DISSERTATION

submitted
to the
Combined Faculty of the Natural Sciences and Mathematics
of
Heidelberg University, Germany
for the degree of
Doctor of Natural Sciences

Put forward by

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Born in: Nürnberg

Oral examination:

Aspherical neighborhoods on arithmetic surfaces

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Abstract

Aspherical neighborhoods on arithmetic surfaces

On arithmetic surfaces over local or global rings of integers this thesis examines whether a geometric point has a basis of étale neighborhoods which are $K(\pi, 1)$ with respect to a full class of finite groups $\mathfrak{c}$. These neighborhoods are also called aspherical neighborhoods. In this thesis we will consider only classes of finite groups $\mathfrak{c}$ such that the order of all groups in $\mathfrak{c}$ is prime to the residue characteristics of the arithmetic surface in question. In the local case we construct a basis of $K(\pi, 1)$-neighborhoods for any geometric point of a normal (but not necessarily regular) arithmetic surface. In the global case the existence of such bases of neighborhoods is proven under additional regularity assumptions and a condition on the l-division points of the Jacobian of the generic fibre. Moreover, we assume in the global case case that $\mathfrak{c}$ is the class of finite l-groups for a prime number l that is invertible on the arithmetic surface.

Asphärische Umgebungen auf arithmetischen Flächen

Die vorliegende Arbeit beschäftigt sich mit der Existenz asphärischer étaler Umgebungsbasen auf arithmetischen Flächen, auch $K(\pi, 1)$-Umgebungsbasen genannt. Genauer wird die $K(\pi, 1)$-Eigenschaft bezüglich einer vollen Klasse endlicher Gruppen $\mathfrak{c}$ untersucht, wobei die Ordnung aller Gruppen in $\mathfrak{c}$ teilerfremd zu den Restklassencharakteristiken der jeweiligen arithmetischen Fläche ist. Das Basisschema der hier behandelten arithmetischen Flächen ist dabei stets ein lokaler oder globaler Zahlring. Im lokalen Fall wird für alle normalen (aber nicht notwendigerweise regulären) arithmetischen Flächen eine $K(\pi, 1)$-Umgebungsbasis konstruiert. Für den globalen Fall sind zusätzliche Reguläritätsannahmen und eine Bedingung an die l-Teilungspunkte der Jacobischen der generischen Faser notwendig. Außerdem beschränkt sich die Untersuchung auf die Klasse $\mathfrak{c}(l)$ der endlichen l-Gruppen für eine auf der arithmetischen Fläche invertierbare Primzahl l.

Acknowledgements

I would like to thank my advisor Alexander Schmidt, firstly for arousing my interest in $K(\pi,1)$-schemes and secondly for many fruitful discussions about this topic. I would like to thank Kay Wingberg for making the effort to be my second advisor. Further thanks go to Armin Holschbach, Johannes Schmidt, Johannes Anschütz, and Malte Witte for answering my various questions. I want to thank David Rohr for proof-reading my thesis although he works in a completely different field. Finally, my thanks go to the members of the mathematical institute for accompanying me through my time as a doctoral student, for interesting seminars, and for many lively conversations during coffee breaks.

Contents

Chapter 1

Introduction

A classical result in topology states that every CW-complex is locally contractible. In other words, for every point x in a CW-complex X we find an open neighborhood U of x such that the inclusion $\{x\} \hookrightarrow U$ is a homotopy equivalence. In the category of CW-complexes this is equivalent to saying that the induced maps of homotopy groups with base point x are isomorphisms, i. e., that $\{x\} \hookrightarrow U$ is a weak homotopy equivalence. Since the homotopy groups of a point are trivial, this amounts to saying that every point of X has a neighborhood with trivial homotopy groups. As a consequence, we can cover X by contractible open subsets. The topological properties of X are then completely encoded in the way these open subsets are patched together to form X.

We would like to have a similar result in étale homotopy theory. Let us sketch shortly what étale homotopy theory is about. In [Fri II] it is explained how to functorially assign to a locally noetherian scheme X with geometric point $\bar{x}$ a pointed pro-CW-complex, the étale topological type. Strictly speaking the étale topological type is not a pro-CW-complex but a pro-simplicial set. After geometric realization, however, we can view it as pro-CW-complex. Its image in the pro-homotopy category of CW-complexes is denoted $X_{\text{ét}}$ and is called étale homotopy type of X. It was already constructed in [AM]. To the pro-CW-complex $X_{\text{ét}}$ we can associate its homotopy pro-groups $\pi_n(X_{\text{ét}})$. In [SGA3], Exp. X, §6 there is already defined a first homotopy pro-group for X, the "pro-groupe fondamentale enlargi" $\pi_1(X, \bar{x})$. In order for étale homotopy theory to be useful, the first homotopy group of the étale homotopy type $X_{\text{ét}}$ should be related to $\pi_1(X, \bar{x})$. And indeed, by [AM], Corollary 10.7 we have $\pi_1(X_{\text{ét}}) \cong \pi_1(X, \bar{x})$. It thus makes sense to define the étale homotopy pro-groups of X as

$$\pi_n(X, \bar{x}) := \pi_n(X_{\text{ét}}).$$

If X is connected, geometrically unibranch, and noetherian, the étale homotopy pro-groups are profinite (see [AM], Theorem 11.1) and thus can be interpreted as topological groups. In this case the first homotopy group coincides with the fundamental group defined in [SGA1]. Furthermore, étale homotopy theory is compatible with étale cohomology in the following sense: Via the isomorphism $\pi_1(X_{\text{ét}}) \cong \pi_1(X, \bar{x})$ the locally constant étale sheaves on X are in one-to-one correspondence with the local systems on $X_{\text{ét}}$. If $\mathcal{A}$ is a locally constant sheaf on X and A its corresponding local system on $X_{\text{ét}}$, we have

$$H^n(X, \mathcal{A}) \cong H^n(X_{\text{ét}}, A)$$

by [AM], Corollary 10.8.

Let us return to the problem of local contractibility. The étale homotopy type $X_{\text{ét}}$ is in general not a CW-complex but a pro-CW-complex. In the category of pro-CW-complexes there is a priori no canonical notion of homotopy equivalence. Yet, we can still speak of a weak homotopy equivalence by saying that it induces isomorphisms on homotopy pro-groups. Following [AM], we will write $\sharp$-isomorphism instead of weak homotopy equivalence. In general, we cannot expect to find étale neighborhoods of a geometric point $\bar{x}$ in X such that all homotopy pro-groups are trivial. This might not even be the case if X is a point, i. e., the spectrum of a field K. If K is not separably closed, its absolute Galois group $\mathcal{G}_K = \mathrm{Gal}(K^{sep}|K)$ is nontrivial and the fundamental group $\pi_1(Spec\ K, Spec\ K^{sep})$ coincides with $\mathcal{G}_K$. The higher homotopy pro-groups of the spectrum of a field, however, are always trivial. This is a consequence of the following criterion, which is a corollary of [AM], Theorem 4.3 (see Corollary 2.5):

Let X be a locally noetherian scheme and $\bar{x}$ a geometric point of X. Then the homotopy pro-groups $\pi_n(X, \bar{x})$ vanish for $n \geqslant 2$ if and only if

$$\varinjlim_{X' \to X} H^n(X', \mathcal{A}) = 0 \tag{1.1}$$

for all locally constant sheaves $\mathcal{A}$ and all integers $n \geqslant 1$. The limit runs over the finite étale coverings $X' \to X$.

When this criterion is satisfied, we say that X is $K(\pi, 1)$, i. e., $X_{\text{ét}}$ is a "pro-Eilenberg MacLane space". In topology these spaces are also called aspherical (which accounts for the title of this thesis). In case X is the spectrum of a field K, the criterion is satisfied as the étale cohomology of X coincides with the group cohomology of the absolute Galois group of K and for group cohomology the equality (1.1) always holds. In fact, the contrary is also true: If (1.1) is satisfied, the étale cohomology of X coincides with the group cohomology of its fundamental group. The $K(\pi, 1)$-schemes are the analogues of contractible CW-complexes in the sense that they have the homotopy type of a point in algebraic geometry, i. e., of the spectrum of a field.

We will need a slightly more specialized notion of $K(\pi, 1)$-spaces. In some situations, it is favorable not to examine all coverings of X but only the $\mathfrak{c}$-coverings for a full class of finite groups $\mathfrak{c}$. There is a notion of $\mathfrak{c}$-completion for the category of pro-CW-spaces, which is universal in the property that all homotopy pro-groups are pro-$\mathfrak{c}$-groups. Denoting by $X_{\text{ét}}(\mathfrak{c})$ the $\mathfrak{c}$-completion of the étale homotopy type $X_{\text{ét}}$, we have $\pi_1(X_{\text{ét}}(\mathfrak{c})) = \pi_1(X, \bar{x})(\mathfrak{c})$, i. e., formation of the fundamental group commutes with $\mathfrak{c}$-completion. However, this is not the case for higher homotopy pro-groups, which boils down to the fact that in general $\mathfrak{c}$-completion of groups is not an exact functor. We say that X is $K(\pi, 1)$ with respect to $\mathfrak{c}$ if only the first homotopy pro-group of $X_{\text{ét}}(\mathfrak{c})$ is nontrivial. We have a cohomological criterion analogous to (1.1), where we take the limit only over the pro-$\mathfrak{c}$-coverings of X and require $\mathcal{A}$ to be contained in $\mathfrak{c}$ and moreover to be a $\pi_1(X, \bar{x})(\mathfrak{c})$-module (instead of only a $\pi_1(X, \bar{x})$-module). This thesis is concerned with finding systems of étale neighborhoods on a scheme which are $K(\pi, 1)$ with respect to a full class of finite groups $\mathfrak{c}$.

