Aspherical neighborhoods on arithmetic surfaces: the local case

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Abstract

On arithmetic surfaces over henselian discrete valuation rings we examine whether a geometric point has a basis of étale neighborhoods whose $\mathfrak{c}$-completed étale homotopy types are of type $K(\pi, 1)$ with respect to a full class $\mathfrak{c}$ of finite groups.

1 Introduction

In [1] Artin proves the comparison theorem of classical with étale cohomology for a variety over the complex numbers. A crucial point in the proof is the construction of a special type of neighborhoods now called good Artin neighborhoods. They are open subschemes which admit a successive fibration into affine curves $X \to B$ with smooth compactification $\bar{X} \to B$ such that the complement $\bar{X} - X$ is étale over B. This type of fibration is called elementary fibration. The construction of a good Artin neighborhood uses Bertini's theorem in order to find a suitable linear subspace of the ambient projective space such that projection along this subspace locally yields an elementary fibration. These neighborhoods are so useful because topologically they are particularly simple. They are examples of $K(\pi, 1)$-spaces, i. e. of spaces whose only nontrivial homotopy group is the fundamental group. This property can be drawn from the long exact homotopy sequence associated with an elementary fibration.

The scenario where X is a smooth variety over an algebraically closed field of positive characteristic was treated by Friedlander in [6]. He examines whether an étale neighborhood is $K(\pi, 1)$ with respect to a prime number ℓ different from the characteristic of X, i. e. if it is $K(\pi, 1)$ after ℓ-completion of the étale homotopy type. He also uses elementary fibrations and the homotopy sequence associated with these fibrations. The major problem he has to deal with is non-exactness of $\mathfrak{c}$-completions for a full class of finite groups $\mathfrak{c}$. If $\mathfrak{c}$ is the class of finite ℓ-groups, he can prove that under certain conditions ℓ-completion is indeed exact by using special features of ℓ-groups.

In the arithmetic setting, i. e. considering schemes flat and of finite type over $\mathbb{Z}$ or $\mathbb{Z}_p$, the above approach is not promising. In fact, étale bases of neighborhoods which admit an elementary fibration never exist (see [11], Chapter 3). In case of arithmetic surfaces $\pi : X \to B$ this is quite obvious because for $U \to X$ étale the restriction of π to U is the only possible fibration into curves (unless the generic fiber is rational but in this case X may be replaced by an étale neighborhood). Even if X is smooth and projective over B, there are open subschemes with no étale neighborhood admitting an elementary fibration. One just has to take the complement of a non-smooth divisor in X.

As a consequence we cannot expect to work with smooth fibrations of arithmetic schemes. This makes it hard to use the machinery of long exact sequences of homotopy groups associated with a fibration. The problem is the lack of a simple relation between the homotopy theoretic fiber and the geometric fibers. Instead, we follow a more explicit approach working directly with the Leray spectral sequence associated with a fibration. Furthermore, we restrict our attention to arithmetic surfaces, the one-dimensional case having been dealt with in [19].

The present work examines arithmetic surfaces which are of finite type over some local ring of integers of residue characteristic p. In contrast to Friedlander we do not need to restrict our attention to ℓ-extensions for a single prime $\ell \neq p$ but can consider more general classes of finite groups. We say that a noetherian scheme X is $K(\pi, 1)$ with respect to a full class of finite groups $\mathfrak{c}$ if the pro-$\mathfrak{c}$-completion of the étale homotopy type of X is $K(\pi, 1)$. Writing $\mathbb{N}(\mathfrak{c})$ for the submonoid of $\mathbb{N}$ consisting of all cardinalities of groups in $\mathfrak{c}$ the main result reads as follows.

Theorem 1.1. *Let B be the spectrum of the ring of integers of the completion K of an algebraic extension of $\mathbb{Q}_p$ with finite ramification index. Let $\mathfrak{c}$ be a full class of finite groups such that the residue characteristic of B is not contained in $\mathbb{N}(\mathfrak{c})$ and for all but finitely many primes $\ell \in \mathbb{N}(\mathfrak{c})$ the extension $K(\mu_\ell)|K$ is a $\mathfrak{c}$-extension. Let Y/B be an arithmetic surface and $\bar{y} \to Y$ a geometric point. Then $\bar{y}$ has a basis of étale neighborhoods which are $K(\pi, 1)$ with respect to $\mathfrak{c}$.*

In particular, there exist $K(\pi, 1)$-neighborhoods with respect to any class of finite groups of the form $\mathfrak{c}(\ell_1, \ldots, \ell_n)$ for prime numbers ℓ_i prime to the residue characteristic of B. Here, $\mathfrak{c}(\ell_1, \ldots, \ell_n)$ denotes the class of finite groups whose order is divisible at most by the primes $\ell_1, \ldots, \ell_n$. If the residue field of B is separably closed, we can take any full class of finite groups with $p \notin \mathbb{N}(\mathfrak{c})$. In particular, we may take the class of all finite groups whose order is prime to p.

Let us explain more closely what a $K(\pi, 1)$-scheme is. Consider a connected, locally noetherian scheme X with geometric point $\bar{x}$. Following [2] we associate with $(X, \bar{x})$ the étale homotopy type X_{et}, which is a pro-object of the homotopy category of pointed, connected CW-complexes. We obtain homotopy pro-groups $\pi_n(X_{\text{ét}})$ and for an abelian group A with a $\pi_1(X_{\text{ét}})$-action cohomology groups $H^n(X_{\text{ét}}, A)$. The first homotopy pro-group of $X_{\text{ét}}$, $\pi_1(X_{\text{ét}})$, coincides with the "pro-groupe fondamentale enlargi" defined in [5], Exp. X, §6 (see [2], Corollary 10.7). If X is geometrically unibranch (e. g. normal), $\pi_1(X_{\text{ét}})$ is profinite and coincides with the usual fundamental group defined in [10], Exp. V. Moreover, for an abelian group A with a $\pi_1(X_{\text{ét}})$-action the cohomology groups $H^n(X_{\text{ét}}, A)$ coincide with the étale cohomology groups $H^n(X, A)$.

Let n be a positive integer and G a pro-group, which is assumed abelian if $n > 1$. There exists a pointed, connected pro-CW-complex whose n^{th} homotopy pro-group is isomorphic to G and whose remaining homotopy pro-groups vanish. It is unique up to $\sharp$-isomorphism (i.e. up to morphisms inducing isomorphisms on homotopy pro-groups) and called Eilenberg MacLane space of type $K(G, n)$. We say that a connected, locally noetherian scheme X with geometric point $\bar{x}$

is $K(\pi, 1)$ if the canonical morphism

$$X_{\text{ét}} \to K(\pi_1(X, \bar{x}), 1).$$

is a $\natural$-isomorphism.

We are interested in a slightly refined version of the $K(\pi, 1)$ property: For a full class of finite groups $\mathfrak{c}$ and a pro-CW-complex Z we denote by $Z(\mathfrak{c})$ the pro-$\mathfrak{c}$-completion of Z (which exists by [2] Theorem 3.4). We say that X is $K(\pi, 1)$ with respect to $\mathfrak{c}$ if $X_{\text{ét}}(\mathfrak{c})$ is $K(\pi, 1)$. Note that in general, being $K(\pi, 1)$ neither implies nor is implied by being $K(\pi, 1)$ with respect to $\mathfrak{c}$. The reason is the following: For any pro-CW-complex Z there is a natural isomorphism

$$\pi_1(Z)(\mathfrak{c}) \xrightarrow{\sim} \pi_1(Z(\mathfrak{c}))$$

but the higher homotopy pro-groups of $Z(\mathfrak{c})$ are not necessarily isomorphic to the $\mathfrak{c}$-completion of the respective homotopy pro-groups of Z.

There is a criterion for a scheme to be $K(\pi, 1)$ with respect to $\mathfrak{c}$ which involves only étale cohomology. In order to explain it let us fix some terminology: A Galois $\mathfrak{c}$-covering of X is a Galois covering with Galois group in $\mathfrak{c}$. A $\mathfrak{c}$-covering is a covering which is dominated by a Galois $\mathfrak{c}$-covering. The étale coverings of X constitute a Galois category by [10], Exp. V, §7. From this it is easy to deduce that the same holds for the $\mathfrak{c}$-coverings of X. We have the following characterization of schemes of type $K(\pi, 1)$ (see [18], Proposition 2.1):

Proposition 1.2. *Let $\mathfrak{c}$ be a full class of finite groups and X a locally noetherian scheme. The following assertions are equivalent:*

(i) *X is $K(\pi, 1)$ with respect to $\mathfrak{c}$.*

(ii) *Let $i \geqslant 1$ and $\Lambda = \mathbb{Z}/\ell\mathbb{Z}$ with $\ell \in \mathbb{N}(\mathfrak{c})$. Then, for every $\mathfrak{c}$-covering $X' \to X$ and every class $\phi \in H^i(X', \Lambda)$ there is a $\mathfrak{c}$-covering $X'' \to X'$ such that ϕ maps to zero under*

$$H^i(X', \Lambda) \to H^i(X'', \Lambda).$$

The reader not familiar with étale homotopy theory may safely take this criterion as a definition of the $K(\pi, 1)$-property. Throughout the rest of this article we will work exclusively with the above cohomological characterization. Note that the condition on the first cohomology group is automatically satisfied as $H^1(X', \Lambda)$ classifies Galois coverings of X' whose Galois group is a quotient of Λ and these coverings are $\mathfrak{c}$-coverings.

Let us now consider the situation on arithmetic surfaces: We fix a base scheme B which is the spectrum of an excellent Dedekind ring of dimension one. In this article we are mainly interested in the case where B is a henselian discrete valuation ring but in view of future work on global arithmetic surfaces we formulate most results in more general terms. It is only in Section 6 and in Section 10 that we restrict our attention to arithmetic surfaces over a henselian base. By an *arithmetic surface* over B we mean an irreducible, normal scheme U of dimension 2 which is flat and of finite type over B with geometrically connected generic fiber. Take a full class of finite groups $\mathfrak{c}$ such that all prime numbers in $\mathbb{N}(\mathfrak{c})$ are invertible on B. Proposition 1.2 leads us to the question whether a given cohomology class $\phi \in H^i(U, \Lambda)$ for $i \geqslant 2$ can be killed by a $\mathfrak{c}$-covering.

Assume there is a compactification $\bar{X}$ of U/B and an open subscheme X of $\bar{X}$ containing U such that $\bar{X} - X$ is the support of a nonempty regular horizontal divisor intersecting all vertical divisors transversally. Setting $Z = X - U$ (with the reduced scheme structure) we have the following exact sequence:

$$\ldots \to H^i_Z(X, \Lambda) \to H^i(X, \Lambda) \to H^i(U, \Lambda) \to H^{i+1}_Z(X, \Lambda) \to \ldots . \qquad (1)$$

This reduces the task of killing cohomology classes in $H^i(U, \Lambda)$ to killing classes in $H^{i+1}_Z(X, \Lambda)$ and $H^i(X, \Lambda)$.

Let us first have a look at $H^{i+1}_Z(X, \Lambda)$. Using resolution of singularities we achieve that Z is a tidy divisor. This property is slightly stronger than being an snc divisor (see Section 2 for a definition). For a tidy divisor Z we can handle $H^{i+1}_Z(X, \Lambda)$ using absolute cohomological purity. Roughly speaking, a cohomology class in $H^{i+1}_Z(X, \Lambda)$ is killed by a $\mathfrak{c}$-covering which is sufficiently ramified along Z (see Section 5).

The cohomology groups $H^i(X, \Lambda)$ are accessible because $\pi : X \to B$ is quite close to being an elementary fibration: There is a base change theorem which asserts that for every geometric point $\bar{b} \to B$ we have

$$(R^j \pi_* \Lambda)_{\bar{b}} \cong H^j(X_{\bar{b}}, \Lambda)$$

(see Proposition 4.3). In particular, $R^j \pi_* \Lambda = 0$ for $j \geqslant 3$. Moreover, $X_{\bar{b}}$ is an affine curve for all $\bar{b}$ where $X_{\bar{b}}$ is regular. As there are only finitely many singular fibers, this implies that $R^2 \pi_* \Lambda$ is a skyscraper sheaf.

Let us specialize to the case we are primarily interested in in this article, namely where B is the spectrum of a henselian discrete valuation ring R. As mentioned before, $H^1(X, \Lambda)$ automatically vanishes in the limit over all $\mathfrak{c}$-coverings. This leaves us to cope with the second cohomology. Unfortunately, $H^2(X, \Lambda)$ is not necessarily killed by a $\mathfrak{c}$-covering. However, a glance at sequence (1) reveals that it suffices to show that

$$\operatorname{coker}(H^2_Z(X, \Lambda) \to H^2(X, \Lambda))$$

is killed by a $\mathfrak{c}$-covering. In Section 6 we give conditions for this to be true.

Having carved out conditions for an arithmetic surface to be $K(\pi, 1)$ with respect to $\mathfrak{c}$ we set out to construct étale neighborhoods U on a given arithmetic surface satisfying these conditions. The main difficulty lies in ensuring that U has enough $\mathfrak{c}$-coverings that are sufficiently ramified along the boundary (see Section 9).

This article is based on parts of my thesis written under the supervision of Alexander Schmidt. I would like to thank him for posing this interesting question and supporting me during the process of answering it. Moreover, my thanks go to the referee whose suggestions helped me a lot in improving the overall structure of the paper.

2 Tidy divisors on arithmetic surfaces

Let X be a noetherian scheme. Throughout this article we identify effective Weil divisors on X with the associated closed subschemes of X whenever this does not lead to confusion. If the ambient scheme is normal, we can do the same for effective Cartier divisors. Remember that an effective Cartier divisor D on X

has *simple normal crossings* at a point $x \in D$ if X is regular at x and there is a local system of parameters $f_1, \dots, f_n$ at x and $m_1, \dots, m_n \in \mathbb{N}_0$ such that $f_1^{m_1} \cdot \dots \cdot f_n^{m_n}$ provides a local equation for D. The effective Cartier divisor D is a simple normal crossings (snc) divisor if it has simple normal crossings at each point. We say that two effective Cartier divisors D and D' *intersect transversally* at a point $x \in D \cap D'$ if they have no common irreducible component passing through x and $D + D'$ has simple normal crossings at x.

Suppose now that X/B is an arithmetic surface. An effective Cartier divisor D on X is *tidy* at a point $x \in D$ if it has simple normal crossings at x and intersects each vertical divisor of X passing through x transversally. A *tidy* divisor on X is an effective Cartier divisor D which is tidy at every point of D. In particular, the horizontal irreducible components of a tidy divisor do not intersect. For a proper closed subscheme $Z \subseteq X$ we say that a closed point $z \in Z$ is a *special point* of Z if either Z is not a tidy divisor at z or Z is tidy at z and Z_{red} is singular at z. The special points of a tidy divisor D are precisely the points where two irreducible components of D intersect. If D is not tidy but only snc, the special points are the singular points of D_{red} and the points where D intersects a vertical divisor non-transversally.

