Introduction

In this paper, we are going to introduce the Farey tessellation. Since it is closely related to the once-punctured torus, we will start with the construction of a hyperbolic once-punctured torus. Then, we will get to know tessellations. Intuitively, a tessellation is some special cover, that is similar to the tiling of a kitchen floor. The Tessellation theorem gives a connection between the once-punctured torus and a special tessellation of the hyperbolic plane $\mathbb{H}^2$. Subsequently, we get to know the Farey circle packing, a collection of circles in $\mathbb{H}^2$. From this circle packing, we construct the so-called Farey Tessellation and show that it is indeed a tessellation of $\mathbb{H}^2$. Finally, we investigate how we can vary certain parameters in the construction of this tessellation, such that the result again is a tessellation.

1. The once-punctured torus

The euclidean torus is obtained by gluing the sides of a square, usually the unit square. If we now consider the one-punctured torus, we just remove one point. Without loss of generality, this point is the point corresponding to the four vertices glued together. However, this surface, equipped with the euclidean metric, is not complete, since we can find a Cauchy-sequence that does not converge.

Fortunately, this is not the only way to construct the once-punctured torus. In this section, we will glue the sides of a hyperbolic square to obtain a surface equipped with a hyperbolic metric that is (homeomorphic to) the once-punctured torus. We will later show that it is complete.
Figure 1: The sides of the ideal rectangle $X$ with vertices $-1, 0, 1$ and $\infty$ can be glued to obtain a once-punctured torus.

Definition 1.1. A vertex of a hyperbolic polygon is an ideal vertex, if it is at infinity of $\mathbb{H}^2$, i.e. if it lies in $\mathbb{R} \cup \{\infty\}$. A hyperbolic polygon is an ideal polygon, if all its vertices are ideal.

We consider the ideal polygon $X$ in $\mathbb{H}^2$ with vertices at $-1, 0, 1$ and $\infty$. We label the edges with $E_1, \ldots, E_4$ and orient them as indicated in Figure 1. We now want to glue opposite edges of $X$ using hyperbolic isometries. There are various ways to do so, we decide to choose the following gluing isometries:

$$
\varphi_1 : E_1 \to E_2, \quad \varphi_1(z) = \frac{z + 1}{z + 2}; \\
\varphi_3 : E_3 \to E_4, \quad \varphi_3(z) = \frac{z - 1}{-z + 2}.
$$

Then $\varphi_1$ sends $-1$ to $0$ and $\infty$ to $1$, $\varphi_3$ sends $1$ to $0$ and $\infty$ to $-1$. We further set $\varphi_2 := \varphi_1^{-1}$ and $\varphi_4 := \varphi_3^{-1}$.

We denote by $d_X$ the hyperbolic metric on the polygon $X$:

$$
d_X(p, q) := \inf\{l_{hyp}(c) \mid c \text{ curve from } p \text{ to } q \text{ in } X\}.
$$

Since $X$ is convex, this is just the restriction of the hyperbolic metric $d_{hyp}$. Now the quotient metric space $(\tilde{X}, \tilde{d}_X)$ obtained from the metric space $(X, d_X)$ by performing the edge gluings is a hyperbolic surface, and it is homeomorphic to the once-punctured torus.

Remark. The metric $\tilde{d}_X$ on the quotient space is defined as follows: Let $\tilde{P}, \tilde{Q} \in \tilde{X}$. A discrete walk from $\tilde{P}$ to $\tilde{Q}$ is a sequence of points $\omega = P_1, Q_1, P_2, Q_2, \ldots, P_n, Q_n$ in $X$ such that $Q_i$ and $P_i$ are glued together and $P_1$ corresponds to $\tilde{P}$, $Q_n$ corresponds to $\tilde{Q}$. The
length of the discrete walk $\omega$ is defined as

$$l_X(\omega) = \sum_{i=1}^{n} d_X(P_i, Q_i).$$

Then one can prove that

$$\tilde{d}_X(\tilde{P}, \tilde{Q}) := \inf \{ l_X(\omega) \mid \omega \text{ discrete walk from } \tilde{P} \text{ to } \tilde{Q} \}$$

defines a semi-metric on $\tilde{X}$. In all cases we consider, it is a metric.

2. Tessellations

In this section, we will first introduce the notion of a tessellation of the hyperbolic plane. The so-called Tessellation theorem gives a connection between tessellations and surfaces obtained by gluing polygon edges. Poincaré’s polygon theorem will give us the means to decide whether such a surface is complete. To explore the meaning of these theorems in practice, we will apply them to the once-punctured torus considered in Section 1.

**Definition 2.1.** A **tessellation** of the hyperbolic plane is a family of tiles $(X_i)_{i \in I}$, with some index set $I$, such that

(i) for all $i \in I$, $X_i$ is a connected polygon in the hyperbolic plane $\mathbb{H}^2$;
(ii) any two $X_i, X_j$ are isometric;
(iii) the union of all $X_i$ covers the whole of $\mathbb{H}^2$;
(iv) the intersection of $X_i$ and $X_j$ for $i \neq j$ consists only of edges and vertices of $X_i$ which are also vertices or edges of $X_j$, i.e. the interiors of $X_i, X_j$ are disjoint;
(v) (Local Finiteness) for all points $P \in \mathbb{H}^2$ there is $\varepsilon > 0$ such that the hyperbolic ball $B_{d_{hyp}}(P, \varepsilon)$ meets only finitely many tiles $X_i$.

Analogously, one can define tessellations of the euclidean space $\mathbb{R}^2$ or the sphere $\mathbb{S}^3$.

Let now $X$ be a polygon in $\mathbb{H}^2$ with edges $E_1, \ldots, E_{2p}$ grouped in pairs $\{E_{2k-1}, E_{2k}\}$ together with gluing isometries $\varphi_{2k-1}: E_{2k-1} \to E_{2k}$ and $\varphi_{2k} := \varphi_{2k-1}^{-1}$. We can extend each $\varphi_i$ to an isometry of $\mathbb{H}^2$ such that $\varphi_i(X)$ is on the side of $\varphi_i(E_i)$ that is opposite of $X$. In other words, $\varphi$ maps a point in the interior of $X$ to a point outside of $X$. For instance, $X$ could be the ideal rectangle considered in Section 1, with gluing isometries as in (1.1).

