A quotient space of representations of triangle reflection groups

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The present paper contains notes of my talk given in the seminar *Hyperbolic geometry, symmetry groups, and more* organized by Prof. Dr. Anna Wienhard and Dr. Gye-Seon Lee in the winter term 2013/2014. It is based on publications of Xin Nie [Nie] and Yves Benoist [Ben01], [Ben02].

Let $\Lambda$ be a group generated by reflections with respect to the faces of a triangle $P \subset \mathbb{H}^2$. The group $\Lambda$ shall admit the presentation

$$\langle r_1, r_2, r_3 | \{(r_i r_j)^{m_{ij}} : i, j \in \{1, 2, 3\}\} \rangle,$$

where $m_{ji} = m_{ij} \in \{2, 3, 4, ...\}$ with $m_{ii} = 1$ and $\frac{1}{m_{12}} + \frac{1}{m_{13}} + \frac{1}{m_{23}} < 1$. Denote this presentation by $\Gamma_0$. We want to study the space

$$\mathcal{F}_{\Gamma_0} = \{ \rho \in \text{Hom}(\Gamma_0, G) \text{ faithful with discrete image } \Gamma := \rho(\Gamma_0) \}
\text{ dividing a properly convex open set } \Omega_{\rho} \subset S^2 \},$$

where $G$ is the group of projective transformations of the projective sphere $S^2$. The group $\Gamma$ divides $\Omega$ if its action on $\Omega$ is proper and cocompact. The group $G$ acts on $\mathcal{F}_{\Gamma_0}$ by conjugation and we can define the quotient

$$X_{\Gamma_0} = \frac{G \backslash \mathcal{F}_{\Gamma_0}}{}.$$

Our goal is to sketch a proof of the following proposition:

**Proposition 1.** If all $m_{ij}$ are not equal to 2 then $X_{\Gamma_0}$ is homeomorphic to $\mathbb{R}^+$. Otherwise it is homeomorphic to a point.

If $X_{\Gamma_0}$ is homeomorphic to $\mathbb{R}^+$ then there is a one-parameter family of representations $\{\rho_t\} \in \mathbb{R}^+$ such that, if $\Omega_t$ is the convex open set associated to $\rho_t$, then $\Omega_t$ converges to $P$ when $t$ tends to 0 or $\infty$. This fact is being illustrated in Figure 1.
Figure 1: Deformation of $\Omega_t$, where $\Gamma_0$ is the (4, 4, 4)-triangle group. The plots were created with SAGE [S+09]. The source code is included in the appendix.
In order to study the space $X_{\Gamma_0}$, we need to know more about projective reflections and find conditions that tell us when the translates of a triangle under the action of the group generated by the reflections with respect to its faces tile a convex subset of $S^2$.

**Definition.** Let $V := \mathbb{R}^3$.

1. The **projective sphere** is the set $S^2 = S(V) := (V - \{0\})/\mathbb{R}_+^*$.
2. The group of its **projective transformations** is $G := SL^\pm(3, \mathbb{R})$.
3. A reflection $\sigma$ is an element of order 2 in $G$ which is the identity on a hyperplane.

All projective reflections are of the form $\sigma = \sigma_{\alpha,v} := \text{Id} - \alpha \otimes v$ for some linear form $\alpha \in V^*$ and $v \in V$ with $\alpha(v) = 2$. We have $\sigma|_{\ker(\alpha)} = \text{Id}_{\ker(\alpha)}$, and $\sigma(v) = -v$. The action on $S^2$ is defined by $\sigma(x) = x - \alpha(x)v$.

**Definition.** A subset $\Omega \subseteq S^2$ is **convex** if its intersection with any great circle is connected. It is **properly convex** if it is convex and its closure $\overline{\Omega}$ does not contain two opposite points.

**Lemma 2.** Let $\sigma_1 = \sigma_{\alpha_1,v_1}$ and $\sigma_2 = \sigma_{\alpha_2,v_2}$ be distinct projective reflections, let $\Delta$ be the group they generate, and define $a_{12} := \alpha_1(v_2)$ and $a_{21} := \alpha_2(v_1)$.