In case X is a smooth variety over $\mathbb{C}$, neighborhoods of this kind were used by Artin in [SGA4] in order to compare the étale cohomology of X with the classical cohomology of $X(\mathbb{C})$. The $K(\pi, 1)$-neighborhoods of X are constructed by locally writing X as successive fibrations by affine curves (see [SGA4], Exp. XI, §3) and restricting these fibrations to the locus where they are particularly well behaved. These well behaved fibrations are called elementary fibrations. The fact that the resulting neighborhoods are $K(\pi, 1)$ can then be drawn from the long exact homotopy sequence associated with an elementary fibration.

The scenario where X is a smooth variety over an algebraically closed field of positive characteristic and $\mathfrak{c}$ is the class of finite p-groups with p prime to the characteristic of X was treated by Friedlander in [Fri I]. He also uses elementary fibrations and the homotopy sequence associated with these fibrations. The major problem he has to deal with is non-exactness of $\mathfrak{c}$-completions for a full class of finite groups $\mathfrak{c}$. If $\mathfrak{c}$ is the class of finite p-groups, he can prove that under certain conditions $\mathfrak{c}$-completion is indeed exact by using special features of p-groups. The difficulties with $\mathfrak{c}$-completions are the reason for the restriction to the case where $\mathfrak{c}$ is the class of finite p-groups.

In the arithmetic setting Schmidt constructed in [Sch II] systems of Zariski neighborhoods of a point x in the spectrum of the ring of integers of a number field, which are $K(\pi, 1)$ with respect to a given prime number p. Notably, p does not need to be prime to the residue characteristic of x.

The aim of this thesis is to obtain similar results for higher dimensional arithmetic schemes, i. e., for schemes which are flat and of finite type over the ring of integers of a local or global number field. One would expect that the existence of $K(\pi, 1)$ neighborhoods on an arithmetic scheme X does not depend on the smoothness of X over a base scheme, which is a relative notion. It rather seems more natural to have $K(\pi, 1)$-neighborhoods as soon as X is regular. The methods at our hands, however, are all relative. Up to now there is no general way to compute cohomology groups of higher dimensional schemes without using in some manner a fibration into curves. We explain in Chapter 3 that even if X is smooth over the base scheme, we cannot expect to work only with smooth fibrations into curves. This makes it impossible to use Friedlander's results about the homotopy fibres, where smoothness plays a prominent role in order to relate the geometric fibres with the homotopy theoretic fibre. Instead, we are forced to work directly with the cohomological criterion (1.1).

This thesis treats the case where the dimension of X is two and the orders of the groups in $\mathfrak{c}$ are prime to the residue characteristics of X. First results in this direction were already obtained by to Baben in his dissertation (see [tB]). The present thesis is partly based on his ideas.

In the context of arithmetic surfaces there is no need to construct a fibration into curves. The surface X already comes with a fibration $X \to B$, where the base scheme B is the spectrum of the ring of integers of a local or global number field. Now, the interesting case is when the geometric point $\bar{x}$ we want to find a $K(\pi, 1)$ neighborhood for maps to a singular fibre of $X \to B$. In general, for an étale neighborhood U of $\bar{x}$ we do not have a base change theorem for the structure map $U \to B$. It is thus favorable to embed U in an arithmetic surface $\bar{U}$ over B where base change holds, e. g. for $\bar{U}/B$ proper. Using resolution of singularities, which is known for two-dimensional excellent curves, one can achieve that the complement of U in $\bar{U}$ is a particularly nice Cartier divisor D.

According to the criterion (1.1) for every étale $\mathfrak{c}$-covering $U' \to U$ and every cohomology class $\phi \in H^n(U', \mathcal{A})$ there has to be found a $\mathfrak{c}$-covering $U'' \to U'$ such that ϕ maps to zero in $H^n(U'', \mathcal{A})$. Unfortunately, the normalization of $\bar{U}$ in U' is not as well behaved as $\bar{U}$ itself. But the singularities arising with such a $\mathfrak{c}$-covering are not too complicated. Chapter 4 shows that these singularities are rational, i. e., that the exceptional fibres of a resolution of the singularities are rational curves. More precisely, there is a desingularization such that the exceptional fibres are chains of $\mathbb{P}^1$'s.

In order to relate the cohomology groups of U with those of $\bar{U}$, we can use the excision sequence

$$\ldots \to H^n(\bar{U}, \mathcal{A}) \to H^n(U, \mathcal{A}) \to H^{n+1}_D(\bar{U}, \mathcal{A}) \to \ldots .$$

In Chapter 5 the cohomology groups with support in D are calculated using Gabber's absolute purity theorem (see [Fuj]). Since in general D is not regular, this has to be done in several steps. Here, among others, the rationality of the singularities arising with a $\mathfrak{c}$-covering of $\bar{U}$ comes into play in order to prove that $H^3_D(\bar{U}, \mathcal{A})$ vanishes in the limit over all $\mathfrak{c}$-coverings.

The next chapter (Chapter 6) treats the cohomology groups $H^n(\bar{U}, \mathcal{A})$. Via the Leray spectral sequence for $\pi : \bar{U} \to B$, the vanishing in the limit over all $\mathfrak{c}$-coverings of $H^n(\bar{U}, \mathcal{A})$ is put down to the vanishing of $H^p(B, R^q\pi_*\mathcal{A})$ in the limit. As mentioned before, $\bar{U}$ is chosen such that there is a base change theorem for the structure map $\pi : \bar{U} \to B$. This makes it easier to treat the cohomology groups $H^p(B, R^q\pi_*\mathcal{A})$. In the local case (i. e., where B is the spectrum of the ring of integers of a local field) the only problematical group is $H^0(B, R^2\pi_*\mathcal{A})$. It is closely related with the cokernel of the intersection matrix of the singular fibres of $\pi : \bar{U} \to B$. In the global case (i. e., where B is the spectrum of the ring of integers of a global number field) additional difficulties arise in the treatment of $H^1(B, R^1\pi_*\mathcal{A})$. Here, the Jacobian of the generic fibre of π enters and we run into the same problems with $\mathfrak{c}$-completion as Friedlander in [Fri I]. This is the reason why we assume in the global case that $\mathfrak{c}$ is the class of finite l-groups for a prime l different from the residue characteristic of $\bar{x}$. Furthermore, also in the treatment of $H^1(B, R^1\pi_*\mathcal{A})$ the cokernel of the intersection matrix plays a prominent role.

The vanishing in the limit of $H_D^{n+1}(\bar{U}, \mathcal{A})$ and $H^p(B, R^q\pi_*\mathcal{A})$ treated in Chapters 5 and 6, respectively, is subject to certain conditions on $\bar{U}$. In Chapter 7 it is explained how to construct U and $\bar{U}$ in order for these conditions to be satisfied. Moreover, it is proven that these conditions are stable under $\mathfrak{c}$-coverings of U.

Finally, the results of the preceding chapters are collected in chapter 8 in order to prove the main theorems of this thesis. They read as follows:

Theorem 1.1: *Let Y/B be an arithmetic surface of local type and $\bar{y} \to Y$ a geometric point. Let $\mathfrak{c}$ be a full class of finite groups such that the residue characteristic of B is not contained in $\mathbb{N}(\mathfrak{c})$ and for all but finitely many primes $l \in \mathbb{N}(\mathfrak{c})$ the extension $B[\mu_l] \to B$ is a $\mathfrak{c}$-extension. Then Y has a basis of étale neighborhoods at $\bar{y}$ which are $K(\pi, 1)$ with respect to $\mathfrak{c}$.*

Theorem 1.2: *Let Y/B be a regular arithmetic surface of global type and $\bar{x} \to Y$ a geometric point lying over a closed point $x \in Y$ mapping to $b \in B$. We assume that x is contained in the regular locus of $(Y_b)_{red}$. Let l be a prime number different from the residue characteristic of x. Let X_0 denote the completion of the generic fibre Y_η of $Y \to B$. Suppose that the action of the inertia group at $\bar{b}$ on the l-division points of the Jacobian of X_0 factors through an l-primary quotient. Then Y has a basis of étale neighborhoods at $\bar{x}$ which are $K(\pi, 1)$ with respect to l.*

Chapter 2

Setup and Notation

2.1 The $K(\pi, 1)$-property

Definition 2.1: *A full class of finite groups is a full subcategory $\mathfrak{c}$ of the category of finite groups satisfying*

(i) $\{1\} \in \mathfrak{c}$

(ii) Any subgroup of a $\mathfrak{c}$-group is in $\mathfrak{c}$. Moreover, if

$$0 \to A \to B \to C \to 0$$

is an exact sequence of groups, then B is a $\mathfrak{c}$-group if and only if A and C are.