**Definition 2.2.** The **tiling group** associated to $X$ and gluing isometries $\varphi_i$ is the group generated by the $\varphi_i$, i.e.

$$\Gamma := \{ \varphi \in \text{Isom}(\mathbb{H}^2) \mid \varphi = \varphi_{i_1} \circ \cdots \circ \varphi_{i_l} \text{ with } i_j \in \{1, \ldots, 2p\} \ \forall j \leq l \}. $$

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Note that the identity is contained in $\Gamma$, as it is the composition of zero gluing maps.

Let $d_X$ be the metric on $X$ as before, and let $(\tilde{X}, \tilde{d}_X)$ be the quotient metric space obtained from $X$ by performing the edge gluings.

**Theorem 2.3 (Tessellation theorem).** Let $X$ be a hyperbolic connected polygon with gluing data as above. Suppose that for each vertex of $X$ the angles of $X$ at all vertices that are glued to a point $P \in \tilde{X}$ sum up to $\frac{2\pi}{n}$ for some $n > 0$ depending on the point $P$. In particular, $(\tilde{X}, \tilde{d}_X)$ then is a hyperbolic surface with cone singularities. Suppose further that $(\tilde{X}, \tilde{d}_X)$ is complete. Then the family of polygons $\{\varphi(X)\}_{\varphi \in \Gamma}$ forms a tessellation of the hyperbolic plane.

Clearly, all $\varphi(X)$ for $\varphi \in \Gamma$ are connected hyperbolic polygons and since all elements of $\Gamma$ are hyperbolic isometries, any two of them are isometric. To prove that their union indeed covers the whole hyperbolic plane, their interiors are mutually disjoint and that they fulfill the Local Finiteness property requires some more thought. The idea behind the proof is to start with one tile, namely $X$, and progressively set one tile after the other - just as one would do tiling a kitchen floor. Potential problems that have to be ruled out are that the tiles do not cover all of $H^2$ or that at some point, they overlap.

**Definition 2.4.** A tile $\varphi(X)$ is called **adjacent** to $X$ at a point $P$, if there exists a sequence $\varphi_{i_1}, \ldots, \varphi_{i_l}$ of gluing maps such that

$$\varphi_{i_{j-1}} \circ \cdots \circ \varphi_{i_1}(P) \in E_{i_j} \forall j \leq l \quad \text{and} \quad \varphi = \varphi_{i_1}^{-1} \circ \cdots \circ \varphi_{i_l}^{-1}.$$  

More generally, two tiles $\varphi(X)$ and $\psi(X)$ are **adjacent** at $P \in \varphi(X) \cap \psi(X)$, if $\psi^{-1} \circ \varphi(X)$ is adjacent to $X$ at $\psi^{-1}(P)$.

Intuitively, the tiles adjacent to $X$ at a point $P$ are just the tiles that are 'neighbours' of $X$ and contain $P$. Let for instance $X$ be the ideal rectangle from Section 1 and let $P \in E_1$ be no vertex. Then the only tiles adjacent to $X$ at $P$ are $X$ itself (with gluing isometry $\varphi = id$) and $\varphi_2(X) = \varphi_1^{-1}(X)$. Note that here, $l = 1$ and $\varphi_{i_1} = \varphi_1$. The first condition in the definition of adjacency is satisfied since $\varphi_1(P) \in E_2$. For $P \in int(X)$, $X$ is the only tile adjacent to $X$ at $P$.

**Lemma 2.5.** There are only finitely many tiles that are adjacent to $X$ at a point $P \in X$.

*Proof.* See [Bon00, p.137ff].

**Corollary 2.6.** Under the hypothesis of the Tessellation theorem 2.3, for every $P \in X$, there is $\varepsilon > 0$ such that the tiles $\varphi(X)$ adjacent to $X$ at $P$ decompose the disk $B_d(P, \varepsilon) \subset H^2$ into finitely many hyperbolic disc sectors with disjoint interiors.
Lemma 2.7. Every $P \in \mathbb{H}^2$ is covered by some $\varphi(X)$ with $\varphi \in \Gamma$.

Proof. We pick a base point $P_0 \in \text{int}(X)$. Let $P \in \mathbb{H}^2$ be an arbitrary point and let $g$ be the unique hyperbolic geodesic joining $P_0$ to $P$ (Figure 2). Denote by $P_1$ the point where $g$ leaves $X$. At $P_1$, $g$ enters one of the finitely many tiles that are adjacent to $X$ at $P_1$. We denote it by $\psi_1(X)$. Repeating this process yields a sequence $(P_n)_{n \in \mathbb{N}}$ of points on $g$ and of tiles $(\psi_n(X))_{n \in \mathbb{N}}$ such that $g$ leaves $\psi_{n-1}(X)$ and enters $\psi_n(X)$ at $P_n$. Note that $\psi_n$ is not always uniquely determined, i.e. when $g$ follows an edge separating two tiles adjacent to $\psi_{n-1}(X)$ at $P_n$. When $g$ enters a tile $\psi_n(X)$ and never leaves it, this process terminates. This happens exactly when $P \in \psi_n(X)$. The proof that this indeed is the case after finitely many steps makes use of the fact that the interior angles glued together sum up to $\frac{2\pi}{n}$, such that $(\bar{X}, \bar{d}_X)$ is a hyperbolic surface, and in particular (and this is the crucial part) uses the compactness of $(\bar{X}, \bar{d}_X)$. For details, see [Bon00, p.141ff].

Definition 2.8. $\varphi(X)$ is called canonical tile for $P$ with respect to the base point $P_0$, if $\varphi(X)$ is adjacent to $\psi_n(X)$ at $P$ and $\psi_n(X)$ is the last tile needed to cover $g$.

In particular, if $P$ is an interior point of $\psi_n(X)$, then $\psi_n(X)$ is the only canonical tile for $P$.

Lemma 2.9. For every $P \in \mathbb{H}^2$, there is $\varepsilon > 0$ such that for all $P' \in B_{d_{hyp}}(P, \varepsilon)$ the canonical tiles for $P'$ are exactly the canonical tiles for $P$ containing $P'$.