1. If $a_{12} > 0$ or $a_{21} > 0$ then the $\delta(L)$, $\delta \in \Delta$, do not tile any subset of $S^2$.
2. Suppose now $a_{12} \leq 0$ and $a_{21} \leq 0$. Consider the following cases:
   (a) $a_{12}a_{21} = 0$. If both $a_{12}$ and $a_{21}$ are equal to 0 then the product is of order 2, and the $\delta(L)$ tile $S^2$. Otherwise they do not tile.
   (b) $0 < a_{12}a_{21} < 4$. The product $\sigma_1\sigma_2$ is a rotation of angle $\theta$ given by $4 \cos(\frac{\theta}{2})^2 = a_{12}a_{21}$. If $\theta = 2\pi/m$ for some integer $m \geq 2$ then $\sigma_1\sigma_2$ is of order $m$, and the $\delta(L)$ tile $S^2$. Otherwise they do not tile.

**Proof.** See Proposition 6 in [Vin].

Let $\sigma_i := \text{Id} - \alpha_i \otimes v_i$, $i = 1, 2, 3$, be projective reflections, let $\Gamma$ be the group they generate and define $a_{ij} := \alpha_i(v_j)$ for all $i, j = 1, 2, 3$. According to Lemma 2, if we want the images $\gamma(P)$ of the triangle $P$, which is the intersection of the half-spheres $\alpha_i \leq 0$, to tile some subset of $S^2$, the following conditions are necessary: For all $i \neq j$ we have

1. $a_{ij}$ and $a_{ji}$ are either both negative or both 0,
2. $a_{ij}a_{ji} = 4 \cos(\frac{\pi}{m_{ij}})^2$ with an integer $m_{ij} \geq 2$.

The next theorem due to Tits and Vinberg is the key to understand $X_{\Gamma_0}$. It says that the conditions given above are not only necessary, but also sufficient. Further information on this theorem can be found in [Ben02].
Theorem 3. Let $P \subset S^2$ be a triangle, and for each face $i$ of $P$, let $\sigma_i = Id - \alpha_i \otimes v_i$ be a projective reflection fixing the face $i$. Suppose that $P$ is the intersection of the half-spheres $\alpha_i \leq 0$ and that the projective reflections satisfy
1. $a_{ij}$ and $a_{ji}$ are either both negative or both 0, and
2. $a_{ij} a_{ji} = 4 \cos(\frac{\pi}{m_{ij}})^2$ with an integer $m_{ij} \geq 2$.

Then
1. the group $\Gamma$ generated by the reflections $\sigma_i$ is discrete,
2. the triangles $\gamma(P)$, $\gamma \in \Gamma$, tile a convex subset $\Omega \subset S^2$, and
3. the morphism $\sigma: \Gamma_0 \to \Gamma$ given by $\sigma(r_i) = \sigma_i$ is an isomorphism.

We outline the idea of the proof of Theorem 3. Define an abstract space $X$ by glueing copies of $P$ indexed by $\Gamma_0$ together along their edges and show that this space is convex. A bijection from $X$ into $\Omega = \bigcup_{\gamma \in \Gamma} \gamma(P)$ yields the desired properties.

Define $X := \Gamma_0 \times P/\sim$. The equivalence relation is generated by
$$(\gamma, p) \sim (\gamma', p')$$
if $p' = p$ and $\gamma^{-1}\gamma' \in \Gamma_p$,
where $\Gamma_p$ is the group generated by the $\sigma_i$ such that $p$ is contained in the face $i$. Furthermore, define
$$\pi: X \to S^2, \ \pi(\gamma, p) = \gamma p.$$ 

We need the notion of a segment in order to determine, whether a space is convex.

Definition. 1. A subset $S \subset S^2$ is called a segment if $\hat{S}$ is a 1-dimensional convex subset.
2. For every $x, y \in X$, a segment $[x, y]$ is a compact subset of $X$ such that the restriction of $\pi$ to $[x, y]$ is a homeomorphism onto some segment of $S^2$ with endpoints $\pi(x)$ and $\pi(y)$.

Lemma 4. For every $x, x' \in X$ there exists at least one segment $[x, x']$.

Lemma 5. The map $\pi: X \to \Omega$ is bijective and $\Omega$ is convex.