Let $Z \subseteq X$ be a proper closed subscheme. We define a *minimal desingularization* of (X, Z) to be a proper morphism of pairs $\phi : (X', Z') \to (X, Z)$ such that X' is regular at all points of Z', the morphism $(X' - Z') \to (X - Z)$ is an isomorphism and ϕ is universal with this property, i. e. any other proper morphism $\phi'' : (X'', Z'') \to (X, Z)$ as above factors through ϕ. Minimal desingularizations of (X, Z) exist by [14] and are unique up to unique isomorphism.

Definition 2.1. *Let X/B be an arithmetic surface and $Z \subseteq X$ a proper closed subscheme. A* tidy desingularization $(X', Z') \to (X, Z)$ *of (X, Z) is a birational morphism $X' \to X$ such that Z' is a tidy divisor of X' and $(X', Z') \to (X, Z)$ factors as*

$$(X', Z') = (X_n, Z_n) \to \dots \to (X_1, Z_1) \to (X_0, Z_0) \to (X, Z),$$

where $X_0 \to X$ is the minimal desingularization of (X, Z) and for $i = 1, \dots, n$ the morphisms $(X_i, Z_i) \to (X_{i-1}, Z_{i-1})$ are blowups of X_{i-1} in special points of Z_{i-1}.

Proposition 2.2. *Let X/B be an arithmetic surface and $Z \subset X$ a proper closed subscheme. Then a tidy desingularization of (X, Z) exists.*

Proof. We may assume that X is regular at all points of $X - Z$ as X is singular in at most a finite set of closed points, which we can remove from X if they do not lie on Z. By [3], Theorems 0.1 and 0.2 there is a desingularization $(X', Z') \to (X, Z)$ which is an isomorphism over the complement of Z such that Z' is an snc-divisor. Moreover, we can assume that $(X', Z') \to (X, Z)$ is obtained from the minimal desingularization by successive blow-ups in singular, hence special, points. Let D' be the union of Z' with the finitely many vertical prime divisors containing the points where Z' intersects a vertical divisor non-transversally. After removing from X' all points of D' which are not contained in Z' and where D' is singular, we may assume that the special points of D' are contained in Z'. By construction, they coincide with the singular points of D'_{red}. Blowing up in singular points of D'_{red}, we achieve that D' is an snc-divisor. This is equivalent to saying that Z' is tidy. □

Let $\mathfrak{c}$ be a full class of finite groups and denote by $\mathbb{N}(\mathfrak{c})$ the submonoid of the positive integers formed by the orders of all groups in $\mathfrak{c}$. For an arithmetic surface X/B such that all elements of $\mathbb{N}(\mathfrak{c})$ are invertible on X and a tidy divisor D on X we want to examine whether $U := X - D$ is $K(\pi, 1)$ with respect to $\mathfrak{c}$. The cohomological criterion spelled out in Proposition 1.2 leads us to study the $\mathfrak{c}$-coverings of U. We can extend a $\mathfrak{c}$-covering $U' \to U$ to a finite morphism $X' \to X$ by taking the normalization of X in U'. We obtain a $\mathfrak{c}$-covering of (X, D), i. e. a finite morphism of pairs $(X_1, D_1) \to (X, D)$ such that $X_1 - D_1 \to X - D$ is an étale $\mathfrak{c}$-covering. Any $\mathfrak{c}$-covering of (X, D) is tame, i. e. the valuations of $K(X)$ associated with the irreducible components of D are tamely ramified in the corresponding function field extension. Otherwise there would be an irreducible component Z of D whose function field is of characteristic $p > 0$ dividing the degree of the covering. However, we assumed the orders of all groups in $\mathfrak{c}$ to be invertible on X, hence not divisible by p.

For a tame covering $(X_1, D_1) \to (X, D)$ the divisor D_1 is not tidy in general. But we find a tidy desingularization $(X', D') \to (X_1, D_1)$ using Proposition 2.2.

Definition 2.3. *A* desingularized tame covering $(X', D') \to (X, D)$ *is the composition of a tame covering* $(X_1, D_1) \to (X, D)$ *and a tidy desingularization* $(X', D') \to (X_1, D_1)$. *In this case we define the exceptional divisor of* $(X', D') \to (X, D)$ *to be the exceptional divisor of* $X' \to X_1$. *A* desingularized $\mathfrak{c}$-covering *is a tame covering* $(X', D') \to (X_1, D_1) \to (X, D)$ *such that* $(X_1, D_1) \to (X, D)$ *is a* $\mathfrak{c}$*-covering.*

3 Setup and Notation

Let $\mathfrak{c}$ be a full class of finite groups. Remember that a profinite group G is called $\mathfrak{c}$-good if for all $G(\mathfrak{c})$-modules $M \in \mathfrak{c}$ and all $i \in \mathbb{N}$ the inflation

$$H^i(G(\mathfrak{c}), M) \to H^i(G, M)$$

is an isomorphism. If G is the absolute Galois group of a field k, this is equivalent to saying that *Spec* k is $K(\pi, 1)$ with respect to $\mathfrak{c}$. We need a slightly stronger version: Denote by H the kernel of the surjection $G \to G(\mathfrak{c})$. We say that G is *strongly* $\mathfrak{c}$*-good* if for all G-modules $M \in \mathfrak{c}$ and all $i \in \mathbb{N}$

$$H^i(G(\mathfrak{c}), M^H) \to H^i(G, M)$$

is an isomorphism. This is equivalent to saying that for all $j \geqslant 1$

$$H^j(H, M) = 0.$$

One example for a strongly $\mathfrak{c}$-good group is $\hat{\mathbb{Z}}$, the absolute Galois group of a finite field. For a prime $p \notin \mathbb{N}(\mathfrak{c})$ take an algebraic extension of $\mathbb{Q}_p$ containing the ℓ^{th} roots of unity for every prime $\ell \in \mathbb{N}(\mathfrak{c})$. Then its absolute Galois group provides another example for a strongly $\mathfrak{c}$-good group.

We now set up the notation for Section 5 and Section 6. We fix an excellent Dedekind scheme of dimension 1. Furthermore, we take a full class of finite groups $\mathfrak{c}$ such that all prime numbers $\ell \in \mathbb{N}(\mathfrak{c})$ are invertible on B and $\mu_\ell \cong \mathbb{Z}/\ell\mathbb{Z}$ on B. We assume that the absolute Galois groups of the residue fields of B at closed points are strongly $\mathfrak{c}$-good.

Over B we fix a proper arithmetic surface $\bar{X}$ with geometric point $\bar{x} \to \bar{X}$ lying over a closed point $x \in \bar{X}$. Let $\bar{D} \subseteq \bar{X}$ be a tidy divisor whose support does not contain x. Let $\bar{D}_h$ be the maximal subdivisor of $\bar{D}$ with support on the isolated horizontal components of $\bar{D}$, i. e., on the horizontal components which do not intersect any other component. Set $X = \bar{X} - \bar{D}_h$ and $U = \bar{X} - \bar{D}$ and denote by $D \subseteq X$ the restriction of $\bar{D}$ to X. We write D_v for the maximal vertical subdivisor of D and D_h for the maximal horizontal subdivisor, such that $D = D_v + D_h$. Notice that D_v is also the maximal vertical subdivisor of $\bar{D}$. The maximal horizontal subdivisor of $\bar{D}$ is given by $\bar{D}_h + D_h$. Let W denote the union of all vertical prime divisors which are contained in a singular fiber of $\bar{X} \to B$ but not in $\bar{D}$. Put differently, W is the Zariski closure of the union of all reduced fibers $(U_b)_{red}$ where $\bar{X}_b$ is singular. Denote by S the finite set of special points of $\bar{D}$, i. e., the set of singular points of $\bar{D}_{red}$.

We denote by $\mathfrak{I}_{(\bar{X},\bar{D})}$ the category of all pointed desingularized $\mathfrak{c}$-coverings of $(\bar{X}, \bar{D})$. We will see in Proposition 7.4 that $\mathfrak{I}_{(\bar{X},\bar{D})}$ is cofiltered. Viewing $\bar{x}$ as geometric point of B we write $\mathfrak{I}_B$ for the category of pointed finite étale $\mathfrak{c}$-coverings of B. By

$$(B' \to B) \mapsto ((\bar{X} \times_B B', \bar{D} \times_B B') \to (\bar{X}, \bar{D}))$$

$\mathfrak{I}_B$ becomes a full subcategory of $\mathfrak{I}_{(\bar{X},\bar{D})}$.

For $(\bar{X}', \bar{D}') \to (\bar{X}, \bar{D})$ in $\mathfrak{I}_{(\bar{X},\bar{D})}$ let

$$\bar{X}' \to B' \to B$$

be the Stein factorization of $\bar{X}' \to \bar{X} \to B$. Then $\bar{X}'$ is an arithmetic surface over B'. We use analogous notation for $(\bar{X}', \bar{D}')$ as for $(\bar{X}, \bar{D})$: We write U' for $\bar{X}' - \bar{D}'$ and $\bar{D}'_h$ for the maximal subdivisor of $\bar{D}'$ with support on the isolated horizontal components of $\bar{D}'$ and so on. Moreover, we write E' for the exceptional divisor of $\bar{X}' \to \bar{X}$.

Lemma 3.1. *Let $\pi : (\bar{X}', \bar{D}') \to (\bar{X}_1, \bar{D}_1) \to (\bar{X}, \bar{D})$ be a desingularized $\mathfrak{c}$-covering. Then $\pi^*\bar{D}_h = \bar{D}'_h$ and D'_h is the horizontal part of $\pi^* D_h$.*

Proof. By the definition of $\bar{D}_h$ and D_h it suffices to show that π maps the irreducible components of $\bar{D}'_h$ to $\bar{D}_h$ and those of D'_h to D_h. By the logarithmic version of Abhyankar's lemma ([9], Thm. 7.3.44) $(\bar{X}_1, \bar{D}_1) \to (\bar{X}, \bar{D})$ is Kummer étale, hence flat. Therefore, $\bar{D}_1 \to \bar{D}$ is flat and moreover proper and of finite presentation. It is thus open and closed. We conclude that a connected component of $\bar{D}_1$ is mapped surjectively onto a connected component of $\bar{D}$. Furthermore, connected components of $\bar{D}'$ are mapped surjectively onto connected components of $\bar{D}_1$. Every irreducible component of $\bar{D}'_h$ is thus mapped to an irreducible component of $\bar{D}_h$.

Since $\bar{D}_h$ is regular and does not intersect the other components of $\bar{D}$, the tame covering $(\bar{X}_1, \bar{D}_1) \to (\bar{X}, \bar{D})$ has regular branch locus in a neighborhood of $\bar{D}_h$. By the generalized Abhyankar lemma (see [10], Exp. XIII, 5.3.0) the preimage $\bar{D}_{h,1}$ of $\bar{D}_h$ in $\bar{X}_1$ is again regular and $\bar{X}_1$ is regular in a neighborhood of $\bar{D}_{h,1}$. In particular, $\bar{D}_{h,1}$ does not contain special points and thus $\pi^*(\bar{D}_h)$ is contained in $\bar{D}'_h$. Hence, the image of every irreducible component of D'_h is contained in D'_h. □

As a consequence of Lemma 3.1 we have $\pi^* X = X'$ and $\pi^* D_v = D'_v$. The preimage of D_h is the sum of D'_h and a divisor with support in E'.

Our objective is to investigate whether U is $K(\pi, 1)$ with respect to $\mathfrak{c}$. In this article we treat the case where B is local henselian. The above setup is more general because we plan another paper on the global case, i.e. where B is an open subscheme of the spectrum of the ring of integers of a finite extension of $\mathbb{Q}$. A great part of the proof is not much more difficult in the global case. Hence, we prove many propositions in a more general setting and specialize to the local case only when this is considerably easier.

For a morphism of schemes $f : Y \to S$, a closed subscheme Z of Y, and a sheaf $\mathcal{F}$ on the étale site of Y we write $R^j_Z f_* \mathcal{F}$ for the higher direct images with support in Z. They are the derived functors of the functor which sends an étale sheaf $\mathcal{F}$ on Y to the sheaf

$$(S' \to S) \mapsto \ker(\mathcal{F}(Y \times_S S') \to \mathcal{F}((Y - Z) \times_S S')) = H^0_{Z \times_S S'}(Y \times_S S', \mathcal{F})$$

on the étale site of S.

Proposition 3.2. *In the above notation assume that B is a henselian discrete valuation ring and that for all primes $\ell \in \mathbb{N}(\mathfrak{c})$ setting $\Lambda = \mathbb{Z}/\ell\mathbb{Z}$ the following conditions are satisfied:*

(1) $\varinjlim_{\mathfrak{I}_{(\bar{X},\bar{D})}} H^i_{D'}(X', \Lambda) = 0$ *for* $i \geqslant 3$ *and*

(2) $\varinjlim_{\mathfrak{I}_{(\bar{X},\bar{D})}} \mathit{coker}\big(H^0(B', R^2_{D'}\pi'_*\Lambda) \to H^0(B', R^2\pi'_*\Lambda)\big) = 0.$

Then U is $K(\pi, 1)$ with respect to $\mathfrak{c}$.