In this thesis $\mathfrak{c}$ will be either the full class of finite l-groups for a prime number l or the slightly more general class $\mathfrak{c}(l_1, \ldots, l_n)$ for prime numbers $l_1, \ldots, l_n$. It is defined as the class of all finite groups G whose order is of the form

$$\#G = l_1^{r_1} \cdot \ldots \cdot l_n^{r_n}$$

for non-negative integers $r_1, \ldots, r_n$.

For a full class $\mathfrak{c}$ of finite groups we define

$$\mathbb{N}(\mathfrak{c}) := \{n \in \mathbb{N} \mid \exists G \in \mathfrak{c} \text{ with } \#G = n\}.$$

By property (ii) of the definition above $\mathbb{N}(\mathfrak{c})$ is a monoid. If $\mathfrak{c}$ is the class of finite p-groups, $\mathbb{N}(\mathfrak{c})$ consists of all powers of p and if $\mathfrak{c}$ is the class $\mathfrak{c}(l_1, \ldots, l_n)$ for prime numbers $l_1, \ldots, l_n$, $\mathbb{N}(\mathfrak{c})$ consists of all products of the form

$$\#G = l_1^{r_1} \cdot \ldots \cdot l_n^{r_n}$$

for non-negative integers $r_1, \ldots, r_n$.

In [AM] to a connected, locally noetherian scheme X with geometric point $\bar{x}$ there is associated a connected, pointed pro-CW-complex X_{et}, the étale homotopy type, such that topological coverings of X_{et} correspond to étale coverings of X. More precisely, $X_{\text{ét}}$ is a pro-object of the homotopy category of pointed, connected CW-complexes. For $X_{\text{ét}}$ we consider the homotopy pro-groups $\pi_n(X_{\text{ét}})$ and for an abelian group A with a $\pi_1(X_{\text{ét}})$-action the cohomology groups $H^n(X_{\text{ét}}, A)$. The first homotopy pro-group of $X_{\text{ét}}$, $\pi_1(X_{\text{ét}})$, coincides with the "pro-groupe fondamentale enlargi" defined in [SGA3], Exp. X, §6 (see [AM], Corollary 10.7). If X is geometrically unibranch (e. g. normal), $\pi_1(X_{\text{ét}})$ is profinite and coincides with the usual fundamental group defined in [SGA1], Exp. V. Moreover, for an abelian group A with a $\pi_1(X_{\text{ét}})$-action the cohomology groups $H^n(X_{\text{ét}}, A)$ coincide with the étale cohomology groups $H^n(X, A)$.

For a full class of finite groups $\mathfrak{c}$ a Galois $\mathfrak{c}$-covering of X is a Galois covering with Galois group in $\mathfrak{c}$ and analogously for X_{et}. A $\mathfrak{c}$-covering is a covering which is dominated by a Galois $\mathfrak{c}$-covering. The étale coverings of X constitute a Galois category by [SGA1], Exp. V, §7. By applying the following lemma to the fundamental group of X, we conclude that the same holds for the $\mathfrak{c}$-coverings of X.

Lemma 2.2: *Let $\mathcal{G}$ be a profinite group*

(i) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be normal subgroups of $\mathcal{G}$ such that for $i = 1, 2$ the quotient $\mathcal{G}/\mathcal{H}_i$ is contained in $\mathfrak{c}$. Then $\mathcal{G}/(\mathcal{H}_1 \cap \mathcal{H}_2) \in \mathfrak{c}$.

(ii) Let $\mathcal{H}_1$ be a normal subgroup of $\mathcal{G}$ such that $\mathcal{G}/\mathcal{H}_1 \in \mathfrak{c}$ and $\mathcal{H}_2$ a normal subgroup of $\mathcal{H}_1$ such that $\mathcal{H}_1/\mathcal{H}_2 \in \mathfrak{c}$. Let $\mathcal{H}$ denote the maximal subgroup of $\mathcal{H}_2$ which is normal in $\mathcal{G}$. Then $\mathcal{G}/\mathcal{H} \in \mathfrak{c}$.

Proof: The first assertion follows from the exact sequence

$$1 \to \mathcal{H}_1\mathcal{H}_2/\mathcal{H}_1 \to \mathcal{G}/(\mathcal{H}_1\mathcal{H}_2) \to \mathcal{G}/\mathcal{H}_2 \to 1.$$

For the second assertion it suffices to prove that $\mathcal{H}_1/\mathcal{H} \in \mathfrak{c}$ as $\mathcal{G}/\mathcal{H}_1 \in \mathfrak{c}$. Note that $\mathcal{H}$ is the intersection of all groups $g\mathcal{H}_2g^{-1}$, where $g \in \mathcal{G}$ runs through a system of representatives of $\mathcal{G}/\mathcal{H}_1$. Furthermore, the groups $g\mathcal{H}_2g^{-1}$ are normal in $\mathcal{H}_1$ and thus the result follows from assertion (i). □

The first statement of the lemma implies that the compositum of two $\mathfrak{c}$-coverings is again a $\mathfrak{c}$-covering and the second one that the composition of two $\mathfrak{c}$-coverings is again a $\mathfrak{c}$-covering.

Let n be a positive integer and G a pro-group, which is assumed abelian if $n > 1$. There exists a pointed, connected pro-CW-complex whose n^{th} homotopy pro-group is isomorphic to G and whose remaining homotopy pro-groups vanish. It is unique up to a $\natural$-isomorphism and called Eilenberg MacLane space of type $K(G, n)$. In case G is a group (not just a pro-group), Eilenberg MacLane spaces exist in the category of CW-complexes (see [EM]). The existence of $K(G, n)$ in the category of pro-CW-complexes for a pro-group G follows by taking inverse systems of $K(\bar{G}, n)$-spaces, where G is represented by an inverse system of groups $\bar{G}$. By abuse of notation we just write $K(G, n)$ for any such pro-space and view it as an object of the category of pointed, connected pro-CW-complexes. For any pointed, connected pro-CW-complex Z there is a canonical morphism (up to homotopy)

$$Z \to K(\pi_1(Z), 1).$$

For a pointed, connected pro-CW-complex Z we denote by $Z(\mathfrak{c})$ the pro-$\mathfrak{c}$-completion of Z (which exists by [AM] Theorem 3.4). There is a natural isomorphism

$$\pi_1(Z)(\mathfrak{c}) \xrightarrow{\sim} \pi_1(Z(\mathfrak{c}))$$

but the higher homotopy groups of $Z(\mathfrak{c})$ are not necessarily isomorphic to the $\mathfrak{c}$-completion of the respective homotopy pro-groups of Z. Following [AM], in the category of pointed, connected pro-CW-complexes a morphism $X \to Y$ is said to be a $\natural$-isomorphism if it induces isomorphisms on all homotopy pro-groups.

Definition 2.3: *Let $\mathfrak{c}$ be a full class of finite groups and X a locally noetherian scheme with geometric point $\bar{x}$. We say that X is $K(\pi, 1)$ with respect to $\mathfrak{c}$ if the canonical morphism*

$$X_{et}(\mathfrak{c}) \to K(\pi_1(X, \bar{x})(\mathfrak{c}), 1)$$

is a $\natural$-isomorphism.

The key method we will apply for examining the $K(\pi, 1)$ property is provided by Theorem 4.3 in [AM]:

Proposition 2.4: *Let $\mathfrak{c}$ be a full class of finite groups and $f : Z \to W$ a morphism of pro-CW-complexes. The following assertions are equivalent:*

(i) $f(\mathfrak{c}) : Z(\mathfrak{c}) \to W(\mathfrak{c})$ *is a $\natural$-isomorphism.*

(ii) $\pi_1(Z)(\mathfrak{c}) \xrightarrow{\sim} \pi_1(W)(\mathfrak{c})$, *and for every $\mathfrak{c}$-twisted coefficient group $M \in \mathfrak{c}$,*

$$H^i(W, M) \xrightarrow{\sim} H^i(Z, M) \qquad \forall i \geqslant 0.$$

(iii) $\pi_1(Z)(\mathfrak{c}) \xrightarrow{\sim} \pi_1(W)(\mathfrak{c})$, *and for every induced map $Z' \to W'$ of corresponding $\mathfrak{c}$-covering spaces, and every (untwisted) abelian $M \in \mathfrak{c}$,*

$$H^i(W', M) \xrightarrow{\sim} H^i(Z', M) \qquad \forall i \geqslant 0.$$

We want to apply this proposition to our situation where f is the classifying map (see [EM])

$$X_{et}(\mathfrak{c}) \to K(\pi_1(X, \bar{x})(\mathfrak{c}), 1).$$

For a full class $\mathfrak{c}$ of finite groups and a locally noetherian scheme X we denote by $\tilde{X}(\mathfrak{c})$ the universal $\mathfrak{c}$-covering of X. Then we have the following characterization of schemes of type $K(\pi, 1)$.