Proof. According to the previous lemma, there is a segment $[x, x']$ for all $x, x' \in X$. Hence, if $\pi(x) = \pi(x')$, then $x = x'$ (see the definition of a segment in $X$). This proves that $\pi: X \to \Omega$ is injective (the map is surjective because of the definition of $\Omega$). Since $\pi: X \to \Omega$ is bijective, all pairs of points in $\Omega$ can be joined by a segment. Therefore $\Omega$ is convex.

This concludes the proof of Theorem 3, since the statements 2. and 3. follow from Lemma 5, and 1. follows from 2.

Lemma 6. The following statements are equivalent:
1. For every vertex $x$ of $P$, the group $\Gamma_x$ is finite.
Note that under these conditions $\Gamma$ divides $\Omega$. Hence, to be sure that the translates $\gamma(P)$ tile some open convex set, it is enough to check local conditions around each vertex of the triangle.

According to [Ben01] the set $\Omega$ is properly convex if the vectors $v_i$ generate $V$ and the linear forms $\alpha_i$ generate $V^*$.

Now, we will identify the quotient space $X_{\Gamma_0}$ with a quotient space of matrices $\bar{M}/\sim$. Let $\rho \in \mathcal{F}_{\Gamma_0}$ and $\Gamma = \rho(\Gamma_0)$, then, according to Theorem 3, $\Gamma$ is generated by projective reflections $\sigma_i$, satisfying the conditions stated in the theorem. Hence, $\rho$ can be identified with a $3 \times 3$ matrix $A = (a_{ij})$, with $a_{ii} = 2$ for $i = 1, 2, 3$ and $a_{ij} = 0$ for $i \neq j$. Two representations $\Gamma_1 = \rho_1(\Gamma_0)$ and $\Gamma_2 = \rho_2(\Gamma_0)$, that are given by $A_1$ and $A_2$, are conjugate by a projective transformation if and only if $A_1$ and $A_2$ are conjugate by a matrix $\text{diag}(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_i > 0$.

Conversely, let $\bar{M}$ be the set of all $3 \times 3$ matrices $A$ such that its entries $a_{ij}$ satisfy the conditions given in Theorem 3. Define an equivalence relation as follows: $A_1$ and $A_2$ in $\bar{M}$ are equivalent if they are conjugate by a matrix $\text{diag}(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_i > 0$. Let $M := \bar{M}/\sim$. As every $A \in \bar{M}$ yields a representation $\rho : \Gamma_0 \to G$ in $\mathcal{F}_{\Gamma_0}$, we have

$$X_{\Gamma_0} \cong M.$$ 

Now, it suffices to prove Proposition 1 for the quotient space $M$. Therefore, we need to introduce cyclic products.

**Definition.** Let $A = (a_{ij})$ be an $n \times n$ matrix and let $1 \leq i_1, \ldots, i_k \leq n$ with $k \geq 1$ be an ordered set of pairwise distinct indices. Then $a_{i_1i_2}a_{i_2i_3}\ldots a_{i_{k-1}i_k}a_{i_ki_1}$ is a **cyclic product** of length $k$.

**Lemma 7.** Let $A = (a_{ij})$ be an $3 \times 3$ matrix satisfying the condition that for any $i$ we have $a_{ii} \neq 0$ and for any $i \neq j$, $a_{ij} = 0$ if and only if $a_{ji} = 0$. Let $B$ satisfy the same condition. We say $A \sim B$ if there are $\lambda_i \neq 0$ such that

$$\text{diag}(\lambda_1, \ldots, \lambda_3)A\text{diag}(\lambda_1, \ldots, \lambda_3)^{-1} = B.$$

Then, $A \sim B$ if and only if for any ordered subset $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, 3\}$ we have

$$a_{i_1i_2}\ldots a_{i_{k-1}i_k}a_{i_ki_1} = b_{i_1i_2}\ldots b_{i_{k-1}i_k}b_{i_ki_1}.$$ 

**Proof.** See Lemma 1 in [Nie]. Suppose $A \sim B$. Then

$$\begin{pmatrix} a_{11} & \lambda_1a_{12}\lambda_2^{-1} & \lambda_1a_{13}\lambda_3^{-1} \\ \lambda_2a_{21}\lambda_1^{-1} & a_{22} & \lambda_2a_{23}\lambda_3^{-1} \\ \lambda_3a_{31}\lambda_1^{-1} & \lambda_1a_{32}\lambda_2^{-1} & a_{33} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

and all cyclic products coincide.