Proof. By Proposition 1.2 it is enough to show that for any $i \geqslant 2$ and $\Lambda = \mathbb{Z}/\ell\mathbb{Z}$ for a prime $\ell \in \mathbb{N}(\mathfrak{c})$

$$\varinjlim_{U' \to U} H^i(U', \Lambda) = 0,$$

where the limit is taken over all $\mathfrak{c}$-coverings of U. Taking the limit of the excision sequences associated with (X', D') for all desingularized $\mathfrak{c}$-coverings $(X', D') \to (X, D)$ we obtain a long exact sequence

$$\ldots \to \varinjlim_{\mathfrak{I}_{(\bar{X},\bar{D})}} H^i(X', \Lambda) \to \varinjlim_{\mathfrak{I}_{(\bar{X},\bar{D})}} H^i(U', \Lambda) \to \varinjlim_{\mathfrak{I}_{(\bar{X},\bar{D})}} H^{i+1}_{D'}(X', \Lambda) \to \ldots.$$

Using condition (1) we obtain

$$\varinjlim_{\mathfrak{I}_{(\bar{X},\bar{D})}} H^i(U', \Lambda) \cong \varinjlim_{\mathfrak{I}_{(\bar{X},\bar{D})}} H^i(X', \Lambda)$$

for $i \geqslant 3$ and an exact sequence

$$\varinjlim_{\mathfrak{I}_{(\bar{X},\bar{D})}} H^2_{D'}(X', \Lambda) \to \varinjlim_{\mathfrak{I}_{(\bar{X},\bar{D})}} H^2(X', \Lambda) \to \varinjlim_{\mathfrak{I}_{(\bar{X},\bar{D})}} H^2(U', \Lambda) \to 0.$$

For a desingularized $\mathfrak{c}$-covering $(X', D') \to (X, D)$ consider the Leray spectral sequence

$$H^i(B', R^j\pi'_*\Lambda) \Rightarrow H^{i+j}(X', \Lambda).$$

Let $\bar{b}' \to B$ be a geometric point of B' lying over the closed point b' of B' (compatible with $\bar{x}$). In Section 4 we will see that absolute cohomological purity and proper base change for $\bar{X}/B$ imply that $(R^j\pi'_*\Lambda)_{\bar{b}'} = H^i(X'_{\bar{b}'}, \Lambda)$ (see Proposition 4.3). Since $X'_{\bar{b}'}$ is a curve over an algebraically closed field, $H^j(X'_{\bar{b}'}, \Lambda)$ vanishes for $j \geqslant 3$. Moreover, for $i \geqslant 1$

$$\varinjlim_{\mathfrak{I}_{B',\bar{x}}} H^i(B', R^j\pi'_*\Lambda) = \varinjlim_{\mathfrak{I}_{B',\bar{x}}} H^i(k(b'), H^j(X'_{\bar{b}'}, \Lambda)) = 0$$

as the absolute Galois group of $k(b')$ is strongly $\mathfrak{c}$-good by assumption. In total the above limit vanishes whenever $i + j \geqslant 3$ and for $(i,j) = (1,1)$ and $(i,j) = (2,0)$. This implies that

$$\varinjlim_{\mathfrak{I}_{(\bar{X},\bar{D})}} H^i(X', \Lambda) = 0$$

for $i \geqslant 3$ and

$$\varinjlim_{\mathfrak{I}_{(\bar{X},\bar{D})}} \ker(H^2(X', \Lambda) \xrightarrow{edge} H^0(B', R^2\pi'_*\Lambda)) = 0. \tag{2}$$

Consider the diagram

$$\begin{tikzcd}
\varinjlim\limits_{\mathfrak{I}_{(\bar{X},\bar{D})}} H^2_{D'}(X', \Lambda) \arrow[r] \arrow[d, "\sim"', "\text{edge}"] & \varinjlim\limits_{\mathfrak{I}_{(\bar{X},\bar{D})}} H^2(X', \Lambda) \arrow[r, two heads] \arrow[d, hook, "\text{edge}"] & \varinjlim\limits_{\mathfrak{I}_{(\bar{X},\bar{D})}} H^2(U', \Lambda) \\
\varinjlim\limits_{\mathfrak{I}_{(\bar{X},\bar{D})}} H^0(B', R^2_{D'}\pi'_*\Lambda) \arrow[r, two heads] & \varinjlim\limits_{\mathfrak{I}_{(\bar{X},\bar{D})}} H^0(B', R^2\pi'_*\Lambda).
\end{tikzcd}$$

The left vertical arrow is an isomorphism because due to purity $R^j_D\pi_*\Lambda = 0$ for $j \leqslant 1$. The surjectivity of the lower horizontal arrow is due to condition (2) and the injectivity of the right vertical arrow is stated above (see (2)). We conclude that the upper left horizontal map is surjective, whence

$$\varinjlim_{\mathfrak{I}_{(\bar{X},\bar{D})}} H^2(U', \Lambda) = 0.$$

$\square$

4 Absolute cohomological purity

Before we go into the details of discussing condition (1) in Proposition 3.2 we draw some conclusions from Gabber's absolute purity theorem that will be crucial in the treatment of cohomology groups with support.

Let X be a noetherian, regular scheme and $Z \subseteq X$ a regular closed subscheme of pure codimension c. Then (X, Z) is called a regular pair of codimension c. Fix a positive integer m invertible on X and set $\Lambda = \mathbb{Z}/m\mathbb{Z}$. The absolute cohomological purity theorem proved by Gabber in [7] provides a canonical isomorphism

$$\underline{H}^n_Z(\Lambda) \cong \begin{cases} 0 & \text{for } n \neq 2c \\ \Lambda_Z(-c) & \text{for } n = 2c. \end{cases}$$

which is induced from the cycle class map sending $1 \in \Lambda$ to the fundamental class $s_{Z/X} \in H^{2c}_Z(X, \Lambda(c))$. Since the étale site of a scheme is equivalent to the étale site of its reduction, the statement also holds if only X_{red} and Z_{red} are regular. We call (X, Z) a *weakly regular pair* if (X_{red}, Z_{red}) is a regular pair. Taking into account that the pullback of the fundamental class $s_{Z_{red}/X_{red}}$ under a morphism $(X', Z') \to (X, Z)$ of weakly regular pairs of codimension c is $e \cdot s_{Z'_{red}/X'_{red}}$, where e denotes the ramification index, we obtain the following compatibility of purity isomorphisms:

Proposition 4.1. *Let $f : (X', Z') \to (X, Z)$ be a morphism of weakly regular pairs of codimension c. Suppose that Z and Z' are irreducible and as cycles on X'_{red} we have $f^*_{red} Z_{red} = e \cdot Z'_{red}$ with a positive integer e (the ramification index). Then, for any $m \in \mathbb{N}$ invertible on X the following diagram commutes*

$$\begin{tikzcd}
H^n_Z(X, \mathbb{Z}/m\mathbb{Z}) \arrow[dd] & H^{n-2c}(Z, \mathbb{Z}/m\mathbb{Z}(-c)) \arrow[l, "\sim"', "purity"] \arrow[d] \\
 & H^{n-2c}(Z', \mathbb{Z}/m\mathbb{Z}(-c)) \arrow[d, "\cdot e"] \\
H^n_{Z'}(X', \mathbb{Z}/m\mathbb{Z}) & H^{n-2c}(Z', \mathbb{Z}/m\mathbb{Z}(-c)). \arrow[l, "\sim"', "purity"]
\end{tikzcd}$$

Corollary 4.2. *Let X be a noetherian, regular scheme and $f : X' \to X$ a tamely ramified covering such that the branch locus $D \subseteq X$ is regular. Let Z be a regular closed subscheme of D and let Z' denote its preimage in X'. Then, for any integer m dividing the ramification index of each irreducible component of Z', the canonical map*

$$H^n_Z(X, \mathbb{Z}/m\mathbb{Z}) \to H^n_{Z'}(X', \mathbb{Z}/m\mathbb{Z})$$

is the zero map for all $n \in \mathbb{N}$.

Proof. Without loss of generality we may assume that Z is integral. The scheme X' and the underlying reduced subscheme of Z' are regular because the branch locus D is regular. Denote by Z'_k for $k = 1, \ldots, r$ the irreducible components of Z'. For each k we can now apply Proposition 4.1 to the morphism

$$X' - \bigcup_{i \neq k} Z'_i \to X$$

to conclude that

$$H^n_Z(X, \mathbb{Z}/m\mathbb{Z}) \to H^n_{Z'_k}(X' - \bigcup_{i \neq k} Z'_i, \mathbb{Z}/m\mathbb{Z})$$

is the zero map. But

$$H^n_{Z'}(X', \mathbb{Z}/m\mathbb{Z}) = \bigoplus_k H^n_{Z'_k}(X' - \bigcup_{i \neq k} Z'_i, \mathbb{Z}/m\mathbb{Z}),$$

and the corollary follows. $\square$

Using Gabber's absolute purity theorem we can prove the following refined version of the proper base change theorem.

Proposition 4.3. *Let (X, Z) be a weakly regular pair of codimension c and set $U = X - Z$. Let $\pi : X \to Y$ be a proper morphism such that Z intersects X_y transversally for any closed point y of Y. Set $\Lambda = \mathbb{Z}/m\mathbb{Z}$ for an integer m prime to the residue characteristics of X. Then for any geometric point $\bar{y} \to Y$ and any integer d the base change morphisms*

$$(R^n(\pi_U)_*\Lambda(d))_{\bar{y}} \to H^n(U_{\bar{y}}, \Lambda(d))$$

are isomorphisms for any $n \geqslant 0$.

Proof. Without loss of generality we may assume Y is the spectrum of a strictly henselian local ring with closed point y. Then, $\mu_m \cong \mathbb{Z}/m\mathbb{Z}$ on X and it suffices to prove the lemma for $d = 0$. We need to show that

$$H^n(U, \Lambda) \to H^n(U_y, \Lambda)$$

is an isomorphism. Consider the following diagram of excision sequences

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & H^n_Z(X,\Lambda) & \longrightarrow & H^n(X,\Lambda) & \longrightarrow & H^n(U,\Lambda) & \longrightarrow & \ldots . \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & H^n_{Z_y}(X_y,\Lambda) & \longrightarrow & H^n(X_y,\Lambda) & \longrightarrow & H^n(U_y,\Lambda) & \longrightarrow & \ldots
\end{array}$$

The homomorphisms $H^n(X, \Lambda) \to H^n(X_y, \Lambda)$ are bijective due to proper base change. Since by assumption Z intersects X_y transversally, $(X_y, Z_y) \to (X, Z)$ is a morphism of weakly regular pairs of codimension c yielding a commutative diagram

$$\begin{array}{ccc}
H^n_Z(X,\Lambda) & \xrightarrow{\sim} & H^{n-2c}(Z,\Lambda(-c)) \\
\downarrow & & \downarrow \\
H^n_{Z_y}(X_y,\Lambda) & \xrightarrow{\sim} & H^{n-2c}(Z_y,\Lambda(-c)).
\end{array}$$

The horizontal maps are purity isomorphisms and the vertical map on the right is an isomorphism by proper base change. Hence, the vertical map on the left is an isomorphism and the lemma follows by applying the five lemma to the above diagram of exact sequences. □

Finally, we prove the following technical result which is a variant of (and follows from) the compatibility of the purity isomorphisms associated with subschemes $Y \subseteq Z \subseteq X$ such that (X, Y), (X, Z) and (Z, Y) are weakly regular pairs:

Proposition 4.4. *Let X/B be an arithmetic surface and $D \subset X$ an snc-divisor. Let $S \subset D$ be a set of closed points containing the set D_{sing} of singular points of D. Denote by D_N the normalization of D and set $S_N = S \times_D D_N$. Then the following diagram of cohomology groups with coefficients in $\Lambda = \mathbb{Z}/m\mathbb{Z}$ (m prime to the residue characteristics of X) commutes*

$$\begin{tikzcd}
H^3_{D-S}(X-S,\Lambda) \arrow[r] \arrow[rr, bend left, "\delta"] & H^3(X-S,\Lambda) \arrow[r, "\delta"] & H^4_S(X,\Lambda) \\
H^1(D-S,\Lambda(-1)) \arrow[u, "purity", "\sim"'] \arrow[d, equal] & & H^0(S,\Lambda(-2)) \arrow[u, "purity", "\sim"'] \\
H^1(D_N-S_N,\Lambda(-1)) \arrow[r, "\delta"] & H^2_{S_N}(D_N,\Lambda(-1)) & H^0(S_N,\Lambda(-2)). \arrow[l, "\sim"', "purity"] \arrow[u, "norm"]
\end{tikzcd}$$

All maps δ are connecting homomorphisms of excision sequences.

Proof. Denote by D_i for $i = 1, \dots r$ the irreducible components of D. Since

$$H^3_{D-S}(X-S,\Lambda) = \bigoplus_i H^3_{D_i-S}(X-S,\Lambda),$$

it suffices to prove the proposition for each component D_i separately. We may thus assume without loss of generality that D is a regular irreducible curve. In this case the above diagram reduces to

$$\begin{tikzcd}
H^3_{D-S}(X-S,\Lambda) \arrow[r] \arrow[rr, bend left, "\delta"] & H^3(X-S,\Lambda) \arrow[r, "\delta"] & H^4_S(X,\Lambda) \\
H^1(D-S,\Lambda(-1)) \arrow[u, "purity", "\sim"'] \arrow[r, "\delta"] & H^2_S(D,\Lambda(-1)) & H^0(S,\Lambda(-2)). \arrow[l, "\sim"', "purity"] \arrow[u, "purity", "\sim"']
\end{tikzcd}$$

Consider the commutative diagram

$$\begin{tikzcd}
H^3_{D-S}(X-S,\Lambda) \arrow[r, "\delta"] & H^4_S(X,\Lambda) \\
H^1(D-S,\underline{H}^2_{D-S}(X-S,\Lambda)) \arrow[u, "\sim"] \arrow[d, "\sim"'] \arrow[r, "\delta"] & H^2_S(D,\underline{H}^2_D(X,\Lambda)) \arrow[u, "\sim"] \arrow[d, "\sim"'] \\
H^1(D-S,\Lambda(-1)) \otimes H^2_{D-S}(X-S,\Lambda(1)) \arrow[r, "\delta\otimes res^{-1}"] & H^2_S(D,\Lambda(-1)) \otimes H^2_D(X,\Lambda(1)) \\
H^1(D-S,\Lambda(-1)) \arrow[u, "\sim", "\otimes s_{D-S/X-S}"'] \arrow[r, "\delta"] & H^2_S(D,\Lambda(-1)). \arrow[u, "\sim", "\otimes s_{D/X}"']
\end{tikzcd}$$

The restriction

$$res : H^2_D(X,\Lambda(1)) \to H^2_{D-S}(X-S,\Lambda(1))$$

is an isomorphism which maps the fundamental class $s_{D/X}$ to $s_{D-S/X-S}$. For this reason, the homomorphism $\delta \otimes res^{-1}$ in the third line of the diagram is well defined and the lowermost square commutes. Commutativity of the middle square follows because $\underline{H}_D(X)$ is a free sheaf which restricts to $\underline{H}_{D-S}(X-S)$

on $D - S$. The upper square commutes due to compatibility of the spectral sequences

$$H^i_S(D, \underline{H}^j_D(X,\Lambda)) \Rightarrow H^{i+j}_S(X,\Lambda),$$
$$H^i(D-S, \underline{H}^j_{D-S}(X-S,\Lambda)) \Rightarrow H^{i+j}_{D-S}(X-S,\Lambda).$$

Furthermore, by [7], Proposition 1.2.1 the following diagram commutes

$$\begin{tikzcd}
H^4_S(X,\Lambda) & \\
H^2_S(D, \underline{H}^2_D(X,\Lambda)) \arrow[u, "\sim"] \arrow[d, "\sim"'] & \\
H^2_S(D,\Lambda(-1)) \otimes H^2_D(X,\Lambda(1)) & \\
H^2_S(D,\Lambda(-1)) \arrow[u, "\sim", "\otimes s_{D/X}"'] & H^0(S,\Lambda(-2)). \arrow[l, "\sim"', "\text{purity}"] \arrow[uuul, "\text{purity}", "\sim"']
\end{tikzcd}$$

Putting the two diagrams together, the assertion of the proposition follows. □

5 Killing cohomology with support

In the setup of Section 3 we derive conditions for hypothesis (1) in Proposition 3.2 to hold. The idea is to kill cohomology classes with support in D by $\mathfrak{c}$-coverings of (X, D) which are sufficiently ramified along D (compare Corollary 4.2). More precisely we need the following notion:

Definition 5.1. *Let Y be an arithmetic surface and $Z \subseteq Y$ a Weil divisor. We say that (Y, Z) has* enough tame coverings *at a closed point y of Z if for every irreducible component C of Z passing through y there is $f \in K(Y)^\times$ with support in Z such that $\deg_C(f) > 0$ and $\deg_W(f) = 0$ for any other prime divisor W passing through y. We say that (Y, Z) has enough tame coverings if it has enough tame coverings at every closed point of Z.*

If (Y, Z) has enough tame coverings at a point y and C is an irreducible component of Z passing through y, we can construct $\mathfrak{c}$-coverings of (Y, Z) of arbitrarily high ramification index in C by taking the normalization of Y in a function field extension $K(Y)(\sqrt[d]{f})|K(Y)$ with f chosen as in Definition 5.1. In a neighborhood of y this covering ramifies only in C.