Corollary 2.5: *Let $\mathfrak{c}$ be a full class of finite groups and X a locally noetherian scheme. The following assertions are equivalent:*

(i) *X is $K(\pi, 1)$ with respect to $\mathfrak{c}$.*

(ii) $H^i(\tilde{X}(\mathfrak{c}), \Lambda) = 0$ *for all $i \geqslant 1$ and all $\Lambda = \mathbb{Z}/l\mathbb{Z}$ for a prime $l \in \mathbb{N}(\mathfrak{c})$.*

(iii) *Let $i \geqslant 1$ and $\Lambda = \mathbb{Z}/l\mathbb{Z}$ with $l \in \mathbb{N}(\mathfrak{c})$. Then, for every $\mathfrak{c}$-covering $X' \to X$ and every class $\phi \in H^i(X', \Lambda)$ there is a $\mathfrak{c}$-covering $X'' \to X'$ such that ϕ maps to zero under*

$$H^i(X', \Lambda) \to H^i(X'', \Lambda).$$

Proof: We first prove the equivalence of (ii) and (iii). Let $\Lambda = \mathbb{Z}/l\mathbb{Z}$ with $l \in \mathbb{N}(\mathfrak{c})$. We have

$$H^i(\tilde{X}(\mathfrak{c}), \Lambda) \cong \varinjlim_{X' \to X} H^i(X', \Lambda),$$

where the limit runs over all $\mathfrak{c}$-coverings $X' \to X$. Assertion (ii) is thus equivalent to the following statement: For any $\mathfrak{c}$-covering $X' \to X$ and every class $\phi \in H^i(X', \Lambda)$ there is a $\mathfrak{c}$-covering $X'' \to X$ which dominates $X' \to X$ such that ϕ maps to zero under

$$H^i(X', \Lambda) \to H^i(X'', \Lambda).$$

Let

$$\begin{tikzcd} X'' \arrow[rr] \arrow[dr] & & X' \arrow[dl] \\ & X & \end{tikzcd}$$

be a commutative diagram of pointed, connected locally noetherian schemes such that all morphisms are étale and $X' \to X$ is a $\mathfrak{c}$-covering. Then, as a consequence of the $\mathfrak{c}$-coverings constituting a Galois category, $X'' \to X'$ is a $\mathfrak{c}$-covering if and only if $X'' \to X$ is. This proves the equivalence of (ii) and (iii).

Let us show that (ii) implies (i). Since every $M \in \mathfrak{c}$ has a decomposition series into simple $\mathfrak{c}$-groups, i. e., groups of the form $\mathbb{Z}/l\mathbb{Z}$, we may assume that for *all* abelian $\mathfrak{c}$-groups M and all $q \geqslant 1$ we have

$$H^q(\tilde{X}(\mathfrak{c}), M) = 0.$$

Therefore, for every $\mathfrak{c}$-twisted abelian $M \in \mathfrak{c}$ the Hochschild-Serre spectral sequence

$$H^i(\pi_1(X, \bar{x})(\mathfrak{c}), H^j(\tilde{X}(\mathfrak{c}), M)) \Rightarrow H^{i+j}(X, M)$$

degenerates implying that the edge homomorphisms

$$H^i(\pi_1(X, \bar{x})(\mathfrak{c}), M) \to H^i(X, M)$$

are isomorphisms for all i. But these edge homomorphisms coincide with the homomorphisms in cohomology induced by the classifying map $X_{et}(\mathfrak{c}) \to K(\pi_1(X, \bar{x})(\mathfrak{c}), 1)$. Moreover,

$$\pi_1(K(\pi_1(X, \bar{x})(\mathfrak{c}), 1))(\mathfrak{c}) = \pi_1(X, \bar{x})(\mathfrak{c})$$

and thus, by the equivalence of (i) and (ii) in Proposition 2.4, X is $K(\pi, 1)$ with respect to $\mathfrak{c}$.

Finally, assume that X is $K(\pi, 1)$ with respect to $\mathfrak{c}$. Let Λ equal $\mathbb{Z}/l\mathbb{Z}$ for $l \in \mathbb{N}(\mathfrak{c})$ and $j \geqslant 1$. By the equivalence of (i) and (iii) in proposition 2.4 we have an isomorphism

$$\varinjlim_{X' \to X} H^j(X', \Lambda) \cong \varinjlim_{X' \to X} H^1(\pi_1(X', \bar{x})(\mathfrak{c}), \Lambda).$$

The right hand side is the limit over all open subgroups of $\pi_1(X, \bar{x})(\mathfrak{c})$ with transition maps the restrictions in group cohomology. But for any profinite group G and G-module Λ the limit over all open subgroups H,

$$\varinjlim_{H \subseteq G} H^j(H, \Lambda)$$

vanishes for all $j \geqslant 1$. We conclude that

$$H^j(\tilde{X}(\mathfrak{c}), \Lambda) = 0.$$

□

2.2 Arithmetic Surfaces

Definition 2.6: *A Dedekind scheme is an excellent, connected, normal, noetherian scheme B of dimension less or equal to 1. We say that a Dedekind scheme B is a local Dedekind scheme if it is the spectrum of a complete discrete valuation ring with finite residue field and a global Dedekind scheme if B is flat and of finite type over $\mathbb{Z}$ or $\mathbb{P}^1_{\mathbb{F}_p}$ for some prime number p.*

A local Dedekind scheme has two points: the generic point η and the special point s. Its fraction field is a local field. The fraction field of a global Dedekind scheme is a global field, i. e., either a number field or a function field.

Definition 2.7: *By an arithmetic scheme we mean an irreducible normal scheme X which is flat and of finite type over a Dedekind scheme B such that the generic fibre is nonsingular and geometrically connected. If the dimension of X is two and the dimension of B is one, we speak of an arithmetic surface. We say that X is of local type if B is a local Dedekind scheme and of global type if B is global.*

Note that if X is an arithmetic surface of local type, it need not be a local scheme.

In this thesis we want to find for each geometric point of a given arithmetic surface X/B a basis of étale neighborhoods with the $K(\pi,1)$-property. Here, an étale neighborhood of a geometric point $\bar{x} \to X$ is an étale morphism of geometrically pointed arithmetic surfaces, i. e., an arithmetic surface X', étale over X together with a geometric point $\bar{x}'$ which maps to $\bar{x}$ on X. We have the following theorem of Nagata (see [Nag] or [Lue] for a more modern exposition):

Theorem 2.8: *Let Y be a noetherian scheme and $Z \to Y$ separated and of finite type. Then there exists a compactification $\bar{Z} \to Y$ of $Z \to Y$, i. e., a proper Y-scheme such that Z admits an open Y-immersion into $\bar{Z}$ with scheme theoretically dense image.*

Therefore, possibly after shrinking B and X (in case X/B is not separated), we have a compactification $\bar{X} \to B$ of $X \to B$. By [Lic], Theorem 2.8, if B is affine and X is regular, $\bar{X}$ is automatically projective over B. By blowing up in closed points we can modify the compactification in order to obtain a particularly well behaved one. More precisely:

Definition 2.9: *Let Y be a normal, noetherian scheme and D an effective Cartier divisor on Y. We say that D has strictly normal crossings at a point $y \in Y$ if Y is regular at y and there is a system of parameters $f_1, \ldots, f_n$ at y such that Zariski locally at y, D is given by div$(f_1^{d_1} \cdots f_n^{d_n})$ for some non-negative integers $d_1, \ldots, d_n$. We say that D has strictly normal crossings if it has strictly normal crossings at every point $y \in Y$. In this case we say that D is an snc-divisor (for strictly normal crossings).*

The divisor D has normal crossings if étale locally it has strictly normal crossings.

Two effective divisors are said to intersect transversally if they do not have a common irreducible component and their sum has normal crossings at every point of the intersection.