Sketch of proof. With the notation as before, let $P$ be contained in the tile $\psi_n(X)$ and let $T$ be the collection of all the tiles $\psi_i(X)$ for $i = 0, \ldots, n$ and all tiles adjacent to them. Then for any point $Q \in g$, there is an $\varepsilon > 0$ such that the ball $B_{d_{hyp}}(Q, \varepsilon)$ is contained in the union $\bigcup_{Y \in T} Y$. If we move $P$ to a point $P'$ that is close to $P$ (for instance at distance $< \varepsilon$), then $g$ moves to a geodesic $g'$ (close to $g$) joining $P_0$ to $P'$ (Figure 2). The tiling process of $g'$ only involves tiles of $T$, so in particular, the final tile $\psi_n'(X)$ is adjacent to $\psi_n(X)$. If $P \in \text{int}(X)$, then (choosing $\varepsilon$ sufficiently small) also $P' \in \text{int}(X)$ and the claim
follows. If $P$ lies on an edge of $\psi_n(X)$, but is not a vertex, there are exactly two tiles adjacent to $\psi_n(X)$ at $P$. Since $P'$ is close to $P$, it lies in one (or both) of these tiles, and the claim follows. If now $P$ is a vertex of $\psi_n(X)$, $P'$ is not a vertex of $\psi'_n(X)$ (for $\varepsilon$ sufficiently small), but is contained in one or two tiles adjacent to $\psi_n(X)$ at $P$, regarding to whether it is interior point of $\psi'_n(X)$ or lies on an edge. In any case, the canonical tiles for $P'$ are exactly the tiles adjacent to $\psi_n(X)$ at $P$ containing $P'$.

Lemma 2.10. Let $\varphi \in \Gamma$ and $P, Q \in \text{int}(\varphi(X))$. If $\varphi(X)$ is canonical for $P$ it is also canonical for $Q$. Further, $\varphi(X)$ is the only canonical tile for $P$ and $Q$.

Proof. See [Bon00, p.144].

Lemma 2.11. Every tile $\varphi(X)$ is canonical for some $P$ in its interior.

Proof. Assume first that the tile $\varphi(X)$ is canonical for some $P$ in its interior, and let $\varphi_i$ be a gluing map. We claim that $\varphi \circ \varphi_i(X)$ is canonical for some $P'$ in its interior. The tiles $\varphi(X)$ and $\varphi \circ \varphi_i(X)$ meet at an edge $\varphi \circ \varphi_i(E_i)$. Let $Q$ be a point on this edge that is not a vertex. Let $P' \in \text{int}(\varphi(X))$ be sufficiently close to $Q$. By Lemma 2.10, $\varphi(X)$ is the only canonical tile for $P''$, and by Lemma 2.9, $\varphi(X)$ is also canonical for $Q$. Any canonical tile for $Q$ now has to be adjacent to $\varphi(X)$. Since $Q$ is not a vertex, the only canonical tiles for $Q$ are $\varphi(X)$ and $\varphi \circ \varphi_i(X)$. Let $P'$ be a point in $\varphi \circ \varphi_i(X)$. If $P'$ is sufficiently close to $Q$, again Lemma 2.9 gives that the canonical tile for $P'$ has to be $\varphi(X)$ or $\varphi \circ \varphi_i(X)$. Since $P' \notin \varphi(X)$, $\varphi \circ \varphi_i(X)$ is canonical for $P'$, as we claimed. Since the tile $X$ is canonical for the point $P_0$, it now follows inductively that any tile $\varphi(X)$ is canonical for some point $P$ in its interior.

Thanks to all the auxiliary lemmata we have just proven, we are now ready to prove our main theorem, the Tessellation theorem.

Proof of Theorem 2.3. We already know that the family $\{\varphi(X)\}_{\varphi \in \Gamma}$ satisfies (i) and (ii) from Definition 2.1. By Lemma 2.7 the union of these tiles covers all of $\mathbb{H}^2$, so (iii) is satisfied as well. Note that for applying Lemma 2.7, we make use of the assumptions that the interior angles sum up to $\frac{2\pi}{n}$ and that the resulting surface is compact. Now suppose that $P \in \text{int}(\varphi(X)) \cap \text{int}(\varphi'(X))$. Then by Lemma 2.10, $\varphi(X) = \varphi'(X)$ since there is a unique canonical tile for $P$. In particular, the interiors of $\varphi(X)$ and $\varphi'(X)$ are disjoint, hence (iv) is satisfied. To prove local finiteness, one considers one point $P \in \mathbb{H}^2$. $P$ has only finitely many canonical tiles, so there exists some ball $B_{\text{hyp}}(P, \varepsilon)$ contained in the union of these tiles. No other tile $\varphi(X)$ can meet this ball, since otherwise, its interior would meet the interior of one of the canonical tiles for $P$. By (iv), this cannot occur. In
total, the family \( \{ \varphi(X) \} \varphi \in \Gamma \) satisfies (i)-(v) from Definition 2.1, so it forms a tessellation of \( \mathbb{H}^2 \).

From Theorem 2.3 we know that, whenever a quotient space \((\hat{X}, \hat{d}_X)\) as above obtained from gluing a polygon is complete (and satisfies the angle-sum condition), it gives rise to a tessellation of \( \mathbb{H}^2 \). Hence, it would be useful to have some easy criterion to check whether or not such a quotient space is complete. If the polygon \( X \) is bounded, then the quotient space \( \hat{X} \) is compact and hence complete. That is why in the following, we only consider unbounded polygons with one or several ideal vertices. A criterion to check whether a quotient space obtained in this way is complete is given by Poincaré’s polygon theorem.

To state it, we first need some notation.

**Definition 2.12.** A horocircle centered at \( \zeta \in \mathbb{R} \cup \{ \infty \} \) is a curve \( C - \{ \zeta \} \), where \( C \) is an euclidean circle in \( \mathbb{H}^2 \) tangent to the real line at \( \zeta \). A horocircle centered at \( \infty \) is just a horizontal line. An isometry \( \varphi \) of \( \mathbb{H}^2 \) is called horocyclic at \( \zeta \) if it respects some horocircle centered at \( \zeta \). For \( \zeta = \infty \), \( \varphi \) then is a horizontal translation \( z \mapsto z + b \) or a reflection across at a vertical line \( z \mapsto \bar{z} + b \).

**Remark.** Since an isometry \( \varphi \) of \( \mathbb{H}^2 \) sends generalized (euclidean) circles to generalized (euclidean) circles, \( \varphi \) sends a horocircle centered at \( \zeta \) to some horocircle centered at \( \varphi(\zeta) \).