Unfortunately, in order to treat condition (1) in Proposition 3.2 we cannot directly apply Corollary 4.2 as D is not necessarily regular. Instead we proceed in two steps using the excision sequence

$$\ldots \to H^i_S(X,\Lambda) \to H^i_D(X,\Lambda) \to H^i_{D-S}(X-S,\Lambda) \to \ldots$$

and applying Corollary 4.2 to $(X - S, D - S)$ and (X, S). In case $i = 3$ the argument is a bit subtle and we need to understand the kernel of the map $H^3_{Z-T}(X - T, \Lambda) \to H^4_T(X, \Lambda)$ for a vertical divisor $Z \subset X$ and a finite set of closed points T containing the singular points of Z. By Proposition 4.4 the purity isomorphisms translate this map to a map $H^1(Z - T, \Lambda(-1)) \to$

$H^1(S,\Lambda(-2))$ which is defined entirely in terms of curves. The following lemma describes its kernel.

Lemma 5.2. *Let C be a projective curve over an algebraically closed field k with only ordinary double points and let Γ_C denote its dual graph. Let $T \subset C$ be a finite set of closed points containing the set C_{sing} of singular points of C. Define $C_N := \coprod_i C_i$, where C_i are the normalizations of the irreducible components of C. Set $T_N = T \times_C C_N$. For $m \in \mathbb{N}$ prime to the characteristic of k consider the homomorphism β of cohomology groups with coefficients in $\mathbb{Z}/m\mathbb{Z}$ defined by*

$$\begin{tikzcd} H^1(C_N - T_N) \arrow[r, "\alpha"] \arrow[d, equal] & H^2_{T_N}(C_N) & H^0(T_N)(-1) \arrow[l, "\sim"', "purity"] \arrow[r, "norm"'] & H^0(T)(-1), \\ H^1(C-T) \arrow[urrr, bend right, "\beta"] & & & \end{tikzcd}$$

where α is the connecting homomorphism of the excision sequence associated with (C_N, T_N). Then

$$\frac{\ker(\beta)}{\ker(\alpha)} \cong H_1(\Gamma_C, \mathbb{Z}/m\mathbb{Z}),$$

where $H_1(\Gamma_C, \mathbb{Z}/m\mathbb{Z})$ denotes singular homology with coefficients in $\mathbb{Z}/m\mathbb{Z}$.

Proof. The group $H_1(\Gamma_C, \mathbb{Z}/m\mathbb{Z})$ can be calculated using a cellular chain complex. The zero-skeleton $(\Gamma_C)_0$ consists of the nodes of the graph which correspond to the irreducible components C_i and the one-skeleton $(\Gamma_C)_1$ is all of Γ_C. Thus, the one-cells are the edges of the graph, which correspond to the points in C_{sing}. We give each edge s a direction by choosing an initial node $C_1(s)$ and an end node $C_2(s)$. Then $H_1(\Gamma_C, \mathbb{Z}/m\mathbb{Z})$ is the first homology of the sequence

$$0 \to H_1((\Gamma_C)_1, (\Gamma_C)_0, \mathbb{Z}/m\mathbb{Z}) \xrightarrow{d} H_0((\Gamma_C)_0, \mathbb{Z}/m\mathbb{Z}) \to 0$$

and the map d can be identified with

$$\begin{aligned} \bigoplus_{s \in C_{sing}} \mathbb{Z}/m\mathbb{Z} \cdot s &\to \bigoplus_i \mathbb{Z}/m\mathbb{Z} \cdot C_i. \\ s &\mapsto C_2(s) - C_1(s). \end{aligned}$$

Let us now compute $\ker(\beta)/\ker(\alpha)$.

$$\begin{aligned} \frac{\ker(\beta)}{\ker(\alpha)} &= \ker\left(\frac{H^1(C-T)}{\ker(\alpha)} \to H^0(T)(-1)\right) \\ &= \ker(Im(\alpha) \to H^0(T)(-1)) \\ &= \ker(\ker(H^2_{T_N}(C_N) \to H^2(C_N)) \to H^0(T)(-1)) \\ &= \ker(H^2_{T_N}(C_N) \to H^2(C_N)) \cap \ker(H^2_{T_N}(C_N) \to H^0(T)(-1)). \end{aligned}$$

We identify the map $H^2_{T_N}(C_N) \to H^2(C_N)$ with

$$\begin{aligned} \bigoplus_{s_N \in T_N} \mathbb{Z}/m\mathbb{Z} \cdot s_N &\to \bigoplus_i \mathbb{Z}/m\mathbb{Z} \cdot C_i, \\ s_N &\mapsto C(s_N) \end{aligned}$$

where $C(s_N)$ is the component of C_N which contains s_N and $H^2_{T_N}(C_N) \to H^0(T)(-1)$ with

$$\bigoplus_{s_N \in T_N} \mathbb{Z}/m\mathbb{Z} \cdot s_N \to \bigoplus_{s \in T} \mathbb{Z}/m\mathbb{Z} \cdot s, \tag{3}$$

$$s_N \to s(s_N) \tag{4}$$

where $s(s_N)$ is the image of s_N in T. In particular, we obtain an isomorphism

$$\bigoplus_{s \in C_{sing}} \mathbb{Z}/n\mathbb{Z} \cdot s \to \ker(\bigoplus_{s_N \in T_N} \mathbb{Z}/n\mathbb{Z} \cdot s_N \to \bigoplus_{s \in T} \mathbb{Z}/n\mathbb{Z} \cdot s),$$
$$s \mapsto (s_N)_2(s) - (s_N)_1(s)$$

where $(s_N)_i(s) \in C_i(s)$ are the two preimages of s in C_N. Therefore, $\ker(\beta)/\ker(\alpha)$ is isomorphic to the kernel of the composition

$$\bigoplus_{s \in C_{sing}} \mathbb{Z}/m\mathbb{Z} \cdot s \to \bigoplus_{s_N \in T_N} \mathbb{Z}/m\mathbb{Z} \cdot s_N \to \bigoplus_i \mathbb{Z}/m\mathbb{Z} \cdot C_i,$$

which maps $s \in C_{sing}$ to $C_2(s) - C_1(s)$. Comparing with the calculation of $H_1(\Gamma_C, \mathbb{Z}/m\mathbb{Z})$ at the beginning of the proof we see that

$$\frac{\ker(\beta)}{\ker(\alpha)} \cong H_1(\Gamma_C, \mathbb{Z}/m\mathbb{Z}). \qquad \square$$

In the following we call a (not necessarily integral) projective curve C over a field k a rational tree if $H^1(C, \mathcal{O}_C) = 0$. By flat base change C is rational if and only if its base change $\bar{C}$ to the separable closure $\bar{k}$ of k is rational. This is the case precisely if every irreducible component of $\bar{C}$ is isomorphic to $\mathbb{P}^1_{\bar{k}}$ and moreover the dual graph of $\bar{C}$ is a tree, i.e. simply connected (see [4], Definition 4.23).

Lemma 5.3. *Let $Z \leqslant D_v$ be a subdivisor whose connected components are rational trees. Suppose that for every geometric point $\bar{b}$ above a closed point $b \in B$ the natural map*

$$\pi_1(b, \bar{b})(\mathfrak{c}) \to \pi_1(B, \bar{b})(\mathfrak{c})$$

is injective. Then

$$\varinjlim_{\mathfrak{I}_B} H^3_{S'}(X', \Lambda) \to \varinjlim_{\mathfrak{I}_B} H^3_{Z' \cup S'}(X', \Lambda)$$

is surjective (Remember that S' is the set of special points of $\bar{D}'$.)

Proof. Using the excision sequence for $S' \subseteq Z' \cup S' \subseteq X'$ we see that the required surjectivity is equivalent to the injectivity of

$$\varinjlim_{\mathfrak{I}_B} H^3_{Z' - S'}(X' - S', \Lambda) \to \varinjlim_{\mathfrak{I}_B} H^4_{S'}(X', \Lambda).$$

In other words, for $B' \to B$ in $\mathfrak{I}_B$ and $\varphi \in H^3_{Z'-S'}(X' - S', \Lambda)$ we have to construct $B'' \to B'$ in $\mathfrak{I}_B$ such that φ maps to zero in $H^4_{S''}(X'', \Lambda)$. As the assumptions are stable under étale base change, we may assume $B' = B$. By Proposition 4.4 we have the following commutative diagram

$$\begin{array}{ccc}
H^3_{Z-S}(X-S,\Lambda) & \longrightarrow & H^4_S(X,\Lambda) \\
\sim\uparrow & & \downarrow \\
H^1(Z-S,\Lambda(-1)) & \xrightarrow{\beta(-1)} & H^0(S,\Lambda(-2)),
\end{array}$$

where $\beta(-1)$ is the (-1)-twist of the map β defined in Lemma 5.2. It thus suffices to show that the kernel of β vanishes in the limit over $\mathfrak{I}_B$. Without loss of generality we may assume that Z is connected. In particular, it is contained in a single closed fiber of $X \to B$ over some point $b \in B$ with residue field $k(b)$. Let $\overline{k(b)}$ be an algebraic closure of $k(b)$ and denote by $\bar{Z}$ and $\bar{S}$ the base change of Z and S, respectively, to $\overline{k(b)}$. Moreover, write Z_N for the normalization of Z and $\bar{Z}_N$ for its base change to $\overline{k(b)}$. Consider the diagram of cohomology groups with coefficients in Λ

$$\begin{array}{ccccccc}
 & & H^2(k(b))^d & \xrightarrow{=} & H^2(k(b))^d & & \\
 & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & H^1(\bar{Z}_N)^{G_{k(b)}} & \longrightarrow & H^1(\bar{Z}-\bar{S})^{G_{k(b)}} & \xrightarrow{\bar{\beta}} & H^0(\bar{S})(-1)^{G_{k(b)}} \\
 & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & H^1(Z_N) & \longrightarrow & H^1(Z-S) & \xrightarrow{\beta} & H^0(S)(-1), \\
 & & \uparrow & & \uparrow & & \\
 & & H^1(k(b))^d & \xrightarrow{=} & H^1(k(b))^d & & \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & &
\end{array}$$

where d is the number of components of Z_N. The vertical sequences are induced by the Hochschild-Serre spectral sequences

$$H^i(k(b), H^j(\bar{Z}_N), \Lambda)) \Rightarrow H^{i+j}(Z_N, \Lambda),$$
$$H^i(k(b), H^j(\bar{Z}_N - \bar{S}_N, \Lambda)) \Rightarrow H^{i+j}(Z_N - S_N, \Lambda).$$

The upper horizontal sequence is exact by the following reason: According to Lemma 5.2, the first homology group $H_1(\Gamma_Z, \Lambda)$ of the dual graph Γ_Z of $\bar{Z}$ is isomorphic to $\ker(\bar{\beta})/\ker(\bar{\alpha})$, where $\bar{\alpha}$ denotes the connecting homomorphism of the excision sequence associated with $\bar{S}_N \hookrightarrow \bar{Z}_N$. As Z is a rational tree, Γ_Z is simply connected, and thus its first homology group vanishes. It follows that the kernel of $\bar{\beta}$ equals the image of the map

$$\gamma : H^1(\bar{Z}_N, \Lambda) \hookrightarrow H^1(\bar{Z}_N - \bar{S}_N, \Lambda) = H^1(\bar{Z} - \bar{S}, \Lambda).$$

Taking $G_{k(b)}$-invariants we obtain the upper sequence of the above diagram, which is therefore exact. A diagram chase now shows the exactness of the lower horizontal sequence.

Again by the rationality assumption on Z, the cohomology group $H^1(\bar{Z}_N)$ vanishes. The above diagram shows that the kernel of β equals $H^1(k(b))^d$. By the assumption on fundamental groups it vanishes in the limit over $\mathfrak{I}_B$. $\square$

Proposition 5.4. *Suppose that the following conditions are satisfied:*

(i) $(\bar{X}, \bar{D})$ *has enough tame coverings.*

(ii) Every connected component of D *has at least one horizontal component.*

(iii) For every geometric point $\bar{b}$ *above a closed point* $b \in B$ *the natural map*

$$\pi_1(b, \bar{b})(\mathfrak{c}) \to \pi_1(B, \bar{b})(\mathfrak{c})$$

is injective.

Then, for any $n \geqslant 3$

$$\varinjlim_{\mathfrak{I}_{(\bar{X},\bar{D})}} H^n_{D'}(X', \Lambda) = 0.$$

Proof. Let $(\bar{X}', \bar{D}')$ be an object of $\mathfrak{I}_{(\bar{X},\bar{D})}$ and φ an element of $H^n_{D'}(X', \Lambda)$. We have to show that there is a desingularized $\mathfrak{c}$-covering $(\bar{X}'', \bar{D}'') \to (\bar{X}', \bar{D}')$ such that the image of φ in $H^n_{D''}(X'', \Lambda)$ is zero. We will see in Proposition 8.1 that the property of having enough tame coverings is stable under desingularized tame coverings. Moreover, it is easy to check that this is true for the remaining assumptions as well. Hence, we may assume $(\bar{X}', \bar{D}') = (\bar{X}, \bar{D})$. We first construct $(\bar{X}', \bar{D}')$ in $\mathfrak{I}_{(\bar{X},\bar{D})}$ such that the image of φ in $H^n_{D'}(X', \Lambda)$ lifts to $H^n_{S'}(X', \Lambda)$.

Let us treat the case $n = 3$. Since $(\bar{X}, \bar{D})$ has enough tame coverings, there is a desingularized $\mathfrak{c}$-covering

$$(X', D') \to (X_1, D_1) \to (X, D),$$

such that m divides the ramification index of each irreducible component of D_1. We have the following commutative diagram of excision sequences with coefficients Λ:

$$\begin{array}{ccccc}
H^3_S(X) & \longrightarrow & H^3_D(X) & \longrightarrow & H^3_{D-S}(X - S) \\
\downarrow & & \downarrow & & \downarrow \\
H^3_{S' \cup E'}(X') & \longrightarrow & H^3_{D'}(X') & \longrightarrow & H^3_{D'-S' \cup E'}(X' - S' \cup E').
\end{array}$$

Let φ' denote the image of φ in $H^3_{D'}(X', \Lambda)$. Applying Corollary 4.2 to $X' - S' \cup E' \to X - S$, we conclude that the rightmost vertical map is the zero map. Consequently, φ' is mapped to zero in $H^3_{D'-S' \cup E'}(X' - S' \cup E', \Lambda)$. Hence, there is $\varphi'_1 \in H^3_{S' \cup E'}(X', \Lambda)$ mapping to φ'. In Proposition 7.1 we will see that the exceptional fibers of a desingularized $\mathfrak{c}$-covering are always rational trees. Therefore, we can apply Lemma 5.3 with $Z = E'$ to obtain a finite étale $\mathfrak{c}$-covering $B'' \to B'$ and thus via base change a finite étale $\mathfrak{c}$-covering $\bar{X}'' \to \bar{X}'$ such that the image of φ'_1 in $H^3_{S'' \cup E''}(X'', \Lambda)$ lifts to an element $\varphi''_2 \in H^3_{S''}(X'', \Lambda)$. Changing notation we may assume that φ lifts to $\varphi_2 \in H^3_S(X, \Lambda)$.