Definition 2.10: *Let X/B be an arithmetic surface. A tidy divisor is an snc-divisor whose horizontal part meets each vertical divisor of X transversally.*

In particular, the horizontal irreducible components of a tidy divisor do not intersect. Tidy divisors have the following local structure:

Lemma 2.11: *Let X/B be an arithmetic surface, $D \subset X$ a tidy divisor and $x \in D$ a closed point with image $b \in B$. Let us fix a uniformizer π at b. Then there is a smooth scheme of finite type Z over B of relative dimension 2, a closed point $z \in Z_b$, and a surjective homomorphism $\mathcal{O}_{Z,z} \to \mathcal{O}_{X,x}$ whose kernel is generated by an element $F \in \mathcal{O}_{Z,z}$ which takes the following form: There is a system of parameters (π, f, g) of $\mathcal{O}_{Z,z}$, a unit $\alpha \in \mathcal{O}_{Z,z}^{\times}$, and non-negative integers j, k with $j + k > 0$ such that*

$$F = f^j g^k - \alpha\pi.$$

Furthermore, if we denote by $\bar{f}$ and $\bar{g}$ the image of f and g in Spec $\mathcal{O}_{X,x}$, we have either

$$(D_{\mathcal{O}_{X,x}})_{red} = div\ \bar{f} + div\ \bar{g} \quad or \quad (D_{\mathcal{O}_{X,x}})_{red} = div\ \bar{f}.$$

Proof: Since D is tidy, X_b has strictly normal crossings at x. Hence, by [Liu], Proposition 2.34 and Remark 2.34 there is a smooth scheme of finite type Z over B of relative dimension 2, a closed point $z \in Z_b$, and a surjective homomorphism $\mathcal{O}_{Z,z} \to \mathcal{O}_{X,x}$ whose kernel is generated by an element $F \in \mathcal{O}_{Z,z}$ which takes the following form: There is a system of parameters (π, f, g) of $\mathcal{O}_{Z,z}$, a unit $\alpha \in \mathcal{O}_{Z,z}^{\times}$, and non-negative integers j, k with $j + k > 0$ such that

$$F = f^j g^k - \alpha\pi.$$

If both j and k are positive, X_b has two components at x and the support of $D_{\mathcal{O}_{X,x}}$ is contained in $(Spec\ \mathcal{O}_{X,x})_b$ as otherwise the horizontal part of D could not intersect X_b transversally at x. If, say, $k = 0$, X_b has one component at x and there are the following three possibilities: If D has two components at x, one of these corresponds to the vertical component $div\ \bar{f}$ and the other one is horizontal. We can choose the system of parameters (π, f, g) such that it is given by $div\ \bar{g}$. If D has a single vertical component at x, it is given by $div\ \bar{f}$. If D has a single horizontal component at x, we can choose the system of parameters (π, f, g) such that it is given by $div\ \bar{g}$. In any case we have either

$$(D_{\mathcal{O}_{X,x}})_{red} = div\ \bar{f} + div\ \bar{g} \quad \text{or} \quad (D_{\mathcal{O}_{X,x}})_{red} = div\ \bar{f} \quad \text{or} \quad (D_{\mathcal{O}_{X,x}})_{red} = div\ \bar{g}$$

and the Lemma follows (possibly interchanging f and g). □

Definition 2.12: *Let X/B be an arithmetic surface and $Z \subseteq X$ a proper closed subscheme. We say that a closed point $z \in Z$ is a special point of Z if either Z is not a tidy divisor at z or Z is tidy at z and Z is singular at z.*

With the same notation as in Lemma 2.11 the special points of D are precisely the points where $(D_{\mathcal{O}_{X,x}})_{red} = div\ \bar{f} + div\ \bar{g}$, i. e., where two irreducible components of D intersect. If D is not tidy but only snc, the special points are the singular points and the points where D intersects a vertical divisor non-transversally.

If Y is a scheme and $Z \subseteq Y$ a closed subscheme, we say (Y, Z) is a pair. By a morphism of pairs $(Y', Z') \to (Y, Z)$ we mean a cartesian diagram

$$\begin{tikzcd} Z' \arrow[r, hook] \arrow[d] & Y' \arrow[d] \\ Z \arrow[r, hook] & Y. \end{tikzcd}$$

Definition 2.13: *Let Y be a normal scheme and $Z \subseteq Y$ a closed subscheme such that $Y - Z$ is dense in Y. By a desingularization in the strong sense $(Y', Z') \to (Y, Z)$ of (Y, Z) we mean a birational morphism $Y' \to Y$ which is an isomorphism over the complement of Z such that Y' is regular at every point of the preimage Z' of Z.*

We say that $(Y_{min}, Z_{min}) \to (Y, Z)$ is the minimal desingularization of (Y, Z), if any other desingularization factors through $(Y_{min}, Z_{min}) \to (Y, Z)$.

If Y is an excellent two-dimensional scheme, e. g. an arithmetic surface, desingularizations in the strong sense exist (see [Lip]). Furthermore, by [Liu], Chapter 9, Proposition 3.32 there exists a unique minimal desingularization of (Y, Z). Desingularizations exist also with the additional requirement that Z' be an snc-divisor (see [CJS]). Moreover, we can require that it be obtained from the minimal desingularization $(Y_{min}, Z_{min}) \to (Y, Z)$ of (Y, Z) by successively blowing up in singular points of Z_{min}. In this thesis we need an even more restrictive type of desingularization:

Definition 2.14: *Let X/B be an arithmetic surface and $Z \subseteq X$ a proper closed subscheme. A tidy desingularization $(X', Z') \to (X, Z)$ of (X, Z) is a birational morphism $X' \to X$ such that Z' is a tidy divisor of X' and $(X', Z') \to (X, Z)$ factors as*

$$(X', Z') = (X_0, Z_0) \to (X_1, Z_1) \to \ldots \to (X_n, Z_n) \to (X, Z),$$

where $(X_n, Z_n) \to (X, Z)$ is the minimal desingularization of (X, Z) and for $i = 1, \ldots, n$ the morphisms $(X_{i-1}, Z_{i-1}) \to (X_i, Z_i)$ are blowups of X_i in special points of Z_i.

It is important that the blowups in the definition of a tidy desingularization are only allowed to take place in special points of Z and not in any closed points of Z. The existence of tidy desingularizations follows from the existence of desingularizations $(X', Z') \to (X, Z)$ such that Z' is an snc-divisor:

Proposition 2.15: *Let X/B be an arithmetic surface and $Z \subset X$ a proper closed subscheme. Then there exists a tidy desingularization of (X, Z).*

Proof: Note that since X is normal and irreducible, the complement of Z is automatically dense, and thus the notion of a desingularization of (X, Z) is defined. Without loss of generality we may assume that X is regular away from Z as X is singular in at most a finite set of closed points, which we can remove from X if they do not lie on Z. By [CJS], Theorems 0.1 and 0.2 there is a desingularization $(X', Z') \to (X, Z)$ which is an isomorphism over the complement of Z such that Z' is an snc-divisor. Moreover, we can assume that $(X', Z') \to (X, Z)$ is obtained from the minimal desingularization by successively blowing up in singular, hence special, points. Let D' be the union of Z' with the finitely many vertical prime divisors containing the points where Z' intersects a vertical divisor non-transversally. After removing from X' all points of D' which are not contained in Z' and where D' is singular, we may assume that the special points of D' are contained in Z'. By construction they coincide with the singular points of D'. Blowing up in singular points of D' we achieve that D' is an snc-divisor. This is equivalent to saying that Z' is tidy. □

Let X/B be an arithmetic surface with compactification $\bar{X} \to B$ and $\bar{x} \to X$ a geometric point of X lying over a closed point x. Let Z be the union of $\bar{X} - X$ with the singular locus of $X - x$ (equipped with the reduced subscheme structure). By the above we can find a tidy desingularization $(\bar{X}', Z') \to (\bar{X}, Z)$. By construction $\bar{X}' \to \bar{X}$ is an isomorphism when restricted to a (Zariski) neighborhood of x. Hence, given an arithmetic surface X/B and a closed point $x \in X$, we can find an open neighborhood $U \subseteq X$ of x and a compactification $\bar{U} \to B$ of U such that $\bar{U} - U$ is a tidy divisor and $\bar{U} - x$ is regular.

2.3 Tame coverings of arithmetic surfaces

Definition 2.16: *Let X be a normal, irreducible scheme and $D \subseteq X$ an snc-divisor. Set $U = X - D$. An étale covering $U' \to U$ is tame, if for any irreducible component C of D the associated extension of function fields $K'|K$ is tamely ramified at the discrete valuation corresponding to C. A tame covering of (X, D) is a morphism of pairs $(X', D') \to (X, D)$ such that $X' \to X$ is finite and $X' - D' \to U$ is a tame étale covering.*

Remark 2.17: *The above definition of tame covering coincides with the one given in [Sch I], Definition 1.4, as the proof of Proposition 1.14, loc. cit. shows.*

For the rest of this section we fix an arithmetic surface X/B, a tidy divisor $D \subseteq X$ and a geometric point $\bar{x}$ lying over a closed point x of $U = X - D$. Moreover, we fix a full class of finite groups $\mathfrak{c}$ such that all integers in $\mathbb{N}(\mathfrak{c})$ are invertible on X. We will mainly be interested in full classes of finite groups of the form $\mathfrak{c}(l_1, \ldots, l_n)$ for prime numbers $l_1, \ldots, l_n$ which are invertible on X.

Definition 2.18: *A $\mathfrak{c}$-covering of (X, D) is a tame covering $(X_1, D_1) \to (X, D)$ such that $X_1 - D_1 \to X - D$ is a finite étale $\mathfrak{c}$-covering.*

If $(X_1, D_1) \to (X, D)$ is a tame covering, in general D_1 is not a tidy divisor of X_1. In fact, X_1 might not even be regular at the points in D_1. But using proposition 2.15 we find a tidy desingularization $(X', D') \to (X_1, D_1)$.