If \( \varphi \) is horocyclic at some \( \zeta \in \mathbb{R} \cup \infty \), then it respects every horocircle centered at \( \zeta \). This can easily be seen in the case \( \zeta = \infty \). For an arbitrary \( \zeta \) the claim follows from this special case by applying an isometry of \( \mathbb{H}^2 \) sending \( \zeta \) to \( \infty \).

Let \( \zeta \) be an ideal vertex of \( X \), endpoint of the edge \( E_i \). The gluing isometry \( \varphi_i \) sends \( \zeta \) to another ideal vertex \( \varphi_i(\zeta) \). We denote the element in the quotient space \( \hat{X} \) corresponding to \( \zeta \) by \( \bar{\zeta} \) and write \( \bar{\zeta} = \{ \zeta_1, \ldots, \zeta_k \} \), meaning that the ideal vertices \( \zeta_i \) are glued to \( \bar{\zeta} \).

**Lemma 2.13.** The indexing of the ideal vertices in \( \bar{\zeta} = \{ \zeta_1, \ldots, \zeta_k \} \) can be chosen such that there exist gluing maps \( \varphi_{ij} : E_{ij} \to E_{ij+1} \) with \( E_{ij+1} := E_{ij}^\prime \) depending on \( \varphi_{ij} \) for \( j = 1, \ldots, k \) satisfying

(i) \( \zeta_j \) is endpoint of \( E_{ij} \), \( \zeta_{j+1} \) is an endpoint of \( E_{ij+1} \) and \( \varphi_{ij}(\zeta_j) = \zeta_{j+1} \);

(ii) the edges \( E_{ij} \) and \( E_{ij}^\prime \) adjacent to \( \zeta_j \) are disjoint, i.e. \( \varphi_{ij} \neq \varphi_{ij-1}^{-1} \) for all \( 1 < j < k - 1 \);

(iii) exactly one of the following holds:

(a) there is a map \( \varphi_{ik} : E_{ik} \to E_{ik}^\prime \) such that \( \varphi_{ik} \) sends \( \zeta_k \) to \( \zeta_1 \), \( \zeta_k \) is an endpoint of \( E_{ik} \) and \( E_{ik}^\prime \) is not the image \( E_{ik}^\prime \) of \( \varphi_{ik-1} \) and such that \( \zeta_1 \) is an endpoint of \( E_{ik}^\prime \) and \( E_{ik}^\prime \) is not the domain \( E_{ik} \) of \( \varphi_{ik} \) or

(b) each of \( \zeta_1, \zeta_k \) is adjacent to a unique edge of \( X \), namely \( E_{i1} \) and \( E_{ik} \) respectively.
Figure 3: These horocircles at the vertices of $X$ fulfill the horocircle-condition. The figure contains an additional edge, the diagonal from 0 to $\infty$ which we will consider later on.

Sketch of proof. In the case that any vertex of $\zeta_i$ is adjacent to exactly two edges, the proof is straight-forward. We start with an arbitrary $\zeta := \zeta_1 \in \bar{\zeta}$ adjacent to some edge $E_{i_1}$ and consider the gluing isometry $\varphi_{i_1}: E_{i_1} \to E'_{i_1}$. We set $\zeta_2 := \varphi_{i_1}(\zeta_1)$ and go on like this. As there are only finitely many edges glued to $\bar{\zeta}$ we eventually reach some $k$ with $\zeta_{j+1} = \zeta_j$ for some $j \leq k$. If $k$ is the smallest such index, assume that $j > 1$. It then follows that $\zeta_1$ is adjacent to only one edge, contradicting our assumption. Hence $j = 1$ and we are in case (iii)(a). The other case, leading to (iii)(b) requires more thought. However, we are only going to use the first case. For details, see [Bon00, p.170f].

Lemma 2.14. The following properties are equivalent:

(i) Horocircle condition: At each ideal vertex $\zeta$ of $X$, one can choose a horocircle $C_\zeta$ centered at $\zeta$ such that whenever $\varphi_i: E_i \to E_{i+1}$ sends $\zeta$ to another ideal vertex $\zeta'$, then $\varphi_i(C_\zeta) = C_{\zeta'}$.

(ii) Edge cycle condition: For every edge cycle around an ideal vertex $\bar{\zeta} = \{\zeta_1, \ldots, \zeta_k\}$ with gluing maps $\varphi_{i_j}: E_{i_j} \to E_{i_{j+1}}$ sending $\zeta_j$ to $\zeta_{j+1}$ for $j = 1, \ldots, k$, the corresponding composition $\varphi_{i_k} \circ \cdots \circ \varphi_{i_1}$ is horocyclic at $\zeta_1$.

Proof. If the horocircle condition is valid, then for any edge cycle around $\bar{\zeta}$ as above, the composition $\varphi := \varphi_{i_k} \circ \cdots \circ \varphi_{i_1}$ sends $\zeta_1$ to itself, in particular it sends $C_\zeta$ to $C_{\zeta}$, so it is horocyclic at $\zeta_1$.

If on the other hand, the edge cycle condition holds and the gluing data around an ideal vertex $\bar{\zeta}$ is arranged as above, pick an arbitrary horocircle centered at $\zeta_1$. Then $C_{\zeta_j} := \varphi_{i_{j-1}} \circ \cdots \circ \varphi_{i_1}(C_{\zeta_1})$ for $j \leq k$ is a horocircle centered at $\zeta_j$ by construction. One easily checks that these horocircles fulfill the horocircle condition. Performing this construction for every ideal vertex proves the claim. □
We are now ready to state the polygon theorem.

**Theorem 2.15** (Poincaré’s polygon theorem). Let $(\bar{X}, \bar{d}_X)$ be the quotient space obtained by gluing the edges of a polygon $(X, d_X)$ in $\mathbb{H}^2$ using gluing maps $\varphi_i: E_i \to E_{i \pm 1}$. Then $(\bar{X}, \bar{d}_X)$ is complete if and only if one of the equivalent conditions of Lemma 2.14 is satisfied.

For the proof, we refer to [Bon00, Chapter 6.8].

To conclude the section on tessellations, we apply our two main theorems on the once-punctured torus. Let $X$ be the hyperbolic polygon with vertices $-1, 0, 1$ and $\infty$ considered in Section 1 with gluing isometries as in (1.1). We show that the horocircle condition from Lemma 2.14 is satisfied in this case.