Now assume that $n \geqslant 4$. By the same argument as for $n = 3$ there is a desingularized $\mathfrak{c}$-covering $(\bar{X}', \bar{D}') \to (\bar{X}, \bar{D})$ such that the image of φ in $H^n_{D'}(X', \Lambda)$ lifts to $H^n_{S' \cup E'}(X', \Lambda)$. In particular, it lifts to $H^n_{D'_v}(X', \Lambda)$ and thus we may assume that φ lifts to $H^n_{D_v}(X, \Lambda)$ right away.

Consider the excision sequence

$$\ldots \to H^n_S(X, \Lambda) \to H^n_{D_v}(X, \Lambda) \to H^n_{D_v - S}(X - S, \Lambda) \to \ldots.$$

By purity we have

$$H^n_{D_v-S}(X-S,\Lambda) \cong H^{n-2}(D_v-S,\Lambda(-1)).$$

For each component Z_i of D_v lying over a closed point $b_i \in B$ with geometric point $\bar{b}_i$ consider the Hochschild-Serre spectral sequence

$$H^r(b_i, H^s((Z_i-S)_{\bar{b}_i},\Lambda)) \Rightarrow H^{r+s}(Z_i-S,\Lambda).$$

Since $(Z_i-S)_{\bar{b}_i}$ is an affine curve over an algebraically closed field, its cohomological dimension is less or equal to one. Moreover, for $r \geqslant 1$, $H^r(b_i, H^s((Z_i - S)_{\bar{b}_i},\Lambda))$ vanishes in the limit over $\mathfrak{I}_B$ as the absolute Galois group of $k(b_i)$ is $\mathfrak{c}$-good and $\pi_1(b,\bar{b})(\mathfrak{c}) \to \pi_1(B,\bar{b})(\mathfrak{c})$ is injective. We conclude that $H^{n-2}(Z_i - S,\Lambda)$ vanishes in the limit over $\mathfrak{I}_B$ for $n \geqslant 4$. As before we replace $\bar{X}$ by $\bar{X}'$ and may assume that φ_1 maps to 0 in $H^n_{D_v-S}(X-S,\Lambda)$. Hence, φ_1 lifts to $\varphi_2 \in H^n_S(X,\Lambda)$.

Having lifted φ to $\varphi_2 \in H^n_S(X,\Lambda)$ for any $n \geqslant 2$ we now construct $(\bar{X}',\bar{D}')$ in $\mathfrak{I}_{(\bar{X},\bar{D})}$ such that φ_2 maps to zero in $H^n_{D'}(X',\Lambda)$. The cohomology group $H^n_S(X,\Lambda)$ is the direct sum of all $H^n_s(X,\Lambda)$ for the finitely many points $s \in S$. For $s \in S$ choose an irreducible component D_s of D passing through s. Since $(\bar{X},\bar{D})$ has enough tame coverings, we find a desingularized $\mathfrak{c}$-covering $(\bar{X}',\bar{D}') \to (\bar{X}_1,\bar{D}_1) \to (\bar{X},\bar{D})$ such that m divides the ramification indices of all irreducible components of $\bar{D}_1$ lying over D_s and is unramified in all other prime divisors passing through s. Since the branch locus is regular in a neighborhood of s, the pair $(\bar{X}_1,\bar{D}_1)$ is regular at all preimage points $s'_1,\dots,s'_r$ of s. Hence, $\bar{X}' \to \bar{X}_1$ is an isomorphism in a neighborhood of $s'_1,\dots,s'_r$. Therefore, by Corollary 4.2, the homomorphism

$$H^n_s(X,\Lambda) \to \bigoplus_i H^n_{s'_i}(X',\Lambda)$$

is the zero map. Take a desingularized $\mathfrak{c}$-covering $(\bar{X}'',\bar{D}'') \to (\bar{X},\bar{D})$ dominating the coverings $(\bar{X}',\bar{D}') \to (\bar{X},\bar{D})$ constructed for each $s \in S$.

We obtain a commutative diagram

$$\begin{array}{ccc} H^n_S(X,\Lambda) & \longrightarrow & H^n_D(X,\Lambda) \\ \downarrow & & \downarrow \\ H^n_{S''\cup E''}(X'',\Lambda) & \longrightarrow & H^n_{D''}(X'',\Lambda), \end{array}$$

where the left vertical homomorphism is the zero map. This implies the assertion. $\square$

6 Killing the Cohomology of higher direct images

In this section we still keep the notation of Section 3 and examine condition (2) of Proposition 3.2, i.e. we strive to kill the cokernel of

$$H^0(B,R^2_D\pi_*\Lambda) \to H^0(B,R^2\pi_*\Lambda)).$$

In the following lemma we explain how to relate this homomorphism with the intersection matrix of the irreducible components of the singular fibers.

Lemma 6.1. *Suppose that B is strictly henselian with closed point s. Denote by ρ the intersection matrix of the components of the special fiber of $\bar{\pi} : \bar{X} \to B$. Then, for any integer c the following diagram commutes*

$$\begin{tikzcd}
H^2_{D_v}(X, \Lambda(c+1)) \arrow[r] & H^2(X, \Lambda(c+1)) \arrow[d, "\sim"', "base\ change"] \\
H^0(D_v, \Lambda(c)) \arrow[u, "purity", "\sim"'] & H^2(X_s, \Lambda(c+1)) \arrow[d, "\sim"', "deg"] \\
\bigoplus_{C \subseteq D_v} \Lambda(c) \cdot C \arrow[u, "\sim"'] \arrow[r, "\rho"] & \bigoplus_{C \subseteq \bar{X}_s, C \cap \bar{D}_h = \varnothing} \Lambda(c) \cdot C.
\end{tikzcd}$$

Proof. It suffices to prove the lemma for $c = 0$. Consider the commutative diagram

$$\begin{tikzcd}
H^2_{D_v}(X, \mu_m) \arrow[r] & H^2(X, \mu_m) \arrow[r, "\sim"] & H^2(X_s, \mu_m) \\
\bigoplus_{C \subseteq D_v} H^1_C(X, \mathbb{G}_m) \otimes \Lambda \arrow[u, "\sim"] \arrow[r] & Pic(X) \otimes \Lambda \arrow[u, "\sim"] \arrow[r] & \bigoplus_{C \cap \bar{D}_h = \varnothing} Pic(C) \otimes \Lambda \arrow[u, "\sim"] \arrow[d, "\sim"', "(deg_C)_C"] \\
\bigoplus_{C \subseteq D_v} \Lambda \cdot C \arrow[u, "\sim"] \arrow[rr] & & \bigoplus_{C \cap \bar{D}_h = \varnothing} \Lambda \cdot C.
\end{tikzcd}$$

The direct sums on the right hand side run only over irreducible components of $\bar{X}_s$ with trivial intersection with $\bar{D}_h$ as these are precisely the components of X_s which are proper over B. The upper right horizontal isomorphism comes from Proposition 4.3. The upper vertical maps are connecting homomorphisms of the Kummer sequence. The concatenation of the left hand side vertical arrows yields the purity isomorphism and the right hand vertical arrows give the degree map on $H^2(X_s, \mu_m)$. The restrictions

$$Pic(X) \to Pic(C)$$

are given by $D \mapsto D \cdot C$ where $D \cdot C$ denotes the intersection product of the divisor D with the curve C. Composition with $\deg_C$ yields the intersection number $(D \cdot C)$. We conclude that the lower horizontal map is indeed given by the intersection matrix $\rho_{C_1, C_2} = (C_1 \cdot C_2)$. □

We set

$$\mathbb{Z}(\mathfrak{c}) = \varprojlim_{n \in \mathbb{N}(\mathfrak{c})} \mathbb{Z}/n\mathbb{Z} = \prod_{\ell \in \mathbb{N}(\mathfrak{c}) \text{ prime}} \mathbb{Z}_\ell.$$

Lemma 6.2. *Assume that $(\bar{X}, \bar{D})$ has enough tame coverings. Then, for every integer $d \in \mathbb{N}(\mathfrak{c})$ there is a desingularized $\mathfrak{c}$-covering $(\bar{X}', \bar{D}') \to (\bar{X}, \bar{D})$ such that the image of*

$$H^2_D(X, \mathbb{Z}(\mathfrak{c})(1)) \to H^2_{D'}(X', \mathbb{Z}(\mathfrak{c})(1))$$

is divisible by d.

Proof. By purity we have

$$\bigoplus_{C\subseteq D} \mathbb{Z}(\mathfrak{c})\cdot C \xrightarrow{\sim} H^2_D(X,\mathbb{Z}(\mathfrak{c})(1)).$$

Moreover, if $(\bar{X}',\bar{D}')\to(\bar{X},\bar{D})$ is a desingularized $\mathfrak{c}$-covering, the induced map

$$\bigoplus_{C\subseteq D} \mathbb{Z}(\mathfrak{c})\cdot C \to \bigoplus_{C'\subseteq D'} \mathbb{Z}(\mathfrak{c})\cdot C'$$

is given by the pull-back of divisors. Let $C\subseteq D$ be an irreducible component and $c\in C$ a closed point of D. Since $(\bar{X},\bar{D})$ has enough tame coverings, there is $f_c\in K(\bar{X})^\times$ with support in $\bar{D}$ such that in a Zariski neighborhood U_c of c we have $div\ f_c = m_c C$ with $m_c>0$. Denote by m_c' the maximal factor of m_c contained in $\mathbb{N}(\mathfrak{c})$. Let $\phi_c:(\bar{X}_c,\bar{D}_c)\to(\bar{X},\bar{D})$ be a desingularized $\mathfrak{c}$-covering with function field extension

$$K(\bar{X}_c)=K(\bar{X})\big(\sqrt[m_c'd]{f_c}\big)|K(\bar{X}).$$

Then $div\ f_c$ is divisible by $m_c'd$ as an element of Div X_c. Thus, $\phi_c^*|_{U_c}(C|_{U_c})$ is divisible by d, i. e., the coefficients of all irreducible components of $\phi_c^*(C)$ whose generic points lie over U_c are divisible by d. This property is conserved by further desingularized coverings.

There are finitely many closed points $c_1,\dots,c_n\in C$ such that the open subschemes $U_{c_1},\dots,U_{c_n}$ cover C. Let $(\bar{X}',\bar{D}')\to(\bar{X},\bar{D})$ be a desingularized $\mathfrak{c}$-covering dominating all coverings $(\bar{X}_{c_i},\bar{D}_{c_i})\to(\bar{X},\bar{D})$ constructed above. Then the pullback of C to $\bar{X}'$ is divisible by d. ☐

Lemma 6.3. *Let B_0 be the strict henselization of B at a geometric point of B over a closed point. Denote by X_0 and D_0 the base change of X and D, respectively, to B_0. Assume that $\bar{D}_h$ is nonempty and meets all irreducible components of W. If $(\bar{X},\bar{D})$ has enough tame coverings, the cokernel of*

$$H^2_{D_0}(X_0,\mathbb{Z}(\mathfrak{c})(1))\to H^2(X_0,\mathbb{Z}(\mathfrak{c})(1))$$

vanishes in the limit over $\mathfrak{I}_{(\bar{X},\bar{D})}$.

Proof. We may replace B by B_0. We just have to check that all tame coverings of $(\bar{X}_0,\bar{D}_0)$ occurring in the proof come from coverings of $(\bar{X},\bar{D})$. It suffices to prove that the cokernel of

$$\phi: H^2_{D_v}(X,\mathbb{Z}(\mathfrak{c})(1))\to H^2(X,\mathbb{Z}(\mathfrak{c})(1))$$

vanishes in the limit over $\mathfrak{I}_{(\bar{X},\bar{D})}$ as $H^2_{D_v}(X,\mathbb{Z}(\mathfrak{c})(1))$ is a direct summand of $H^2_D(X,\mathbb{Z}(\mathfrak{c})(1))$. Taking the inverse limit over all $\Lambda\cong\mathbb{Z}/n\mathbb{Z}$ with $n\in\mathbb{N}(\mathfrak{c})$ of the diagrams in Lemma 6.1 and setting $c=0$, we obtain

$$\begin{tikzcd}
H^2_{D_v}(X,\mathbb{Z}(\mathfrak{c})(1)) \arrow[r, "\phi"] & H^2(X,\mathbb{Z}(\mathfrak{c})(1)) \arrow[d, "\sim"', "\text{base change}"] \\
H^0(D_v,\mathbb{Z}(\mathfrak{c})) \arrow[u, "\text{purity}", "\sim"'] & H^2(X_s,\mathbb{Z}(\mathfrak{c})(1)) \arrow[d, "\sim"', "\deg"] \\
\bigoplus_{C\subseteq D_v}\mathbb{Z}(\mathfrak{c})\cdot C \arrow[u, "\sim"'] \arrow[r, "\rho"] & \bigoplus_{C\subseteq D_v}\mathbb{Z}(\mathfrak{c})\cdot C.
\end{tikzcd}$$

Since we assumed that $\bar{D}_h$ meets all irreducible components of W, we have that $C \cap \bar{D}_h = \varnothing$ if and only if $C \subseteq D_v$. By [15], Theorem 9.1.23 the intersection matrix of the components of the special fiber is negative semidefinite and its radical is generated by a rational multiple of the special fiber. Since we assumed that $\bar{D}_h$ is nonempty, the support of D does not comprise all irreducible components of the special fiber. Hence, the restriction ρ of the intersection matrix to the components of D_v is negative definite. We conclude that

$$\phi \otimes \mathbb{Q} : H^2_D(X, \mathbb{Z}(\mathfrak{c})(1)) \otimes \mathbb{Q} \to H^2(X, \mathbb{Z}(\mathfrak{c})(1)) \otimes \mathbb{Q}$$

is an isomorphism and thus the cokernel of ϕ is $\mathfrak{c}$-torsion. Take $d \in \mathbb{N}(\mathfrak{c})$ such that $d \cdot \operatorname{coker} \phi = 0$. By Lemma 6.2 there is a desingularized $\mathfrak{c}$-covering $(\bar{X}', \bar{D}') \to (\bar{X}, \bar{D})$ of $(\bar{X}, \bar{D})$ such that the image of

$$H^2_D(X, \mathbb{Z}(\mathfrak{c})(1)) \to H^2_{D'}(X', \mathbb{Z}(\mathfrak{c})(1))$$

is divisible by d. Taking into account that multiplication by d is injective on $H^2(X', \mathbb{Z}(\mathfrak{c})(1))$ this proves the result. □

We can now specify sufficient conditions for assertion (2) in Proposition 3.2 to hold:

Proposition 6.4. *Assume that $\bar{D}_h$ is nonempty and intersects all irreducible components of W and that $(\bar{X}, \bar{D})$ has enough tame coverings. Then*

$$\text{coker}\Big(\varinjlim_{\mathfrak{I}(\bar{X},\bar{D})} H^0(B', R^2_{D'}\pi'_*\Lambda) \to \varinjlim_{\mathfrak{I}(\bar{X},\bar{D})} H^0(B', R^2\pi'_*\Lambda)\Big) = 0.$$

Proof. Since the assumptions are stable under desingularized tame coverings (see Proposition 8.1), it suffices to show that the cokernel of

$$H^0(B, R^2_D\pi_*\Lambda) \to H^0(B, R^2\pi_*\Lambda)$$

is killed by a desingularized $\mathfrak{c}$-covering. We have a direct sum decomposition indexed by the irreducible components D_i of D:

$$R^2_D\pi_*\Lambda = \bigoplus_i R^2_{D_i}\pi_*\Lambda.$$

It is sufficient to prove that the cokernel of the vertical part vanishes after a desingularized $\mathfrak{c}$-covering. Both $R^2_{D_v}\pi_*\Lambda$ and $R^2\pi_*\Lambda$ are skyscraper sheaves with support in the singular locus of $X \to B$. We can treat each singular fiber separately and thus assume that B is a henselian discrete valuation ring. We only have to make sure that the constructed desingularized $\mathfrak{c}$-covering extends to a desingularized $\mathfrak{c}$-covering above the initial base scheme. We have the following diagram of exact sequences

$$\begin{tikzcd} H^0(B, R^2_{D_v}\pi_*\mathbb{Z}(\mathfrak{c})) \arrow[r, hook, "\cdot m"] \arrow[d, "\rho"] & H^0(B, R^2_{D_v}\pi_*\mathbb{Z}(\mathfrak{c})) \arrow[r, two heads] \arrow[d, "\rho"] & H^0(B, R^2_{D_v}\pi_*\Lambda) \arrow[d, "\rho"] \\ H^0(B, R^2\pi_*\mathbb{Z}(\mathfrak{c})) \arrow[r, hook, "\cdot m"] & H^0(B, R^2\pi_*\mathbb{Z}(\mathfrak{c})) \arrow[r, two heads] & H^0(B, R^2\pi_*\Lambda). \end{tikzcd} \tag{5}$$

The exactness of the above sequences can be checked by using the explicit description of the cohomology groups involved.

