cong O^+(1, n)/O(n)$

7. Anti-de Sitter space: $AdS_{n+1} \cong O(2, n)/O(1, n)$

In particular these examples can be endowed with the special structure of a "symmetric space" which will be explained in detail in the next lecture.

Remark: As a last remark, we have to make clear that not every smooth manifold can become a homogeneous space under some special action! There are, most importantly, topological constraints for becoming a homogeneous space and in particular it has been proven by G.Mostow in 2005 that a compact homogeneous space isn't allowed to have negative Euler characteristic χ. Example: for the case of an orientable surface it is $\chi = 2 - 2g$, where g the denotes genus of the surface, that would mean that a double torus($g = 2, \chi = -2$) is directly out of the game for becoming homogeneous, although the torus $\mathbb{T}$ ($g = 1, \chi = 0$) is trivially homogeneous under $\mathbb{T} = \mathbb{C}/\mathbb{Z}^2$.