Let $C_{\infty}$ be the horizontal line given by $\text{Im}(z) = 1$, and let $C_\zeta$ be the horocircle centered at $\zeta$ with radius $\frac{1}{2}$ for $\zeta = -1, 0, 1$ (Figure 3). Then $\varphi_1$ sends $\infty$ to 1 and the point $-2 + i \in C_{\infty}$ to $1 + i$. Hence, $\varphi_1(C_{\infty}) = C_1$. A similar computation gives the same result for the other gluing isometries. So $X$ satisfies the horocircle condition and therefore, by Poincaré’s polygon theorem, the once-punctured torus $(\bar{X}, \bar{d}_X)$ is complete. Since the inner angle at all ideal vertices of $X$ is 0, we can apply the Tessellation theorem and obtain: The family $\{\varphi(X)\}_{\varphi \in \Gamma}$, where $\Gamma$ is generated by the gluing isometries $\varphi_i$, forms a tessellation of the hyperbolic plane (Figure 4).

### 3. The Farey circle packing and tessellation

In this section, we get to know the Farey circle packing and the corresponding Farey tessellation and discover that it is closely related to the once-punctured torus. In the following, $X$ will denote the ideal hyperbolic polygon considered in Section 1.
For any \( \frac{p}{q} \in \mathbb{Q} \) considered, let \( p, q \) be coprime and \( q > 0 \). For any such \( \frac{p}{q} \) draw in \( \mathbb{R}^2 \) the circle \( C_{\frac{p}{q}} \) of diameter \( \frac{1}{q} \) that is tangent to the x-axis at \( (\frac{p}{q}, 0) \) and lies in the upper half-plane (Figure 5). We make the following observations:

(i) The circles \( C_{\frac{p}{q}} \) have disjoint interiors.

(ii) \( C_{\frac{p}{q}} \) and \( C_{\frac{p'}{q'}} \) are tangent if and only if \( pq' - p'q = \pm 1 \). We say that \( \frac{p}{q} \) and \( \frac{p'}{q'} \) form a Farey pair.

(iii) \( C_{\frac{p}{q}}, C_{\frac{p''}{q''}} \) and \( C_{\frac{p'}{q'}} \) with \( \frac{p}{q} < \frac{p''}{q''} < \frac{p'}{q'} \) are pairwise tangent to each other if and only if

\[
\frac{p''}{q''} = \frac{p}{q} \oplus \frac{p'}{q'},
\]

where \( \frac{p}{q} \oplus \frac{p'}{q'} := \frac{p + p' + q + q'}{q + q'} \) is called the Farey sum of \( \frac{p}{q} \) and \( \frac{p'}{q'} \).

The same holds if we consider \( \infty = \frac{1}{0} = -\frac{1}{0} \) and set \( C_{\infty} := \{(x, y) \in \mathbb{R}^2 \mid y = 1\} \) with interior points \( (x, y) \) with \( y > 1 \).

If \( \frac{p}{q} \) and \( \frac{p'}{q'} \) form a Farey pair, i.e. if \( C_{\frac{p}{q}} \) and \( C_{\frac{p'}{q'}} \) are tangent, we now connect \( (\frac{p}{q}, 0) \) and \( (\frac{p'}{q'}, 0) \) by a semi-circle. Erasing the circles \( C_{\frac{p}{q}} \), we are left with a collection of hyperbolic geodesics that looks similar to the tessellation belonging to the once-punctured torus. We call this the Farey tessellation of \( \mathbb{H}^2 \) of the hyperbolic plane - even if we do not know yet if it actually is a tessellation (Figure 7).

Let us get back to the ideal polygon \( X \). We split \( X \) along the diagonal from 0 to \( \infty \) into two triangles \( T^+ \) and \( T^- \). By the isometry \( z \mapsto -\bar{z} \), both triangles are isometric. Hence,
Figure 7: Connection the end-points of circles that are tangent to each other, we obtain a family of geodesics in $\mathbb{H}^2$, the so-called Farey-tessellation.

since the family $\varphi(X)_{\varphi \in \Gamma}$ forms a tessellation of $\mathbb{H}^2$, also

$$\mathcal{T} := \{ \varphi(T^+), \varphi(T^-) \}_{\varphi \in \Gamma}$$

forms a tessellation of $\mathbb{H}^2$.

**Theorem 3.1.** $\mathcal{T}$ is equal to the Farey tessellation, i.e. its edges are hyperbolic geodesics joining $\frac{p}{q}$ to $\frac{p'}{q'}$ whenever $pq' - p'q = \pm 1$.

For the proof of this theorem, we make use of two lemmata. Remember that $\varphi \in \text{PSL}_2(\mathbb{Z})$ is the group of linear fractional maps of the form

$$\varphi(z) = \frac{az + b}{cz + d} \text{ with } a, b, c, d \in \mathbb{Z}, \ ad - bc = 1.$$  

**Lemma 3.2.** Let $\frac{p}{q}, \frac{p'}{q'} \in \mathbb{Q} \cup \{ \infty \}$ form a Farey pair. Then for any $\varphi \in \text{PSL}_2(\mathbb{Z})$, the points $\varphi(\frac{p}{q})$ and $\varphi(\frac{p'}{q'})$ form a Farey pair as well.

**Proof.** Simple computation. \( \square \)

The pairs $\{0, \infty\}, \{0, 1\}$ and $\{1, \infty\}$ form Farey pairs, and similar the vertices of $T^-$. It follows that the endpoints of each edge of $\mathcal{T}$ form a Farey pair, since they are all images of edges of $T^+$ or $T^-$ under elements of $\Gamma \subseteq \text{PSL}_2(\mathbb{Z})$.

**Lemma 3.3.** Let $g_1, g_2$ be distinct geodesics in $\mathbb{H}^2$ whose endpoints form Farey pairs. Then $g_1$ and $g_2$ are disjoint.

**Proof.** Let the endpoints of $g_1$ be $\frac{p_1}{q_1}$ and $\frac{p'_1}{q_1}$ with indexing chosen such that $p'_1q_1 - p_1q'_1 = 1$. Then

$$\varphi(z) := \frac{q_1z - p_1}{-q'_1z + p_1}$$

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sends \( g_1 \) to the geodesic with endpoints 0 and \( \infty \). Let \( \varphi(g_2) \) have endpoints \( \frac{p_2}{q_2} \) and \( \frac{p'_2}{q'_2} \). Suppose that \( g_1 \) and \( g_2 \) meet. Then \( \varphi(g_1) \) and \( \varphi(g_2) \) meet as well, in particular \( \varphi(g_2) \) crosses the line from 0 to \( \infty \), so \( p_2 \) and \( p'_2 \) have different signs, as \( q_2, q'_2 > 0 \) by assumption. From Lemma 3.2 we know that \( \frac{p_2}{q_2} \) and \( \frac{p'_2}{q'_2} \) satisfy the relation \( p_2 q'_2 - p'_2 q_2 = \pm 1 \), what is not possible if \( p_2, p'_2 \) have different signs. Hence, \( g_1 \) and \( g_2 \) have to be disjoint.