In order to show that the cokernel of the right hand side vertical map in diagram (5) vanishes after a desingularized $\mathfrak{c}$-covering it suffices to show that the cokernel of the middle vertical map does so. The stalk of the morphism $R^2_{D_v}\pi_*\mathbb{Z}(\mathfrak{c}) \to R^2\pi_*\mathbb{Z}(\mathfrak{c})$ at $\bar{b}$ is

$$H^2_{D_{\bar{b}}}(X^{sh}, \mathbb{Z}(\mathfrak{c})) \to H^2(X^{sh}, \mathbb{Z}(\mathfrak{c})).$$

By Lemma 6.1 it is given by the intersection matrix ρ of the components of $D_{\bar{b}}$. Since $D_{\bar{b}}$ does not contain all components of the geometric special fiber, ρ is injective. Denote by $\mathscr{F}$ the cokernel. By Lemma 6.3 there is a desingularized $\mathfrak{c}$-covering $(\bar{X}', \bar{D}') \to (\bar{X}, \bar{D})$ such that $\mathscr{F} \to \mathscr{F}'$ is the zero map (where $\mathscr{F}'$ is the respective cokernel defined on X'). We have an exact sequence

$$0 \to H^0(B, R^2_{D_v}\pi_*\mathbb{Z}(\mathfrak{c})) \to H^0(B, R^2\pi_*\mathbb{Z}(\mathfrak{c})) \to H^0(B, \mathscr{F}).$$

So the cokernel of $H^0(B, R^2_{D_v}\pi_*\mathbb{Z}(\mathfrak{c})) \to H^0(B, R^2\pi_*\mathbb{Z}(\mathfrak{c}))$ is a subgroup of $\mathscr{F}$. This shows the result. □

7 Exceptional fibers

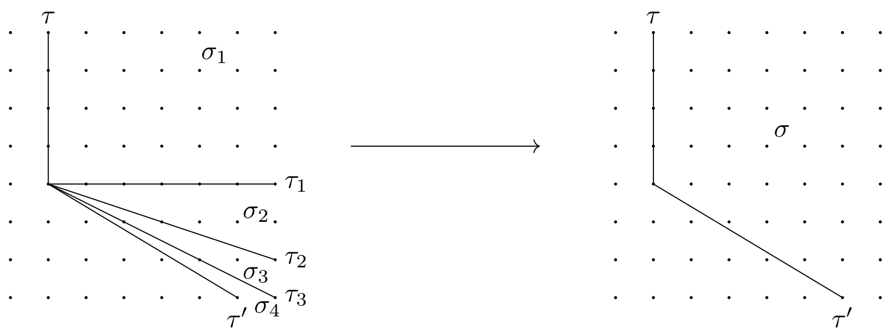
Remember that we postponed the discussion of three issues: Firstly, we have to show that the category $\mathfrak{I}_{(\bar{X},\bar{D})}$ is cofiltered. Secondly, we have yet to see that the dual graph of the exceptional divisor of a desingularized tame covering is simply connected. Finally, we need that the property of having enough tame coverings is stable under desingularized $\mathfrak{c}$-coverings. All three assertions rely upon an examination of the exceptional fibers of a desingularized tame covering. In this section we describe the structure of these exceptional fibers and answer the first two questions. The treatment of the third question is completed in Section 8.

Let us call curve a noetherian scheme whose irreducible components are one-dimensional. We say that a curve C is a *chain of* $\mathbb{P}^1$*'s* if it is a scheme of finite type over a field k whose irreducible components $C_1, \ldots, C_n$ are isomorphic to $\mathbb{P}^1_k$, for $i = 1, \ldots, n-1$ the curve C_i intersects C_{i+1} in exactly one point, which is moreover k-rational, and $C_i \cap C_j$ is empty for $|i-j| \geqslant 2$. If C is a closed subscheme of another curve C_0, we say that C is a *bridge of* $\mathbb{P}^1$*'s* in C_0 if C is a chain of $\mathbb{P}^1$'s and C intersects exactly two of the remaining irreducible components of C_0 and this intersection takes place in two k-rational points $c_1 \in C_1$ and $c_n \in C_n$.

The following result was proved in case B is the spectrum of a discrete valuation ring with algebraically closed residue field by Viehweg (see [20]). In general, it boils down to the fact that logarithmic singularities on a surface are of type A_n. This should be well known. However, the author was not able to find a good reference. Therefore, we include a proof.

Proposition 7.1. *Let X/B be an arithmetic surface and $D \subseteq X$ a tidy divisor. Let $(X_1, D_1) \to (X, D)$ be a tame covering of (X, D) and $(X'_{min}, D'_{min}) \to (X_1, D_1)$ the minimal desingularization of (X_1, D_1). Then D'_{min} is a tidy divisor and the exceptional fibers of $X'_{min} \to X_1$ are bridges of $\mathbb{P}^1$'s in D'_{min} (i. e. the singularities of X_1 at points in D_1 are of type A_n, or Hirzebruch-Jung singularities). In particular, $(X'_{min}, D'_{min}) \to (X, D)$ is a tidy desingularization of (X, D). Moreover, for any other desingularized tame covering $(X', D') \to (X, D)$ the exceptional fibers are bridges of $\mathbb{P}^1$'s in D', as well.*

Proof. We view (X, D) as a log scheme with log structure given by the divisor D. Since D has normal crossings, (X, D) is log-regular and the corresponding log structure is toric. The tame covering $(X_1, D_1) \to (X, D)$ is Kummer étale by the logarithmic version of Abhyankar's lemma (see [9], Thm. 7.3.44). In particular, it is log-smooth and thus (X_1, D_1) is log-regular and the corresponding log structure $\mathscr{M}_{D_1} \to \mathcal{O}_{X_1}$ toric, as well. In section 10 of [13] Kato associates a fan F_{D_1} to the log scheme (X_1, D_1). In this context a fan is a monoidal space which is locally isomorphic to *Spec* P for a sharp monoid P. As a topological space the fan F_{D_1} is the subspace $\{x \in X_1 | I(x, \mathscr{M}_{D_1}) = \mathfrak{m}_x\}$ of X_1, where $I(x, \mathscr{M}_{D_1})$ is the ideal generated by $\mathscr{M}_{D_1,x} \backslash \mathscr{M}^{\times}_{D_1,x}$. The structure sheaf is given by the inverse image of $\mathscr{M}_{D_1} \backslash \mathcal{O}^{\times}_{X_1}$. Since the log structure of (X_1, D_1) is toric, the fan F_{D_1} corresponds to a classical fan Δ, i.e. a fan of convex polyhedral cones in a two-dimensional lattice L as described in [8]. We may work locally and thus assume that Δ consists of one two-dimensional cone σ together with its two one-dimensional faces τ and τ' and $\{0\}$. The faces τ and τ' correspond to prime divisors P and P' on X_1 constituting the irreducible components of D_1 (see [13], Corollary 11.8). They intersect in one point $x_1 \in X_1$, which is the only possibly singular point of X_1. By [8], section 2.6 we can find a subdivision Δ' of Δ in cones which are isomorphic to $\mathbb{N}^2$. In dimension 2 a subdivision of σ is given by inserting additional rays $\tau_1, \ldots, \tau_{n-1}$ forming the faces of cones $\sigma_1, \ldots, \sigma_n$.



By [13], 10.4. this provides us with a resolution of singularities $(X', D') \to (X_1, D_1)$ such that D' has strictly normal crossings. The exceptional fiber consists of prime divisors $E_1, \ldots, E_{n-1}$ corresponding to the rays $\tau_1, \ldots, \tau_{n-1}$ and E_i intersects E_{i+1} in one point corresponding to the cone σ_i. The strict transforms of P and P' correspond to the rays τ and τ'. Hence, P intersects τ_1 in one point and P' intersects τ_{n-1} in one point. It remains to see that the E_i are rational. By the proof of [13], Proposition 9.9 we have

$$X' = X_1 \times_{\mathbb{Z}[\Delta]} \mathit{Spec}\, \mathbb{Z}[\Delta'].$$

The exceptional fiber is thus given by

$$\mathit{Spec}\, k(x_1) \times_{\mathbb{Z}[\Delta]} \mathit{Spec}\, \mathbb{Z}[\Delta'].$$

Locally this is the spectrum of $k(x_1)[\sigma_i^\vee / \sigma^\vee]$, which is readily checked to be rational. □

Corollary 7.2. *In the situation of Proposition 7.1 let x_1 be a special point of D_1. Let Z_1 be an irreducible component of D_1 containing x_1. Denote by Z'*

its strict transfer in X'_{min} *and by* Z *its image in* X. *Let* $E_1, \ldots, E_n$ *be the irreducible components of the exceptional fiber of* $X'_{min} \to X_1$ *above* x_1 *such that* E_i *intersects* E_{i+1} *and* Z' *intersects* E_1. *Then above an open neighborhood of* x_1 *the pullback of* Z *to* X'_{min} *is given by*

$$a_0 Z' + a_1 E_1 + \ldots a_n E_n$$

with $a_0 > a_1 > \ldots > a_n > 0$.

Proof. Denote by b the image of x in B and by φ the morphism $X'_{min} \to X$. In order to simplify notation, we set $E_0 := Z'$. By the projection formula and Proposition 7.1 we have

$$0 = \varphi^* Z \cdot E_n = (a_0 E_0 + a_1 E_1 + \ldots a_n E_n) \cdot E_n = [k(x_1) : k(b)] a_{n-1} + a_n E_n^2.$$

Since the desingularization $X'_{min} \to X_1$ is minimal, E_n cannot be a (-1)-curve and thus $E_n^2 < -[k(x_1) : k(b)]$. (The self-intersection of E_n has to be negative by [15], chapter 9, Theorem 1.27.) Hence,

$$a_{n-1} = -a_n E_n^2 > a_n.$$

By induction we may assume that $a_{i+1} < a_i$ for $0 < k \leqslant i < n$. Again by the projection formula we obtain

$$0 = \varphi^* Z \cdot E_k = [k(x_1) : k(b)](a_{k-1} + a_{k+1}) + a_k E_k^2.$$

By induction and using $E_k^2 \leqslant -2[k(x_1) : k(b)]$ we conclude that

$$a_{k-1} = -a_{k+1} - \frac{a_k}{[k(x_1) : k(b)]} E_k^2 \geqslant -a_{k+1} + 2a_k > a_k. \qquad \square$$

Corollary 7.3. *Let* X/B *be an arithmetic surface and* $D \subseteq X$ *a tidy divisor. Let* $(X_1, D_1) \to (X, D)$ *be a tame covering of* (X, D) *and* $(X', D') \to (X_1, D_1)$ *a desingularization of* (X_1, D_1). *Assume that every irreducible component of an exceptional fiber of* $(X', D') \to (X_1, D_1)$ *intersects the other irreducible components of* D' *in at least two points. Then* $(X', D') \to (X_1, D_1)$ *is a tidy desingularization.*

Proof. We can factor $(X', D') \to (X_1, D_1)$ as

$$(X', D') := (X'_n, D'_n) \to \ldots \to (X'_1, D'_1) \to (X'_0, D'_0) \to (X_1, D_1),$$

where $(X'_0, D'_0) \to (X_1, D_1)$ is the minimal desingularization of (X_1, D_1) and for $i = 1, \ldots, n$ the morphism $(X'_i, D'_i) \to (X'_{i-1}, D'_{i-1})$ is the blowup of X'_{i-1} in a closed point d'_{i-1} of D'_{i-1}. By Proposition 7.1 the minimal desingularization $(X'_0, D'_0) \to (X_1, D_1)$ is a tidy desingularization. Moreover, blowing up in closed points does not destroy the tidiness of a divisor. Hence, D'_i is a tidy divisor of X'_i for all $i = 0, \ldots n$. Suppose that $(X', D') \to (X_1, D_1)$ is not a tidy desingularization. Then there is an index i such that d'_{i-1} is not a special point of D'_{i-1}, i. e., d'_{i-1} is a regular point of D'_{i-1}. Let i_0 be the biggest such index. Then the exceptional fiber of $(X'_{i_0}, D'_{i_0}) \to (X'_{i_0-1}, D'_{i_0-1})$ has only one intersection point with the other irreducible components of D'_{i_0}. This does not change by blowing up D'_{i_0} in special points. We thus obtain a contradiction. $\square$

Let D be a tidy divisor on an arithmetic surface X and $\bar{x} \to U = X - D$ a geometric point. The explicit description of the exceptional fibers in Proposition 7.1 enables us to prove that the category $\mathfrak{I}_{(X,D)}$ is cofiltered:

Proposition 7.4. *The following assertions hold:*

(i) If $(X', D') \to (X, D)$ and $(X'', D'') \to (X', D')$ are both desingularized $\mathfrak{c}$-coverings, the composite $(X'', D'') \to (X, D)$ is again a desingularized $\mathfrak{c}$-covering.