Now, suppose that all cyclic products with the same indices coincide. We say that a matrix $A$ is reducible if $A$ can be put into block-diagonal form (one may have to reorder the basis). Otherwise $A$ is irreducible.
The hypothesis on $A$ and $B$ implies that $a_{ii} = b_{ii}$ and $a_{ij} = 0$ if and only if $b_{ij} = 0$ for any $i \neq j$ (consider cyclic products of length 1 or 2). Hence, if necessary after a reordering of basis, the matrices $A$ and $B$ can be put into block-diagonal form with irreducible blocks, and the $r$th block of $A$ has the same size as the $r$th block of $B$. Therefore, $A$ and $B$ are conjugate via diagonal matrices if and only if their blocks are. Hence, in the following we can assume that $A$ and $B$ are irreducible.

Let $\lambda_1 = 1$. Irreducibility implies that for $\lambda_2$ and $\lambda_3$ we can choose

$$\lambda_2 = \begin{cases} \frac{b_{23}b_{31}}{a_{23}a_{31}} & \text{or} \quad \lambda_3 = \begin{cases} \frac{b_{23}b_{31}}{a_{23}a_{31}} & \text{or} \end{cases} \end{cases}$$

Consider $\lambda_2$: If $a_{12} = 0$ then $a_{13} \neq 0$ and $a_{32} \neq 0$, because otherwise $A$ can be put into block-diagonal form with more than one block. If $a_{13} = 0$ or $a_{32} = 0$, then $a_{12} \neq 0$ for the same reason.

It remains to check that the value of $\lambda_2$ and $\lambda_3$ does not depend on our choice. For $\lambda_2$ we have

$$b_{23}b_{31} a_{23} a_{31} = \frac{a_{12}}{b_{12}} = \frac{b_{21}}{a_{21}},$$

where we used the fact that cyclic products of the same set of indices coincide. The proof for $\lambda_3$ works in the same way. It is a straightforward calculation to show that $A \sim B$ for this choice of $\lambda_i$.

**Proof of Proposition 1.** Confer [Nie]. Let $A, B \in \tilde{M}$. We differ two cases:

1. There are indices $i, j$ such that $m_{ij} = 2$. Then, cyclic products of length
   - one are diagonal entries, which are equal to 2,
   - two are determined by the conditions given in Theorem 3, and
   - three are equal to 0.

   Hence, according to Lemma 7, it follows that $A \sim B$.

2. For all $i, j$ we have $m_{ij} \neq 2$. Again, cyclic products of length one or two coincide. There are two cyclic products of length three:

   $$\phi(A) = a_{12}a_{23}a_{31} \quad \text{and} \quad \tilde{\phi}(A) = a_{13}a_{32}a_{21}$$

   The product

   $$\phi(A)\tilde{\phi}(A) = 4^3 \cos \left( \frac{\pi}{m_{12}} \right)^2 \cos \left( \frac{\pi}{m_{23}} \right)^2 \cos \left( \frac{\pi}{m_{13}} \right)^2,$$

   which determined by Theorem 3, is constant. By Lemma 7, we have $A \sim B$ if and only if $\phi(A) = \phi(B)$. The value of $\phi(A)$ is always negative. We show that the map

   $$\psi: M \to \mathbb{R}_+, \quad [A] \mapsto |\phi(A)|.$$

   is a homeomorphism. Because of the argument above, the map is well-defined. It is clear that $\psi$ is continuous. Let

   $$A_t := \begin{pmatrix} 2 & -t \cos \left( \frac{\pi}{m_{12}} \right) & -t \cos \left( \frac{\pi}{m_{12}} \right) \cos \left( \frac{\pi}{m_{23}} \right) \cos \left( \frac{\pi}{m_{13}} \right)^2 \\
   -t \cos \left( \frac{\pi}{m_{12}} \right) & 2 & -t \cos \left( \frac{\pi}{m_{12}} \right) \cos \left( \frac{\pi}{m_{23}} \right) \cos \left( \frac{\pi}{m_{13}} \right)^2 \\
   -t \cos \left( \frac{\pi}{m_{12}} \right) \cos \left( \frac{\pi}{m_{23}} \right) & -t \cos \left( \frac{\pi}{m_{12}} \right) \cos \left( \frac{\pi}{m_{23}} \right) \cos \left( \frac{\pi}{m_{13}} \right)^2 & 2 \end{pmatrix}.$$
Then \( \phi(A_t) = -t \). Define

\[
\tilde{\psi}: \mathbb{R}_+ \to M, \quad t \mapsto [A_t].
\]