Lemma 3.3 shows that any geodesic whose endpoints form a Farey pair must be an edge of the Farey tessellation: Since the tiles of \( T \) are ideal triangles, their interiors cannot contain any complete geodesic. If now \( g \) is a hyperbolic geodesic whose endpoints form a Farey pair and \( g \) meets an edge \( g' \) of \( T \), then \( g = g' \) by Lemma 3.3. As a result, the edges of the tessellation \( T \) are exactly the complete geodesics whose endpoints form a Farey pair, so \( T \) coincides with the Farey tessellation. This proves Theorem 3.1. The denomination Farey tessellation is now justified, since it is indeed a tessellation of \( \mathbb{H}^2 \).

**Remark.** We now may ask if, given the tessellation of \( \mathbb{H}^2 \) corresponding to the once-punctured torus, we can reconstruct the Farey circle packing. This indeed is possible: Recall the horocircles \( C_{\infty}, C_{-1}, C_0 \) and \( C_1 \) from the end of Section 2. Two of these horocircles meet if and only if they are tangent and this is exactly when their centers are the ends of an edge of the Farey tessellation. In the tessellation of the hyperbolic plane by the tiles \( \varphi(X), \varphi \in \Gamma \), the images of these horocircles \( C_\zeta \) for \( \zeta = -1, 0, 1, \infty \) form a family of horocircles, all centered at the images of \( -1, 0, 1 \) and \( \infty \) under \( \varphi \) (Figure 8). These are points \( \frac{p}{q} \in \mathbb{Q} \), and, just as their pre-images, two of these horocircles meet only when they are tangent, i.e. when their centers are the ends of an edge of the Farey tessellation. Computing the image of \( C_{\infty} \) under an isometry \( \varphi(z) = \frac{az + b}{cz + d} \in \text{PSL}_2(\mathbb{Z}) \) exactly, one finds that they are horocircles centered of \( \frac{p}{q} \) with diameter \( \frac{1}{\varphi} \). Hence, this family of horocircles coincides with the Farey circle packing.
4. Shearing the Farey tessellation

In Section 1, we glued the ideal polygon \( X \) using gluing isometries \( \varphi_i \) as in (1.1). We are now going to modify these gluing isometries slightly and check if this construction again gives rise to a tessellation of \( \mathbb{H}^2 \). We start with a basic result from hyperbolic geometry.

**Lemma 4.1.** Given a triple \( \zeta_1, \zeta_2, \zeta_3 \in \mathbb{R} \cup \{ \infty \} \) of distinct points and another such triple \( \zeta'_1, \zeta'_2, \zeta'_3 \in \mathbb{R} \cup \{ \infty \} \), there is a unique isometry of \( \mathbb{H}^2 \) sending each \( \zeta_i \) to the corresponding \( \zeta'_i \). Also, at each \( \zeta_i \) there is a unique horocircle \( C_i \) centered at \( \zeta_i \) such that any two \( C_i, C_j \) are tangent to each other and meet at a point of the complete hyperbolic geodesic going from \( \zeta_i \) to \( \zeta_j \).

**Proof.** For the first part, without loss of generality, we can assume that \( \zeta'_1 = 0, \zeta'_2 = 1 \) and \( \zeta'_3 = \infty \). The general case then is given by composition of isometries obtained in this special case. If \( \infty \notin \{ \zeta_1, \zeta_2, \zeta_3 \} \), an isometry \( \varphi \) satisfying \( \varphi(\zeta_i) = \zeta'_i \) is

\[
\varphi(z) = \frac{z - \zeta_1}{z - \zeta_3} \cdot \frac{\zeta_2 - \zeta_3}{\zeta_2 - \zeta_1}.
\]

If one of the \( \zeta_i \) is equal to \( \infty \), for instance \( \zeta_1 = \infty \), \( \varphi \) can be defined by

\[
\varphi(z) = \frac{\zeta_2 - \zeta_3}{z - \zeta_3}.
\]

The cases that \( \zeta_2 \) or \( \zeta_3 \) are equal to \( \infty \) are similar. For the second claim, it suffices to check for the points 0, 1 and \( \infty \), since by the first part, we can always get back to this case using a unique hyperbolic isometry. Here, we first consider the horocircle \( C_\infty \) - it is a straight line given by the equation \( \text{Im}(z) = k \) for some \( k > 0 \). Since the horocircles \( C_0 \) and \( C_1 \) are supposed to be tangent to \( C_\infty \), we conclude that both have radius \( \frac{k}{2} \). The only \( k \) for which \( C_0 \) and \( C_1 \) are tangent then is \( k = 1 \), hence the \( C_i \) are uniquely determined. The point where \( C_\infty \) and \( C_0 \) meet clearly lies on the geodesic from 0 to \( \infty \), the same holds for \( C_\infty \) and \( C_1 \). \( C_0 \) and \( C_1 \) meet at the point \( \frac{1}{2} + \frac{1}{2}i \), which is on the geodesic from 0 to 1 as well. \( \square \)

For any edge of an ideal triangle, we can now by Lemma 4.1 fix a base point, namely the point where the unique horocircles centered at its endpoints meet.