(ii) If $(X', D') \to (X, D)$ and $(X'', D'') \to (X, D)$ are desingularized $\mathfrak{c}$-coverings, there is a commutative diagram of desingularized $\mathfrak{c}$-coverings

$$\begin{tikzcd} & (X', D') \arrow[dr] & \\ (X''', D''') \arrow[ur] \arrow[dr] & & (X, D). \\ & (X'', D'') \arrow[ur] & \end{tikzcd}$$

Proof. (i). Let X_1 be the normalization of X in $K(X')$ and X_2 its normalization in $K(X'')$. Furthermore, denote by X_1' the normalization of X' in $K(X'')$. We obtain a Cartesian diagram

$$\begin{tikzcd} D'' \arrow[r] \arrow[d, hook] & D_2 \arrow[r] \arrow[d, hook] & D \arrow[d, hook] \\ X'' \arrow[r] & X_2 \arrow[r] & X. \end{tikzcd}$$

Since $U' = X' - D'$ is the normalization of $U = X - D$ in $K(X')$ and $U'' = X'' - D''$ is the normalization of U' in $K(X'')$, we conclude that U'' is also the normalization of U in $K(X'')$. It is thus an open subscheme of X_2 and $U'' \to U$ is a finite étale $\mathfrak{c}$-covering as finite étale $\mathfrak{c}$-coverings are stable under composition. Hence, $X'' \to X_2$ is birational and an isomorphism on U''. Moreover, $D'' \subseteq X''$ is a tidy divisor. The only remaining question is whether $X'' \to X_2$ is obtained from the minimal desingularization of (X_2, D_2) by successively blowing up in special points. By Corollary 7.3 it suffices to show that every irreducible component of an exceptional fiber of $X'' \to X_2$ meets the other irreducible components of D'' in at least two points. The morphisms $X'' \to X'$ and $X' \to X$ factor as

$$\begin{aligned} (X', D') &= (Y_m, Z_m) \to \ldots \to (Y_0, Z_0) &&= (X_1, D_1) \to (X, D), \\ (X'', D'') &= (Y_n, Z_n) \to \ldots \to (Y_{m+1}, Z_{m+1}) &&= (X_1', D_1') \to (X', D'), \end{aligned}$$

where $(Y_0, Z_0) \to (X_1, D_1)$ and $(Y_{m+1}, Z_{m+1}) \to (X_1', D_1')$ represent the minimal desingularizations of (X_1', D_1') and (X_1, D_1), respectively, and for $i = 1, \ldots, m$ and $i = m+2, \ldots, n$ the morphism $(Y_i, Z_i) \to (Y_{i-1}, Z_{i-1})$ is the blowup of Y_{i-1} in a special point z_{i-1} of Z_{i-1}. Let E be an irreducible component of an exceptional fiber of $X'' \to X_2$. There is $i \in \{1, \ldots, m\} \cup \{m+2, \ldots n\}$ such that the image of E in Y_i is one-dimensional and its image in Y_{i-1} is a closed point. This closed point is precisely the point z_{i-1} and we obtain a finite morphism from E to the exceptional fiber of $Y_i \to Y_{i-1}$. Since $X_i \to X_{i-1}$ is the blowup of X_{i-1} in z_{i-1} and z_{i-1} is a special point, its exceptional fiber intersects the other irreducible components of Z_i in two points. The intersection points of E contain the preimages of these two points and thus there are at least two intersection points.

(ii). Let K''' be the compositum of $K(X')$ and $K(X'')$ and X_3 the normalization of X in K'''. This defines a $\mathfrak{c}$-covering $(X_3, D_3) \to (X, D)$. We obtain rational maps $X_3 \dashrightarrow X'$ and $X_3 \dashrightarrow X''$, which, restricted to $U_3 = X_3 - D_3$, are finite étale $\mathfrak{c}$-coverings of $U' = X' - D'$ and $U'' = X'' - D''$, respectively. Using elimination of indeterminacies and the existence of tidy desingularizations we find a desingularization $(X''', D''') \to (X_3, D_3)$ dominating (X', D') and (X'', D'') such that D''' is tidy. Suppose there is an irreducible component E of an exceptional fiber of X''' with only one intersection point with the other irreducible components of D'''. Let us write

$$(X''', D''') = (X'''_n, D'''_n) \to \ldots \to (X'''_0, D'''_0) \to (X_3, D_3),$$

where $(X'''_0, D'''_0) \to (X_3, D_3)$ is the minimal desingularization of (X_3, D_3) and for $i = 1, \ldots, n$ the morphism $X'''_i \to X'''_{i-1}$ is the blowup of X'''_{i-1} in a closed point $d_{i-1} \in D'''_{i-1}$. There is $i \in \{1, \ldots, n\}$ such that the image of E is the point d_{i-1} and the image of E in X'''_i is the exceptional fiber E_i of $X'''_i \to X'''_{i-1}$. Since E has only one intersection point, the same holds for E_i. Furthermore, the blowup points d_{k-1} for $k = i+1, \ldots, n$ must not lie above E_i except possibly above the intersection point z_{i-1} of E_{i-1} with the other irreducible components. One checks that after blowing up in z_{i-1} the strict transform of E_i is still a (-1)-curve. Therefore, we can contract E. Moreover, by similar arguments as in the proof of part (i) the image of E in X' as well as in X'' is a point. Hence, the contraction still factors through $X' \to X$ and $X'' \to X$. After finitely many contractions we may assume that all irreducible components of exceptional fibers of $X''' \to X_3$ have at least two intersection points. Then the same holds for the exceptional fibers of $X''' \to X'$ and of $X''' \to X''$ as these are contained in the exceptional fibers of $X''' \to X_3$. The assertion now follows from Corollary 7.3. $\square$

8 Stability of enough tame coverings

Let us fix an arithmetic surface X/B and a tidy divisor $D \subseteq X$. The aim of this section is to show:

Proposition 8.1. *Let $\pi : (X', D') \to (X_1, D_1) \to (X, D)$ be a desingularized $\mathfrak{c}$-covering. If (X, D) has enough tame coverings, the same holds for (X', D').*

For the proof of Proposition 8.1 we need to investigate the multiplicities of the irreducible components of the pullback to X' of a prime divisor on X.

Definition 8.2. *Let $f : (X', D') \to (X, D)$ be a desingularized $\mathfrak{c}$-covering. Let $x' \in D'$ be a closed point and denote by $x \in D$ the image of x' in X. Let us call $D_1, \ldots, D_n$ (necessarily $n = 1$ or $n = 2$) the irreducible components of D passing through x and $D'_1, \ldots, D'_m$ ($m \leqslant n$) the irreducible components of D' passing through x'. Restricting f to a suitable neighborhood of x', the pullback* *of Cartier divisors via f induces a homomorphism*

$$\mathbb{Q} \cdot D_1 \oplus \ldots \oplus \mathbb{Q} \cdot D_n \to \mathbb{Q} \cdot D'_1 \oplus \ldots \oplus \mathbb{Q} \cdot D'_m.$$

We call this morphism multiplicity homomorphism *at x' and its transformation matrix with respect to the above bases* multiplicity matrix *at x'.*

Multiplicity homomorphisms are compatible with composition. If $(X'', D'') \to (X', D')$ is another morphism as above and x'' a closed point of D'' mapping to $x' \in D'$, the multiplicity homomorphism of $(X'', D'') \to (X', D')$ at x'' is the composition of the multiplicity homomorphism of $(X'', D'') \to (X', D')$ at x'' and the multiplicity homomorphism of $(X', D') \to (X, D)$ at x'.

Lemma 8.3. *Let $(X', D') \to (X, D)$ be the blowup of X in a special point x of D. Then all multiplicity homomorphisms are surjective.*

Proof. Denote by D_1 and D_2 the irreducible components of D passing through x and by D'_1 and D'_2 their strict transforms in X'. Furthermore, let E denote the singular fiber of $X' \to X$. On $E \subseteq D'$ there are two points x'_1 and x'_2 where D' is singular, namely the respective intersection points with D'_1 and D'_2. The pullback of D_i is given by $D'_i + E$. Hence, the intersection matrix at x'_1 as well as at x'_2 (with respect to the bases $\{(D_1, D_2), (D'_1, E)\}$ and $\{(D_1, D_2), (E, D'_2)\}$, respectively) is

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

which is invertible. If $x' \in E$ is a nonsingular point of D', its multiplicity matrix is

$$\begin{pmatrix} 1 & 1 \end{pmatrix},$$

which is nonzero and thus its multiplicity homomorphism is surjective. The multiplicity homomorphism at any other closed point of D' is the identity. $\square$

Lemma 8.4. *Let $\varphi : (X', D') \to (X_1, D_1) \to (X, D)$ be a desingularized $\mathfrak{c}$-covering. Then all multiplicity homomorphisms are surjective.*

Proof. By Lemma 8.3 we may assume that $X' \to X_1$ is the minimal desingularization of X_1. Let $x' \in D'$ be a closed point and denote by x_1 and x the image of x' in X_1 and X, respectively. If x' is a regular point of D', there is only one irreducible component of D' passing through x'. Hence, the multiplicity homomorphism at x' is surjective if and only if it is nonzero, which is clear by taking the pullback of any irreducible component of D passing through x.

Suppose that x' is a singular point of D'. Then also x_1 and x are singular points of D_1 and D, respectively. There are two irreducible components Z_1 and W_1 of D_1 passing through x_1 mapping to the irreducible components W and Z of D passing through x. According to Corollary 7.2 we have in a neighborhood of x'

$$\varphi^* Z = a_0 Z' + a_1 E_1 + \ldots a_n E_n$$

with $a_0 > a_1 > \ldots a_n > 0$ and

$$\varphi^* W = b_1 E_1 + \ldots b_n E_n + b_{n+1} W'$$

with $b_1 < \ldots < b_n < b_{n+1}$, and where Z' and W' denote the strict transforms of Z_1 and W_1, respectively, in X'. Setting $E_0 := Z'$ and $E_{n+1} := W'$ we know that there is an integer i with $0 \leqslant i \leqslant n$ such that x' is the intersection point of E_i with E_{i+1}. The multiplicity matrix at x' is

$$\begin{pmatrix} a_i & b_i \\ a_{i+1} & b_{i+1} \end{pmatrix}$$

and

$$det \begin{pmatrix} a_i & b_i \\ a_{i+1} & b_{i+1} \end{pmatrix} = a_i b_{i+1} - a_{i+1} b_i > a_i b_i - a_i b_i = 0$$

as $a_{i+1} < a_i$ and $b_{i+1} > b_i$. Therefore, also in this case the multiplicity homomorphism is surjective. □

Proof of Proposition 8.1. Assume that (X, D) has enough tame coverings. Let $x' \in D'$ be a closed point and Z' an irreducible component of D' passing through x'. We have to find $f' \in K(X')^\times$ with support in D' such that $\deg_{Z'}(f') > 0$ and $\deg_{C'}(f') = 0$ for all other irreducible components C' of D' passing through x'. Let $Z_1, \ldots, Z_r$ (for $r = 1$ or $r = 2$) denote the irreducible components of D passing through the image point $x \in D$ of x'. Since (X, D) has enough tame coverings, for $i = 1, \ldots, r$ there is $f_i \in K(X)^\times$ with support in D such that $\deg_{Z_i}(f_i) > 0$ and $\deg_{Z_j}(f_i) = 0$ for $i \neq j$. The projections of $div\ f_i$ to

$$\mathbb{Q} \cdot Z_1 \oplus \ldots \oplus \mathbb{Q} \cdot Z_r$$

constitute a basis of this vector space. Let $Z' = Z'_1, \ldots, Z'_s$ denote the irreducible components of D' passing through x'. Lemma 8.4 provides the surjectivity of the multiplicity homomorphism

$$\phi_{x'} : \mathbb{Q} \cdot Z_1 \oplus \ldots \oplus \mathbb{Q} \cdot Z_r \to \mathbb{Q} \cdot Z'_1 \oplus \ldots \oplus \mathbb{Q} \cdot Z'_s$$

at x' induced by pullback. We obtain integers $d, k_1, \ldots, k_r$ with $d > 0$ such that

$$d \cdot Z'_1 = \phi_{x'}(k_1 div\ f_1 + \ldots k_r div\ f_r).$$

In other words, setting $f = f_1^{k_1} \cdot \ldots f_r^{k_r}$ we have in a neighborhood of x'

$$div\ f = d \cdot Z',$$

what we wanted to prove. □

9 Neighborhoods with enough tame coverings

At this point we have completed the discussion of conditions (1) and (2) in Proposition 3.2. As a result we know that under the assumptions listed in Proposition 5.4 and Proposition 6.4 the arithmetic surface U is $K(\pi, 1)$ with respect to $\mathfrak{c}$. The remaining task is to construct neighborhoods on a given arithmetic surface satisfying these assumptions. The property of having enough tame coverings is the most difficult to realize. It is the aim of this section to explain how to construct neighborhoods with enough tame coverings. We need the following variant of prime avoidance.

Lemma 9.1. *Let A be a noetherian ring and $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ and $\mathfrak{q}_1, \dots, \mathfrak{q}_s$ prime ideals such that for $i \neq j$ $\mathfrak{q}_i$ is not contained in $\mathfrak{q}_j$. For $j \leqslant s$ define the integer m_j by*

$$\mathfrak{p}_1 \cdot \dots \cdot \mathfrak{p}_r \subseteq \mathfrak{q}_j^{m_j} \backslash \mathfrak{q}_j^{m_j+1}.$$

Then there is $a \in A$ such that for $i \leqslant r$ we have $a \in \mathfrak{p}_i$ and for $j \leqslant s$ we have $a \notin \mathfrak{q}_j^{m_j+1}$.

For the rest of this section we use the following notation: For an integral closed subscheme Z of an affine scheme *Spec* A we denote by $\mathfrak{p}_Z$ the prime ideal of A corresponding to the generic point of Z. Moreover, we write $m_x(Z)$ for the multiplicity of a closed subscheme Z in a point x.

Lemma 9.2. *Let X/B be a quasi-projective arithmetic surface such that B is a discrete valuation ring with finitely generated quotient field. Let $x_1, \dots, x_n$ be finitely many points of X. Then there are horizontal prime divisors $G_1, \dots, G_s$, $G_{s+1}, \dots, G_r$ such that $G_1, \dots, G_s$ and $G_{s+1}, \dots, G_r$ each generate the Weil divisor class group $CH^1(X)$ of X. Furthermore, the supports of G_i for $i = 1, \dots, r$ do not contain x_j for $j = 1, \dots, n$ and the supports of G_i and G_j for $i \leqslant s$ and $j > s$ are disjoint.*

Proof. The generic fiber X_η of $X \to B$ is a smooth curve over a finitely generated field. By a generalization of the Mordell-Weil theorem due to Néron (see [16]) its Weil divisor class group is finitely generated. Denote by $C_1, \dots, C_l$ the irreducible components of the special fiber. The Weil divisor class group of X is generated by the Weil divisor class group of X_η and by $C_1, \dots, C_l$. It is therefore also finitely generated, by prime divisors $D_1, \dots, D_m$, say.