Definition 2.19: *A desingularized $\mathfrak{c}$-covering $(X', D') \to (X, D)$ is a cartesian diagram*

$$\begin{tikzcd} D' \arrow[r] \arrow[d, hook] & D_1 \arrow[r] \arrow[d, hook] & D \arrow[d, hook] \\ X' \arrow[r] & X_1 \arrow[r] & X, \end{tikzcd}$$

such that $(X_1, D_1) \to (X, D)$ is a $\mathfrak{c}$-covering and $(X', D') \to (X_1, D_1)$ is a tidy desingularization.

If $\mathfrak{c}$ is not specified, we tacitly assume it to be the class of all finite groups with order prime to the residue characteristics of X and speak of desingularized tame coverings instead of desingularized $\mathfrak{c}$-coverings . Furthermore, we call the strict transform of D_1 in X' the generalized strict transform of D and we call the exceptional divisor of $X' \to X_1$ the generalized exceptional divisor of $(X', D') \to (X, D)$.

Remark 2.20: *The factorization $(X', D') \to (X_1, D_1) \to (X, D)$ in the above definition is uniquely determined by the morphism $X' \to X$. Namely, $X_1 \to X$ is the normalization of X in X' and the morphism $X' \to X_1$ comes from the universal property of the normalization. Therefore, we can exclude (X_1, D_1) from the notation and just write $(X', D') \to (X, D)$.*

Lemma 2.21: *The following assertions hold:*

(i) If $(X', D') \to (X, D)$ and $(X'', D'') \to (X', D')$ are both desingularized $\mathfrak{c}$-coverings, the composite $(X'', D'') \to (X, D)$ is again a desingularized $\mathfrak{c}$-covering.

(ii) If $(X', D') \to (X, D)$ and $(X'', D'') \to (X, D)$ are desingularized $\mathfrak{c}$-coverings, there is a commutative diagram of desingularized $\mathfrak{c}$-coverings

$$\begin{tikzcd} & (X', D') \arrow[dr] & \\ (X''', D''') \arrow[ur] \arrow[dr] & & (X, D). \\ & (X'', D'') \arrow[ur] & \end{tikzcd}$$

(iii) Let X'/B' be another arithmetic surface with tidy divisor $D' \subseteq X'$. Let $\bar{x} \to X' - D'$ be a geometric point. There is at most one desingularized tame covering $(X', D') \to (X, D)$ such that $\bar{x} \to X' - D' \to X - D$ coincides with the fixed geometric point $\bar{x} \to X - D$.

In order to show this lemma, we need to know more about the local structure of desingularized tame coverings. Therefore, we postpone its proof to the end of section 4.3 in the next chapter.

We denote by $\mathfrak{I}_{X,D,\bar{x}}$ the category of all desingularized $\mathfrak{c}$-coverings $(X', D') \to (X, D)$ together with a geometric point $\bar{x} \to X'$ such that $\bar{x} \to X' \to X$ coincides with the fixed geometric point $\bar{x} \to X$. By the above lemma $\mathfrak{I}_{X,D,\bar{x}}$ is a cofiltered category.

Lemma 2.22: *Let A be a $\mathfrak{c}$-twisted $\pi_1(U, \bar{x})$-module and i a non-negative integer. Then*

$$\varinjlim_{(X',D') \in \mathfrak{I}_{X,D,\bar{x}}} H^i(X' - D', A) \cong \varinjlim_{U' \to U} H^i(U', A), \tag{2.1}$$

where the limit on the right is over the étale $\mathfrak{c}$-coverings of U.

Proof: Any étale $\mathfrak{c}$-covering $U' \to U$ can be lifted to a desingularized $\mathfrak{c}$-covering $(X', D') \to (X, D)$. □

Let $\mathcal{C}$ be a category in which direct limits exist. Let $F : \mathfrak{I}_{X,D,\bar{x}} \to \mathcal{C}$ be a contravariant functor and (X', D') an object of $\mathfrak{I}_{X,D,\bar{x}}$. We say that $F(X', D')$ vanishes in the limit if the natural map

$$F(X', D') \to \varinjlim_{\mathfrak{I}_{X,D,\bar{x}}} F(X'', D'')$$

is the zero map. If $\mathcal{C}$ is the category of abelian groups (or a subcategory thereof) and ϕ is an element of $F(X', D')$, we say that ϕ vanishes in the limit if the image of ϕ in $\varinjlim_{\mathfrak{I}_{X,D,\bar{x}}} F(X'', D'')$ is zero.

For instance, we will use this terminology for the functors $\mathfrak{I}_{X,D,\bar{x}} \to \mathcal{Ab}$ defined by

$$(X', D') \mapsto H^i(X', A),$$
$$(X', D') \mapsto H^i_{D'}(X', A).$$

for a locally constant sheaf A on X_{et}. Note that if $F(X, D)$ vanishes in the limit, this does not yet imply that

$$\varinjlim_{\mathfrak{I}_{X,D,\bar{x}}} F(X', D') = 0.$$

The vanishing of the limit is equivalent (by definition) to saying that for every object (X', D') of $\mathfrak{I}_{X,D,\bar{x}}$ we have that $F(X', D')$ vanishes in the limit.

Chapter 3

Non-existence of good Artin neighborhoods

Before we start the construction of $K(\pi,1)$-neighborhoods on arithmetic surfaces we explain why Artin's and Friedlander's method using elementary fibrations is not promising in the arithmetic setting. Let us review their line of reasoning.

Definition 3.1: *An elementary fibration is a morphism of schemes $f : X \to B$ that can be embedded in a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & \bar{X} \arrow[d, "\bar{f}"] & Y \arrow[l, hook', "i"'] \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

satisfying the following conditions:

(i) j is an open immersion which is dense in every fibre and $X = \bar{X} - Y$.

(ii) $\bar{f}$ is smooth and projective with geometrically irreducible fibres of dimension 1.

(iii) g is étale with nonempty fibres.

Remark 3.2: *Note that if $X \to B$ is an elementary fibration embedded in a diagram as in the above definition and if $\bar{X}' \to \bar{X}$ is a finite étale morphism, then the pullback*

$$f_{\bar{X}'} : X \times_{\bar{X}} \bar{X}' \to B$$

is again an elementary fibration. Moreover, the pullback of $X \to B$ via any morphism $B' \to B$ is again an elementary fibration as smoothness is stable under base change.

In [SGA4] Artin defines good neighborhoods (now called good Artin neighborhoods) over an algebraically closed field as successive elementary fibrations. More precisely, the definition is as follows:

Definition 3.3: *Let k be an algebraically closed field. A good Artin neighborhood over k is a scheme X over k such that there exist k-schemes*

$$X = X_n, \dots, X_0 = Spec\ k$$

and elementary fibrations $f_i : X_i \to X_{i-1}, i = 1, \dots, n$.

In case k is the field $\mathbb{C}$ of complex numbers, denote by $(X_i)_{\mathrm{cl}}$ the analytification of X_i endowed with the classical topology. As explained in [SGA4], Exp. XI, Variante 4.6 the elementary fibrations $f_i : X_i \to X_{i-1}, i = 1, \dots, n$ induce locally trivial fibrations $(f_i)_{\mathrm{cl}} : (X_i)_{\mathrm{cl}} \to (X_{i-1})_{\mathrm{cl}}$ whose fibres are non-complete curves C_i. If x is a closed point of X, it determines points x_i of $(X_i)_{\mathrm{cl}}$ and c_i of C_i. Since C_i is non-complete, we have

$$\pi_n(C_i, c_i) = 0$$

for $i > 1$ and $\pi_1(C_i, c_i)$ is a finitely generated free group. By induction on i and using the long exact sequence of homotopy groups associated to a fibration, we conclude that

$$\pi_n(X_{\mathrm{cl}}, x) = 0$$

for $i > 1$ and that $\pi_1(X_{\mathrm{cl}}, x)$ is a successive extension of free groups. In particular, X_{cl} has the $K(\pi, 1)$-property. Being a successive extension of free groups, $\pi_1(X_{\mathrm{cl}}, x)$ is a good group, i. e.,

$$K(\pi_1(X_{\mathrm{cl}}, x), 1)^\wedge \xrightarrow{\sim} K(\pi_1(X_{\mathrm{cl}}, x)^\wedge, 1),$$

where "( )^" denotes profinite completion. Hence, $X_{\mathrm{cl}}^\wedge$ is a $K(\pi, 1)$ space, as well. By the generalized Riemann existence theorem (see [AM], Theorem 12.9 and Corollary 12.10) the same is true for $X_{\text{ét}}$. Actually, the argument is the other way round: In the proof of the comparison theorem of étale and classical cohomology ([SGA4], Exp. XI, Théorème 4.4), Artin implicitly shows that if X/k is a good Artin neighborhood, X is $K(\pi, 1)$ with respect to the class of finite groups. Artin's comparison theorem is then used in the proof of the generalized Riemann existence theorem.