The function \( \tilde{\psi} \) is continuous and we have \( \psi\tilde{\psi} = \text{id}_{\mathbb{R}_+} \) and \( \tilde{\psi}\psi = \text{id}_M \). Hence, \( \psi \) is a homeomorphism.

\[
\square
\]

References


### Set Parameters

```python
s = 2
p = 4
q = 4
r = 4
N = 1000
```

```python
def is_in_interval(x):
    temp_var = 0
    temp_var_2 = False
    m = 0
    k = var('k')
    while temp_var_2 is False:
        if sum(3^k, k, 0, m) <= x < sum(3^k, k, 0, m + 1):
            temp_var_2 = True
        if mod(m, 2) == 1:
            temp_var = 1
        m = m + 1
    return temp_var
```

```python
xRotation = matrix([[1, 0, 0],
                    [0, cos(pi/4), -sin(pi/4)],
                    [0, sin(pi/4), cos(pi/4)]])
```

```python
yRotation = matrix([[cos(pi/4), 0, sin(-pi/4)],
                    [0, 1, 0],
                    [-sin(-pi/4), 0, cos(pi/4)]])
```

```python
Rotation = yRotation * xRotation
```

```python
e_1 = vector([1, 0, 0])
e_2 = vector([0, 1, 0])
e_3 = vector([0, 0, 1])
```

```python
e1 = matrix([1, 0, 0]).T
e2 = matrix([0, 1, 0]).T
e3 = matrix([0, 0, 1]).T
```

```python
I = matrix([[1, 0, 0], [0, 1, 0], [0, 0, 1]])
```
List_of_Unit_Vectors = [e1, e2, e3]

### Define Reflections

Bilinear_Form_s =
matrix([[1, -s*cos(pi/p), -cos(pi/q)/s],
       [-cos(pi/p)/s, 1, -s*cos(pi/r)],
       [-cos(pi/q)*s, -cos(pi/r)/s, 1]])

Generators = [I] * 3

for i in range(3):
    Generators[i] =
        I - 2 * Bilinear_Form_s *
        List_of_Unit_Vectors[i] *
        List_of_Unit_Vectors[i].T

Reflections = [I] * N
Reflections[0] = Generators[0]

for i in range(3, N):
    Reflections[i] =
        Generators[mod(i, 3)] * Reflections[int(i/3)-1]

Remember_Repetition = [0] * N

for i in range(0, N):
    for j in range(i+1, N):
        if Reflections[i] == Reflections[j]:
            Remember_Repetition[j] = 1

### Plot

Convex_Set = polygon(
    [[Rotation[0][0], Rotation[1][0]],
     [Rotation[0][1], Rotation[1][1]],
     [Rotation[0][2], Rotation[1][2]]],
    color='blue')

for i in range(N):
if Remember_Repetition[i] == 0:
    L_1 = Rotation * Reflections[i] * e_1
    L_2 = Rotation * Reflections[i] * e_2
    L_3 = Rotation * Reflections[i] * e_3
    L_1 = L_1 / sqrt(L_1 * L_1)
    L_2 = L_2 / sqrt(L_2 * L_2)
    L_3 = L_3 / sqrt(L_3 * L_3)

select_color = is_in_interval(i+1)
if select_color == 0:
    Convex_Set +=
    polygon([[L_1[0], L_1[1]],
             [L_2[0], L_2[1]],
             [L_3[0], L_3[1]]],
            color='yellow')
if select_color == 1:
    Convex_Set +=
    polygon([[L_1[0], L_1[1]],
             [L_2[0], L_2[1]],
             [L_3[0], L_3[1]]],
            color='blue')
show(Convex_Set, axes=0)