Since X is quasi-projective over an affine scheme, there is an affine open subscheme *Spec* $A \subseteq X$ containing $x_1, \dots, x_n$, as well as the generic points of $D_1, \dots, D_m$ and of $C_1, \dots, C_l$ (see [15], Proposition 3.3.36). By Lemma 9.1 we can choose $f_1, \dots, f_m \in A$ such that for $i = 1, \dots, m$

$$\begin{aligned} &f_i \in \mathfrak{p}_{D_i} \backslash \mathfrak{p}_{D_i}^2, \\ &f_i \notin \mathfrak{p}_{C_j}^{m_{C_j}(D_i)+1} \text{ for } j = 1, \dots, l, \\ &f_i \notin \mathfrak{p}_{x_j}^{m_{x_j}(D_i)+1} \text{ for } j = 1, \dots, n. \end{aligned}$$

Viewing f_i as elements of $K(X)^\times$ we obtain divisors $D_1 - div\ f_1, \dots, D_m - div\ f_m$ generating the Weil divisor class group. The supports of the divisors $D_i - div\ f_i$ do not contain $x_1, \dots, x_n$ and the coefficients of $C_1, \dots, C_l$ are zero, i. e., $D_i - div\ f_i$ are horizontal. Denote by $G_1, \dots, G_s$ the prime divisors in the support of $D_1 - div\ f_1, \dots, D_m - div\ f_m$. Then $G_1, \dots, G_s$ are horizontal prime divisors generating the Weil divisor class group whose supports do not contain $x_1, \dots, x_n$.

Denote by $z_1, \dots, z_t$ the intersection points of $G_1, \dots, G_s$ with the special fiber. By the same argument as above we find horizontal prime divisors $G_{s+1}, \dots, G_r$ generating $\mathrm{CH}^1(X)$ whose supports do not contain $x_1, \dots, x_n$ nor $z_1, \dots, z_t$. Hence, the support of G_i for $i \leqslant s$ is disjoint from the support of G_j for $j > s$ as $z_1, \dots, z_t$ are the only possible intersection points of G_i with another divisor. □

Lemma 9.3. *Let X/B be an arithmetic surface and let $G_1, \dots, G_s$ be horizontal prime divisors generating the Weil divisor class group $CH^1(X)$ of X. Let x be a closed point of X of codimension 2 such that X is regular at x and x is not contained in any G_j for $j = 1, \dots, s$. Denote by $X' \to X$ the blowup of X in x. Let G be a horizontal prime divisor disjoint from $G_1, \dots, G_s$ with nontrivial intersection with the exceptional locus E. Then $G_1, \dots, G_s, G$ generate $CH^1(X') \otimes \mathbb{Q}$.*

Proof. The Weil divisor class group of X' is generated by $G_1, \dots, G_s$ and E. Let G_0 denote the image of G in X. Since $G_1, \dots, G_s$ generate the Weil divisor class group of X, there are $n_j \in \mathbb{Z}$ such that

$$G_0 = \sum_{j=1}^{s} n_j G_j$$

in $\mathrm{CH}^1(X)$. By [15], Chapter 9, Proposition 2.23 the pullback of G_0 to X' is given by

$$G + m_x(G_0) \cdot E.$$

Since $x \in G_0$, the multiplicity $m_x(G_0)$ is positive. In $\mathrm{CH}^1(X') \otimes \mathbb{Q}$ we thus have

$$E = \frac{1}{m_x(G_0)} (\sum_{j=1}^{s} n_j G_j - G). \qquad \square$$

Lemma 9.4. *Let X be a projective arithmetic surface over a discrete valuation ring B and D a tidy divisor on X. Let $x, z_1, \dots, z_k \in X$ be closed points such that X and the reduced special fiber $X_{s,red}$ are regular at x. Assume moreover that x is not a special point of D. Then there is a horizontal prime divisor D_x passing through x and disjoint from $z_1, \dots, z_k$ such that $D_x + D$ is tidy.*

Proof. Denote by $s = Spec\, k$ the special point of B and by $\eta = Spec\, K$ the generic point. Choose an embedding $X \hookrightarrow \mathbb{P}^N_B$. This induces embeddings $X_s \hookrightarrow \mathbb{P}^N_k$ and $X_\eta \hookrightarrow \mathbb{P}^N_K$. Let T be the finite subscheme of $\mathbb{P}^N_k$ which is the disjoint union of all singular points of X, all singular points of $X_{s,red}$ and all special points of D (they are all contained in the special fiber). In order to prove the lemma it suffices to find a hyperplane H of $\mathbb{P}^N_B$ intersecting X transversally, passing through x, and disjoint from T such that $D_x := H \times_{\mathbb{P}^N_B} X$ is regular and $D_x + D$ is tidy. By [12], Lemma 1.3 a hyperplane H satisfies these conditions if

(i) H_s intersects X_s transversally, passes through x, and is disjoint from T,

(ii) H_η intersects X_η transversally.

Assume first that k is finite. By [17], Thm. 1.2 there is a hypersurface H_s of $\mathbb{P}^N_k$ intersecting $X_{s,red}$ transversally, passing through x, and disjoint from T. Changing the projective embedding we may assume that H_s is a hyperplane. If k is infinite, the existence of the hyperplane H_s follows by the classical Bertini theorem.

Let H be any hyperplane of $\mathbb{P}^N_B$ with special fiber the hyperplane H_s constructed above. We claim that the generic fiber H_η intersects X_η transversally. Let $y \in X_\eta$ be a closed point in the intersection and choose a point y_s of the special fiber which is a specialization of y. Then y_s is not contained in T as H_s is disjoint from T. Hence, H_s intersects X_s transversally at y_s. Since y is a generalization of y_s this implies that H_η intersects X_η transversally at y. $\square$

Proposition 9.5. *Let Y/B be an arithmetic surface such that B is a discrete valuation ring with finitely generated quotient field and $x \in Y$ a closed point in the special fiber. Then there is an open neighborhood $V \subseteq Y$ of x and a compactification $\bar{X}/B$ of V such that $\bar{D} = \bar{X} - V$ is a tidy divisor and such that the following assertion holds: For every closed point $y \in \bar{X}$ and every prime divisor Z of $\bar{X}$ passing through y there is $f \in K(\bar{X})^\times$ with support in $Z \cup \bar{D}$ such that $\deg_Z(f) > 0$ and $\deg_W(f) = 0$ for all other prime divisors W passing through y.*

Proof. Take an affine open neighborhood V' of x such that the complement contains all singular points except x and all vertical prime divisors not passing through x. Since V' is affine, we can choose a projective compactification $\bar{V}'$ of V' over B. Set $D' = \bar{V}' - V'$ with the reduced scheme structure. By [14] we can replace $(\bar{V}, D')$ by a desingularization (in the strong sense) and thus assume that x is the only possible singular point of $\bar{V}'$ and D' is a Cartier divisor. Choose prime divisors $G_1, \ldots, G_r$ of $\bar{V}'$ not passing through x as in Lemma 9.2. Making V' smaller we may assume that $G_1, \ldots, G_r$ are contained in D'.

Let $(\bar{X}, D_0) \to (\bar{V}', D')$ be a tidy desingularization, which exists by Proposition 2.2. Since $\bar{V}'$ is regular at every point in D', the morphism $\bar{X} \to \bar{V}'$ is a consecutive blowup in closed points over D'. Moreover, the exceptional fiber of each blowup in a closed point z is isomorphic to $\mathbb{P}^1_{k(z)}$ (see [15], Chapter 8, Theorem 1.19). Denote by $E_1, \ldots, E_n$ the irreducible components of the exceptional divisor of $\bar{X} \to \bar{V}'$. For each $i = 1, \ldots, n$ choose two different closed points $y_i, z_i \in E_i$ in the regular locus of D_0. By Lemma 9.4 there is a (horizontal) prime divisor D_1 intersecting E_1 transversally at y_1 and disjoint from $y_2, \ldots, y_n, z_1, \ldots, z_n$ such that $D_0 + D_1$ is tidy. By the same argument there is a prime divisor D_2 intersecting E_2 transversally at y_2 and disjoint from $y_3, \ldots, y_n, z_1, \ldots, z_n$ such that $D_0 + D_1 + D_2$ is tidy. Continuing this way we obtain for $i = 1, \ldots, n$ horizontal prime divisors D_i and K_i intersecting E_i transversally at y_i and z_i, respectively, and such that

$$\bar{D} := D_0 + D_1 + \ldots + D_n + K_1 + \ldots + K_n$$

is tidy. We set $V = \bar{X} - \bar{D}$.

We claim that $(\bar{X}, \bar{D})$ has the required properties. Let $y \in \bar{X}$ be a closed point and Z a prime divisor of $\bar{X}$ passing through y. Either $G_1, \ldots, G_s$ or $G_{s+1}, \ldots, G_r$ do not pass through y, say $G_1, \ldots, G_s$. Similarly, either $D_1, \ldots, D_n$ or $K_1, \ldots, K_n$ do not pass through y, say $D_1, \ldots, D_n$. By Lemma 9.3 the prime divisors $G_1, \ldots, G_s, D_1, \ldots, D_n$ generate the first Chow group $\mathrm{CH}^1(\bar{X}) \otimes \mathbb{Q}$. Hence, there are $m, m_1, \ldots, m_n, n_1, \ldots, n_s \in \mathbb{Z}$ with $m > 0$ and $f \in K(\bar{X})^\times$ such that

$$mZ = \sum_{j=1}^{n} m_j D_j + \sum_{j=1}^{s} n_j G_j + div\ f.$$

The prime divisors $D_1, \ldots, D_n$ and $G_1, \ldots, G_s$ do not pass through y. Therefore, $\deg_W(f) = 0$ for all prime divisors W different from Z passing through y and $\deg_Z(f) = m > 0$. Furthermore, $D_1, \ldots, D_n, G_1, \ldots, G_s$ are contained in $\bar{D}$ and thus f has support in $Z \cup \bar{D}$. □

As a direct consequence of Proposition 9.5 we obtain:

Corollary 9.6. *In the situation of Proposition 9.5 let $U \subseteq V$ be a neighborhood of x such that $D' = \bar{X} - U$ is the support of a tidy divisor. Then $(\bar{X}, D')$ has enough tame coverings.*

10 The main result

We are now in the position to construct neighborhoods on an an arithmetic surface Y/B satisfying all assumptions made in Proposition 5.4 and Proposition 6.4. Note that the assumption on the fundamental group of B is automatic in the local case.

Theorem 10.1. *Let B be the spectrum of a henselian discrete valuation ring R which is formally smooth over a discrete valuation ring with finitely generated quotient field. Let $\mathfrak{c}$ be a full class of finite groups such that the residue characteristic of R is not contained in $\mathbb{N}(\mathfrak{c})$. Assume that $\mu_\ell \subseteq R$ for all primes $\ell \in \mathbb{N}(\mathfrak{c})$ and that the absolute Galois group of the residue field of R is $\mathfrak{c}$-good. Let $\pi : Y \to B$ be an arithmetic surface and $x \in Y$ a point. Then there is an open neighborhood U of x and a compactification $U \subseteq \bar{X}$ of $U \to B$ such that the complement $\bar{D}$ of U in $\bar{X}$ is a tidy divisor with the following properties.*

(i) The horizontal part of $\bar{D}$ has nontrivial intersection with all vertical prime divisors on $\bar{X}$.

(ii) $(\bar{X}, \bar{D})$ has enough tame coverings.

As a consequence U is $K(\pi, 1)$ with respect to $\mathfrak{c}$.

Proof. Without loss of generality we may assume that x is a closed point lying over the closed point b of B. The arithmetic surface Y/B is of finite presentation. Hence, it is the base change to B of an arithmetic surface Y_0/B_0 such that B_0 is a discrete valuation ring with finitely generated quotient field and B is formally smooth over B_0. Formally smooth base change does not affect the tidiness of a divisor, nor does it disturb properties (i) and (ii). Therefore, it suffices to construct U with properties (i) and (ii) for B local with finitely generated quotient field.

Choose an open neighborhood V of x and a compactification $\bar{X}/B$ as in Proposition 9.5. Denote by D' the complement of V (with the reduced scheme structure). On every irreducible component C of $\bar{X}_b$ take a closed point $c_C \neq x$ in the smooth locus of C and not contained in any other irreducible component of $\bar{X}_b$. Using Lemma 9.4 we construct a horizontal divisor D'' passing through c_C for every vertical prime divisor C such that $\bar{D} := D' + D''$ is tidy. Then $(\bar{X}, \bar{D})$ has enough tame coverings by Corollary 9.6. Moreover, $(\bar{X}, \bar{D})$ has properties (i) and (ii).

Setting $U = \bar{X} - \bar{D}$ we conclude that U is $K(\pi, 1)$ with respect to $\mathfrak{c}$ by combining Proposition 3.2, Proposition 5.4, and Proposition 6.4. Note that the assumptions on the roots of unity and the residue field of R are part of the general setup described in Section 3. They are thus implicit in Proposition 5.4 and Proposition 6.4. $\square$

Notice that the absolute Galois group of an algebraic extension of a finite field is $\mathfrak{c}$-good for any class of finite groups $\mathfrak{c}$. Moreover, completion in formally smooth. Therefore, Theorem 10.1 implies Theorem 1.1 as stated in the introduction. The following corollaries give more explicit examples of situations where Theorem 10.1 applies. Remember that for given primes $\ell_1, \dots, \ell_n$, we denoted by $\mathfrak{c}(\ell_1, \dots, \ell_n)$ the full class of finite groups whose orders are contained in the submonoid of $\mathbb{N}$ generated by $\ell_1, \dots, \ell_n$.

Corollary 10.2. *Let Y be an arithmetic surface over the spectrum B of a discrete valuation ring and $y \in Y$ a point. Let $\mathfrak{c}$ be a full class of finite groups such that the residue characteristic of B is not contained in $\mathbb{N}(\mathfrak{c})$. Then there is a basis of Zariski neighborhoods of y which are $K(\pi, 1)$ with respect to $\mathfrak{c}$ in the following cases:*

(i) B is the spectrum of the ring of integers of a finite extension K of $\mathbb{Q}_p$ and $\mathfrak{c}$ is of the form $\mathfrak{c}(\ell_1, \dots, \ell_n)$ for primes $\ell_1, \dots, \ell_n \neq p$ such that $\mu_{\ell_i} \subseteq K$.

(ii) B is the spectrum of the ring of integers of the completion of the maximal unramified extension of a finite extension of $\mathbb{Q}_p$.

Corollary 10.3. *Let Y be an arithmetic surface over the spectrum B of a discrete valuation ring and $\bar{y} \to Y$ a geometric point. Let $\mathfrak{c}$ be a full class of finite groups such that the residue characteristic of B is not contained in $\mathbb{N}(\mathfrak{c})$. Then there is a basis of étale neighborhoods of $\bar{y}$ which are $K(\pi, 1)$ with respect to $\mathfrak{c}$ in the following cases:*

(i) B is the spectrum of the ring of integers of a finite extension of $\mathbb{Q}_p$ and $\mathfrak{c}$ is of the form $\mathfrak{c}(\ell_1, \dots, \ell_n)$ for primes $\ell_1, \dots, \ell_n \neq p$.

(ii) B is the spectrum of the ring of integers of the completion of the maximal unramified extension of a finite extension of $\mathbb{Q}_p$.

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