In [Fri I] Friedlander extends this result to smooth schemes over algebraically closed fields of positive characteristic p. In this setting, he examines the $K(\pi, 1)$-property with respect to a prime number $l \neq p$. Let us explain his key theorem. For an elementary fibration $f : X \to S$ denote by $\mathfrak{f}(f_{\text{ét}}(l))$ the homotopy theoretic fibre of the associated morphism $f_{\text{ét}}(l) : X_{\text{ét}}(l) \to S_{\text{ét}}(l)$ of l-completed étale homotopy types. Using the long exact homotopy sequence associated to the fibre triple $\mathfrak{f}(f_{\text{ét}}(l)) \to X_{\text{ét}}(l) \to S_{\text{ét}}(l)$, Friedlander shows (see [Fri I], Theorem 9):

Theorem 3.4: *Let $f : X \to S$ be an elementary fibration of connected, normal, noetherian schemes, pointed by a geometric point $\bar{x}$. Let l be a prime not occurring as a residue characteristic of S. If $R^1 f_*(\mathbb{Z}/l\mathbb{Z})$ is a $\pi_1(S, \bar{x})(l)$-module and if $\pi_2(S_{\acute{e}t}(l), \bar{x}) = 0$, then the natural maps $(X_{\bar{x}})_{\acute{e}t} \to \mathfrak{f}(f_{\acute{e}t})$ and $\mathfrak{f}(f_{\acute{e}t}) \to \mathfrak{f}(f_{\acute{e}t}(l))$ induce a $\natural$-isomorphism*

$$(X_{\bar{x}})_{\acute{e}t}(l) \to \mathfrak{f}(f_{\acute{e}t}(l)).$$

Consequently, $\pi_1(\mathfrak{f}(f_{\acute{e}t}(l)))$ is free, pro-l and

$$\mathfrak{f}(f_{\acute{e}t}(l)) \to K(\pi_1(\mathfrak{f}(f_{\acute{e}t}(l))), 1)$$

is a $\natural$-isomorphism.

In particular, under the assumptions of the above theorem, X is $K(\pi, 1)$ with respect to l. Friedlander concludes the existence of $K(\pi, 1)$-neighborhoods in the following manner: By [SGA4], Exp. XI, Proposition 3.3 any closed point $\bar{x}$ of a smooth scheme over an algebraically closed field k has a basis of étale neighborhoods which are good Artin neighborhoods. Fix one such good Artin neighborhood and write

$$X = X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \dots \xrightarrow{f_1} X_0 = Spec\ k$$

with elementary fibrations $f_i : X_i \to X_{i-1}$. The property that $R^1(f_i)_*(\mathbb{Z}/l\mathbb{Z})$ is a $\pi_1(X_{i-1}, \bar{x})(l)$-module is étale local, so we can assume that it holds for all i. The condition that $\pi_2((X_{i-1})_{\text{ét}}(l), \bar{x})$ is trivial holds for $i = 1$. If it holds for i, Friedlander's theorem asserts that X_i is $K(\pi, 1)$ with respect to l and in particular $\pi_2((X_i)_{\text{ét}}(l), \bar{x}) = 0$. Therefore, by induction on i, X is $K(\pi, 1)$ with respect to l.

Let us now consider the arithmetic situation. The natural analogue of a good Artin neighborhood is the following.

Definition 3.5: *An arithmetic good Artin neighborhood is an arithmetic scheme X/B such that there exist B-schemes*

$$X = X_n, \dots, X_0 = B$$

and elementary fibrations $f_i : X_i \to X_{i-1}, i = 1, \dots, n$.

Note that the base scheme B is part of the datum of a good Artin neighborhood. If X/B is an arithmetic scheme with geometric point $\bar{x}$, an étale neighborhood of $\bar{x}$ is a commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

such that X'/B' is an aritmetic scheme and $X' \to X$ is étale. We have to allow for a change of the base scheme because we require the generic fibre of an arithmetic scheme to be geometrically connected. Furthermore, if we hope to find étale neighborhoods that are arithmetic good Artin neighborhoods, we have to admit that B can be replaced by an open subscheme in order to get rid of unwanted singular fibres.

Fix a smooth arithmetic scheme X/B and a geometric point $\bar{x}$ of X inducing a geometric point $\bar{b}$ of B. Pick a prime l different from the residue characteristic of $\bar{x}$. Suppose that B has a basis of étale neighborhoods that are $K(\pi, 1)$ with respect to l. This is the case if B is a local or global Dedekind scheme (see [Sch II] for the global case, the local case is easy). If $\bar{x}$ has a basis of étale neighborhoods that are arithmetic good Artin neighborhoods, the same reasoning as above shows that $\bar{x}$ has a basis of $K(\pi, 1)$-neighborhoods with respect to l. Unfortunately, bases of étale neighborhoods that are arithmetic good Artin neighborhoods do not exist for any arithmetic scheme. In the remaining part of this chapter we explain why they do not exist. We first treat the case of relative dimension one.

Lemma 3.6: *Let B be a strictly henselian Dedekind scheme and X/B a proper, smooth arithmetic surface with geometric point $\bar{x}$ over a closed point x of X. There is an étale neighborhood $X'/B \to X/B$ of $\bar{x}$ such that any étale neighborhood $X''/B'' \to X'/B'$ of $\bar{x}$ is* not *an arithmetic good Artin neighborhood.*

Proof: Let $Y \to X$ be the blow-up in a closed point $x_0 \neq x$ of X. The special fibre of Y/B has two irreducible components C_1 and C_2. For $i = 1, 2$ choose three closed points c_i^1, c_i^2, c_i^3 in C_i different from x and different from the intersection point of C_1 with C_2. For each (i, j) choose a horizontal divisor D_i^j intersecting C_i transversally at c_i^j. Let *Spec* $A \subseteq Y$ be an affine open subscheme containing x, c_i^j, D_i^j. This is possible as Y/B is projective by [Lic], Theorem 2.8. Denote by $\mathfrak{p}_x, \mathfrak{p}_{c_i^j}, \mathfrak{p}_{D_i^j}$ the prime ideals corresponding to x, c_i^j, D_i^j, respectively. Using prime evasion (see Lemma 7.6 for a version that applies in our situation) we find $f \in A$ such that

$$\begin{aligned} &f \in \mathfrak{p}_{D_i^j} \text{ for } i = 1, 2 \text{ and } j = 1, 2, 3, \\ &f \notin \mathfrak{p}_{c_i^j}^2 \text{ for } i = 1, 2 \text{ and } j = 1, 2, 3, \\ &f \notin \mathfrak{p}_x. \end{aligned}$$

In other words we have

$$div\ f = \sum_{i,j} D_i^j + D,$$

where the support of D contains neither x nor c_i^j. Choose an integer $m > 1$ prime to the residue characteristic of B. Let Y' be the normalization of Y in the function field extension

$$K(Y)[\sqrt[m]{f}]|K(Y).$$

and $Y' \to B' \to B$ the Stein factorization of $Y' \to B$. The morphism $Y' \to Y$ is purely ramified in the divisors D_i^j. Therefore, the special fibre of Y' has two irreducible components C_1' and C_2' dominating C_1 and C_2, respectively. Moreover, by the Hurwitz-formula C_1' and C_2', as well as the generic fibre of Y' have genus at least one. The morphism $Y' \to X$ is étale in a neighborhood of x. We can thus find an open subscheme $X' \subseteq Y'$ such that X'/B' is an étale neighborhood of $\bar{x}$.

We claim that any étale neighborhood $X''/B'' \to X'/B'$ of $\bar{x}$ is *not* an arithmetic good Artin neighborhood. It suffices to show that X'' does not posess a smooth compactification over B''. Denote by Y'' the normalization of Y' in X''. Using resolution of singularities we find a regular compactification $\bar{X}''$ of X''. It can be obtained from Y'' by blowing up in the singular locus. Since Y'' is normal, the singular locus is zero-dimensional. By the Hurwitz formula the strict transforms of the irreducible components of the special fibre of $\bar{X}''$ are again curves of genus at least one. By [Liu], Theorem 3.21 there exists a unique minimal model $\bar{X}''_{\min}$ of the generic fibre of Y''. We have a birational morphism

$$\bar{X}'' \to \bar{X}''_{\min}$$

which is a successive blow-up in regular points. The exceptional fibres are thus rational. We conclude that curves of genus at least one cannot be contracted. Hence, the special fibre of $\bar{X}''_{\min} \to B''$ has at least two components. In particular, $\bar{X}''_{\min}$ is not smooth over B''. Therefore, there does not exist a smooth compactification of X''. □

Corollary 3.7: *Let X/B be an arithmetic surface and $\bar{x}$ a geometric point above a closed point of X. There is no basis of étale neighborhoods of $\bar{x}$ that are arithmetic good Artin neighborhoods.*