Recall the hyperbolic polygon \( X \) from Section 1 with gluing isometries \( \varphi_i \) as in (1.1). We can modify the gluing isometries as follows:

\[
\varphi_1: E_1 \to E_2, \quad \varphi(z) = \frac{z + 1}{z + a}, \quad \varphi_3: E_3 \to E_4, \quad \varphi(z) = \frac{z - 1}{-z + b}. \quad (4.1)
\]
Until now, we considered the special \( a = b = 2 \) and obtained a complete hyperbolic surface. However, for any \( a, b \in \mathbb{Z} \), \( \varphi_1 \) sends the edge \( E_1 \) going from \(-1\) to \( \infty \) to the edge \( E_2 \) going from 0 to 1, similarly \( \varphi_3 \) sends \( E_3 \) to \( E_4 \). We can therefore work with these more general gluing isometries; in this case, we do not know yet if the resulting surface is complete. We split \( X \) again as in Section 3 and obtain triangles \( T^+ \) and \( T^- \). Using Lemma 4.1, we find base points on each of the edges of \( X \). The base points \( P_i \) on the edges \( E_i \) are given as

\[
P_1 = -1 + i, \quad P_2 = \frac{1}{2} + \frac{1}{2} i, \quad P_3 = 1 + i \quad \text{and} \quad P_4 = -\frac{1}{2} + \frac{1}{2} i.
\]

One easily checks that for the case \( a = b = 2 \), the gluing isometries send base points to base point. This is why the horocircles corresponding to the tessellation fit nicely side-by-side (Figure 8). For arbitrary \( a, b \), this property does not hold. To see this, fix \( a \) and \( b \). Recall that \( \varphi_2(z) := \varphi_1^{-1}(z) = \frac{z-1}{z+1} \). Then the basepoint of \( E_1 = \varphi_2(E_2) \) corresponding to horocircles of \( \varphi_2(X) \) is \( \varphi_2(P_2) = -1 + i(a-1) \). Note that \( \varphi_2(P_2) \neq P_1 \) for \( a \neq 2 \). Since \( E_1 \) is a vertical line, by the formula for hyperbolic distance, we find

\[
d_{\text{hyp}}(\varphi_2(P_2), P_1) = |\log(a-1)|.
\]

Seen from the interior of \( X \), \( \varphi_2(P_2) \) is at signed distance \( s_1 := -\log(a-1) \) to the left of \( P_1 \) (compare Figure 9). Since \( \varphi_1 \) is a hyperbolic isometry, also \( P_2 = \varphi_1(\varphi_2(P_2)) \) is at distance \(|s_1|\) from \( \varphi(P_1) \). If we consider signed distances, we have to keep in mind that \( \varphi_1 \) sends the interior of \( X \) to the side of \( E_2 \) that is opposite to \( X \). Hence, as isometries are orientation-preserving, \( P_2 \) is at signed distance \( s_1 \) to the left of \( \varphi_1(P_1) \) seen from \textit{outside} of \( X \). Seen from the interior, \( \varphi_1(P_1) \) is at signed distance \( s_1 \) to the left of \( P_2 \). Similarly, the basepoint \( \varphi_4(P_4) \) on \( E_3 \) determined by \( \varphi_4(X) \) is at signed distance \( s_3 := \log(b-1) \) to the left of \( P_3 \) and on \( E_3 \), the base point \( \varphi_3(P_3) \) is at signed distance \( s_3 \) to the left of \( P_4 \), both seen from the interior of \( X \). We now consider any edge \( E \) of the partial tessellation of \( \mathbb{H}^2 \) associated to \( X \) and gluing isometries \( \varphi_1, \varphi_3 \). Then \( E = \varphi(E_1) \) or \( E = \varphi(E_3) \) for some element \( \varphi \in \Gamma \), and \( E \) separates the tile \( \varphi(X) \) from a tile \( \psi(X) \) for some \( \psi \in \Gamma \). If we transport our considerations from above to the tile \( \varphi(X) \), we see that, seen from the interior of \( \varphi(X) \), the base point determined by \( \psi(X) \) is at signed distance \( s_1 \) to the left of the base point determined by \( \varphi(X) \) if \( E = \varphi(E_1) \) and at signed distance \( s_3 \) to the left of the base point determined by \( \varphi(X) \) if \( E = \varphi(E_3) \).

Figure 9 illustrates the case \( s_1 = 0.25 \) and \( s_3 = -1 \). We denote the partial tessellation of \( \mathbb{H}^2 \) obtained in this way by \( \mathcal{T}_{s_1,s_3} \). We do not know yet if \( \mathcal{T}_{s_1,s_3} \) is a tessellation, as it does not necessarily need to cover the whole of \( \mathbb{H}^2 \). Every tile of \( \mathcal{T}_{s_1,s_3} \) corresponds to a tile of the tessellation \( \mathcal{T} = \mathcal{T}_{0,0} \). To obtain \( \mathcal{T}_{s_1,s_3} \) from \( \mathcal{T} \), we progressively slide all tiles to the...
Figure 9: If we shear the Farey tessellation according to shear parameters $s_1 = 0.25$ and $s_3 = -1$, the result is a partial tessellation of $\mathbb{H}^2$.

left along the edges by signed distance $s_1$ or $s_3$, according to whether the edge considered is an image of $E_1$ or $E_3$.

**Definition 4.2.** The partial tessellation $\mathcal{T}_{s_1,s_3}$ is obtained by shearing $\mathcal{T}$ according to the shear parameters $s_1$ and $s_3$.

We can generalize this construction once more by introducing an additional edge $E_5$, the diagonal from 0 to $\infty$. As before, we obtain two triangles $T^+$ and $T^-$. We replace $T^-$ by its image under the isometry $\varphi_5$ defined by $z \mapsto \exp^{-s_5}$ for some shear parameter $s_5$. Seen from the interior of $T^+$, we slide $T^-$ to the left along $E_5$ by distance $s_5$. We obtain a new ideal polygon $\tilde{X} := \varphi_5(T^-) \cup T^+$ (Figure 10).

Starting with this sheared polygon, we construct a partial tessellation of $\mathbb{H}^2$ as before, using the shear parameters $s_1$ and $s_3$. Remember that $s_1 = -\log(a-1)$ and $s_3 = \log(b-1)$, hence $a = e^{-s_1} + 1$ and $b = e^{s_3} + 1$. For gluing the sides of $\tilde{X}$, we use the isometries

$$\tilde{\varphi}_1(z) := \varphi_1 \circ \varphi_5^{-1}(z) = \frac{e^{s_5}z + 1}{e^{s_5}z + e^{-s_1} + 1},$$

$$\tilde{\varphi}_3(z) := \varphi_5 \circ \varphi_3(z) = e^{-s_5} \frac{z - 1}{-z + e^{s_3} + 1}.$$

**Lemma 4.3.** The images of $\tilde{X}$ under the tiling group $\tilde{\Gamma}$ generated by $\tilde{\varphi}_1$ and $\tilde{\varphi}_3$ cover the whole hyperbolic plane if and only if $s_1 + s_3 + s_5 = 0$.

**Proof.** By Poincaré’s polygon theorem 2.15, the quotient space $(\tilde{X}, \tilde{d}_X)$ is complete if and only if the horocircle condition from Lemma 2.14 holds. The only edge cycle around an ideal vertex $\tilde{\zeta}$ in $\tilde{X}$ consists of the vertex $\infty = \{\infty, 1, 0, -e^{-s_5}\}$. The composition
\( \tilde{\varphi}_4 \circ \tilde{\varphi}_2 \circ \tilde{\varphi}_3 \circ \tilde{\varphi}_1 \) sends \( \infty \) to \( \infty \):

\[
\begin{align*}
\tilde{\varphi}_4 \circ \tilde{\varphi}_2 \circ \tilde{\varphi}_3 \circ \tilde{\varphi}_1(\infty) &= \tilde{\varphi}_4 \circ \tilde{\varphi}_2(0) \\
&= \tilde{\varphi}_4(-e^{-s_5}) \\
&= \infty,
\end{align*}
\]

so the gluing maps corresponding to this edge cycle are \( \tilde{\varphi}_1 : E_1 \to E_2 \), \( \tilde{\varphi}_3 : E_3 \to E_4 \), \( \tilde{\varphi}_2 = \tilde{\varphi}_1^{-1} : E_2 \to E_1 \) and \( \tilde{\varphi}_4 = \tilde{\varphi}_3^{-1} : E_4 \to E_3 \). By Poincaré’s polygon theorem 2.15, the quotient space \( \tilde{X}, \tilde{d}_{\tilde{X}} \) is complete if and only if the composition \( \tilde{\varphi}_4 \circ \tilde{\varphi}_2 \circ \tilde{\varphi}_3 \circ \tilde{\varphi}_1 \) is horocyclic at \( \infty \).

Computing the composition explicitly, we find

\[
\tilde{\varphi}_4 \circ \tilde{\varphi}_2 \circ \tilde{\varphi}_3 \circ \tilde{\varphi}_1(z) = e^{2(s_1+s_3+s_5)}z + (1 + e^{s_3} + e^{s_3+s_1} + e^{s_3+s_1+s_5} + e^{2s_3+s_1-s_5} + e^{2s_3+2s_1+s_5}).
\]

Remember that an isometry \( \varphi \) is horocyclic at \( \infty \) if is a translation \( z \mapsto z + b \) or a reflection \( z \mapsto -\bar{z} + b \). The second case cannot occur here, hence the composition map is horocyclic at \( \infty \) if and only if \( s_1 + s_3 + s_5 = 0 \).

Let now \( s_1 + s_3 + s_5 = 0 \). Then \( (\tilde{X}, \tilde{d}_{\tilde{X}}) \) is complete by Poincaré’s polygon theorem, and since \( \tilde{X} \) is an ideal quadrangle, it follows by the Tessellation theorem 2.3 that the family \( \{\varphi(\tilde{X})\}_{\varphi \in \tilde{\Gamma}} \) forms a tessellation of \( \mathbb{H}^2 \). Conversely, let the images of \( \tilde{X} \) under \( \tilde{\Gamma} \) tessellate \( \mathbb{H}^2 \). By Theorem A.1, the space \( (\tilde{X}, \tilde{d}_{\tilde{X}}) \) is isometric to the quotient \( (\mathbb{H}^2/\tilde{\Gamma}, \tilde{d}_{\tilde{\Gamma}}) \) of \( \mathbb{H}^2 \) under the action of \( \tilde{\Gamma} \). Since \( \tilde{\Gamma} \) is a subgroup of \( \text{PSL}_2(\mathbb{Z}) \), and elements of \( \text{PSL}_2(\mathbb{Z}) \) act discontinuously on the complete metric space \( (\mathbb{H}^2, d_{\text{hyp}}) \), the quotient space \( (\mathbb{H}^2/\tilde{\Gamma}, d_{\tilde{\Gamma}}) \) (and therefor also \( (\tilde{X}, \tilde{d}_{\tilde{X}}) \)) is complete (Lemma A.2). Hence, by Poincaré’s polygon theorem 2.15 and the considerations above, \( s_1 + s_3 + s_5 = 0 \). \( \square \)

Figure 10: Replacing the triangle \( T^- \) by \( \varphi_5(T^-) \) with \( \varphi_5(z) = e^{-s_5} \), we obtain a new hyperbolic polygon \( \tilde{X} \).
Figure 11: Using shear parameters $s_1, s_3, s_5$, we obtain another partial tessellation of $\mathbb{H}^2$. Here, $s_1 = 0.25$, $s_3 = -0.75$ and $s_5 = 0.25$, so by Lemma 4.3 it is not a tessellation.

Let us recall what we just did. We started with the once-punctured torus from Section 1, that gives rise to a tessellation of $\mathbb{H}^2$. We modified the gluing isometries to obtain a partial tessellation $\mathcal{T}_{s_1,s_3}$, but did not know yet if it covered the whole hyperbolic plane. Introducing a third shear parameter $s_5$, we just showed that the tessellation corresponding to shear parameters $s_1, s_3, s_5$ is complete if and only if the shear parameters sum up to 0. Hence, we now know infinitely many tessellations of $\mathbb{H}^2$, that can all be obtained from our starting point, the Farey tessellation, by shearing.

A. Appendix

Here, we state (without proof) two properties concerning quotient spaces used above. They can be found in Chapter 7 of [Bon00].

**Theorem A.1.** Let the group $\Gamma$ act by isometries and discontinuously on $(\mathbb{H}^2, d_{hyp})$, and let $\Delta$ be a fundamental domain for the action of $\Gamma$, i.e. a connected polygon in $\mathbb{H}^2$ such that all images of $\Delta$ under elements of $\Gamma$ are distinct and form a tessellation of $\mathbb{H}^2$. Then the space $(\bar{\Delta}, \bar{d}_\Delta)$ obtained from $\Delta$ by gluing edges is isometric to the quotient space $(\mathbb{H}^2/\Gamma, \bar{d}_{hyp})$ of $\mathbb{H}^2$ by the action of $\Gamma$.

**Lemma A.2.** Let $\Gamma$ be a group acting discontinuously on the complete metric space $X$. Then the quotient space $(X/\Gamma, \bar{d}_X)$ is complete.

References