Proof: Assume there is a basis of étale neighborhoods of $\bar{x}$ that are arithmetic good Artin neighborhoods. Its base change to the strict henselization of B at $\bar{x}$ is again a basis of arithmetic good Artin neighborhoods. Replacing X/B by an étale neighborhood we may assume that X/B is itself a good Artin neighborhood. The same holds for its base change to the strict henselization of B at $\bar{x}$. By Lemma 3.6 this is not possible. □

Proposition 3.8: *Let X/B be an arithmetic scheme and $\bar{x}$ a geometric point above a closed point of X. There is no basis of étale neighborhoods of $\bar{x}$ that are arithmetic good Artin neighborhoods.*

Proof: Suppose the contrary. In particular, there is an étale neighborhood X'/B' of $\bar{x}$ which admits an elementary fibration $X' \to S'$ over B' such that S' is regular. Denote by s the image of $\bar{x}$ in S'. Choose a curve C' passing through s' which is regular at s' and flat over B'. The strict henselization of C' at $\bar{x}$ is a Dedekind scheme B'' and the base change X'' of X' to B'' is an arithmetic surface which is an elementary fibration over B''. By assumption X has a basis of étale neighborhoods of $\bar{x}$ that are arithmetic good Artin neighborhoods. Its pullback to B'' is a basis of étale neighborhoods of $\bar{x}$ in X'' that are arithmetic good Artin neighborhoods. But by Corollary 3.7 such a basis of neighborhoods does not exist, a contradiction. □

The proof of Proposition 3.8 shows that the existence of arithmetic good Artin neighborhoods fails already at the first step. It fails in such a way that we cannot even expect to find a basis of étale neighborhoods with smooth compactifications over some regular scheme of one dimension less. Even if the initial arithmetic scheme X/B is smooth and has a smooth compactification, we cannot hope to work with smooth fibrations. This makes it hard to determine the homotopy theoretic fibre in order to use the long exact homotopy sequence. Instead, we will pursue a more explicit approach using the cohomological criterion for the $K(\pi, 1)$-property provided by Corollary 2.5.

Chapter 4

Exceptional fibres

In this chapter we begin to work on the problem of constructing $K(\pi,1)$-neighborhoods on arithmetic surfaces. For an arithmetic surface X/B and a tidy divisor $D \subseteq X$ we want to investigate whether $U = X - D$ is $K(\pi,1)$ with respect to a given class of finite groups $\mathfrak{c}$. By the cohomologial criterion for the $K(\pi,1)$-property (see Corollary 2.5) we have to show that for a geometric point $\bar{x}$ of U

$$\varinjlim_{U' \to U} H^i(U, \Lambda) = 0$$

for $i \geqslant 1$ and $\Lambda = \mathbb{Z}/m\mathbb{Z}$ with $m \in \mathbb{N}(\mathfrak{c})$. The limit runs over all pointed finite étale $\mathfrak{c}$-coverings $U' \to U$. By Lemma 2.22 this amount to showing that

$$\varinjlim_{(X',D') \in \mathfrak{I}_{X,D,\bar{x}}} H^i(X' - D', \Lambda) = 0.$$

Remember that the category $\mathfrak{I}_{X,D,\bar{x}}$ is the category of all desingularized $\mathfrak{c}$-coverings of (X, D) together with a lift of the geometric point $\bar{x}$. The reason why we have to desingularize the $\mathfrak{c}$-coverings of (X, D) is the following. Let $(X_1, D_1) \to (X, D)$ be a $\mathfrak{c}$-covering. Since $X_1 \to X$ is étale over U, the open subscheme $U_1 = X_1 - D_1$ of X_1 has essentially the same regularity properties as U. However, the covering $X_1 \to X$ can ramify outside of U. Therefore, the divisor D_1 might not be tidy and in particular, X_1 might not be regular at the points of D_1. With other words, new singularities arise when we replace (X, D) by (X_1, D_1). But in order to calculate the cohomology groups of U_1, which is the task of Chapter 5 and Chapter 6, it would be very helpful if the complement of U_1 were a tidy divisor.

At this point the reader might ask why we have to embed U in a bigger scheme at all. The reason is that, in general, we do not have a base change theorem for the morphism $U \to B$ but we would like to compute the cohomology of U via the Leray spectral sequence associated with $U \to B$. We circumvent this problem by first lifting comology classes on U to cohomology classes on a bigger scheme X where base change holds. We could take for X a compactification of U but it turns out to be more favorable to remove from the compactification of U a regular horizontal divisor in order to obtain X. This makes $X \to B$ a non-smooth analogue of an elementary fibration with the advantage that the general fibres are *affine* curves.

Let us return to the subject of this chapter. We consider a desingularized $\mathfrak{c}$-covering $(X', D') \to (X_1, D_1) \to (X, D)$. In this thesis we only examine the case where all elements of $\mathbb{N}(\mathfrak{c})$ are invertible on X. Hence, the $\mathfrak{c}$-covering $(X_1, D_1) \to (X, D)$ is tame. We want to understand the singularities of X_1 at the points in D_1. This amounts to understanding the exceptional fibres of $X' \to X_1$. It turns out that only rational singularities can occur meaning that the exceptional fibres of $X' \to X$ are rational curves.

4.1 The local structure of tame coverings

The local structure of a tame covering is described in [SGA1], Exp. XIII, 5.3.0 by the generalized Abhyankar lemma:

Theorem 4.1: *Let Y be a strictly henselian, regular, local scheme and $D = \sum_{i=1}^{r} div\ f_i$ a normal crossing divisor on X. Then every connected tame covering of (Y, Z) is a quotient of a tame covering of the form*

$$Y_1 = X[T_1, \ldots, T_r]/(T_1^{n_1} - f_1, \ldots, T_r^{n_r} - f_r),$$

where n_i are positive integers prime to the residue characteristic of Y.

In the two-dimensional situation we are interested in, there are three possibilities: Either D is zero meaning that every tame covering of (X, D) is étale, i. e., trivial, or $D = div\ a$ is a prime divisor, or $D = div\ a + div\ b$ has two irreducible components. In case D is a prime divisor the situation is quite simple as in this case D is regular and we have:

Lemma 4.2: *Let Y be a regular, noetherian scheme and $D \subset Y$ a divisor whose underlying reduced scheme is regular. Let $(Y_1, D_1) \to (Y, D)$ be a tame covering of (Y, D). Then Y_1 as well as the underlying reduced subscheme of D_1 in Y_1 are regular. If moreover (Y, D) is smooth over some regular, noetherian scheme B, so is $(Y_1, (D_1)_{red})$*

Proof: The problem is étale local so we may assume $Y = Spec\ R$ with R a strictly henselian local ring. In this setting the divisor D_{red} is either empty or given by a regular element $u \in R$. In the former case Y_1 is étale over Y and thus regular. In the latter case, by the generalized Abhyankar lemma Y_1 is a disjoint union of schemes of the form $Spec\ R[\sqrt[d]{u}]$ with d prime to the residue characteristic of Y. We conclude that Y_1 and $(D_1)_{red}$ are regular.

Assume now that (Y, D) is smooth over some regular, noetherian scheme B. We may assume that B is the spectrum of a strictly henselian ring A. Then Y is étale locally isomorphic to the spectrum of $A[T_1, \ldots, T_n]$ such that either D is empty or $D = div\ T_n$. In the former case there is nothing to prove and we thus assume $D = div\ T_n$. By the generalized Abhyankar lemma Y_1 is a disjoint union of schemes of the form

$$Spec\ A[T_1, \ldots, T_n, T]/(T^d - T_n) \cong Spec\ A[T_1, \ldots, T_{n-1}, T]$$

and the reduced scheme underlying D_1 is given by $div\ T$. This proves the result. □

Let us return to the situation where X/B is a strictly henselian arithmetic surface. If $D = div\ a + div\ b$ and $(X_1, D_1) \to (X, D)$ is a tame covering, X_1 might be singular. By the generalized Abhyankar lemma (Theorem 4.1) it is a quotient of

$$X_0 = X[T_1, T_2]/(T_1^{n_1} - a, T_1^{n_2} - b)$$

with positive integers n_1 and n_2 prime to the residue characteristic of X. It is quite complicated to write down a general quotient of X_0 explicitly. However, X_1 can be described as the normalization of X in the function field extension $K(X_1)|K(X)$ and the subextensions of $K(X_0)|K(X)$ are determined by the following lemma.

Lemma 4.3: *Let K be a field and d a positive integer prime to the characteristic of K. Suppose that $\mu_d \subseteq K$ and let $a, b \in K^\times$. Any subextension of $K(\sqrt[d]{a}, \sqrt[d]{b})|K$ is of the form $K[\sqrt[n]{a}, \sqrt[m]{a^r b^s}]$ with $m, n|d$, $0 \leqslant r, s \leqslant m - 1$ and $gcd(r, s, m) = 1$.*

Proof: Let $\bar{K}$ be a separable closure of K and denote by $\mathcal{G}_K$ the absolute Galois group of K. Choose a primitive d^{th} root of unity. This provides us with an identification $H^1(\mathcal{G}_K, \mu_d) \cong Hom(\mathcal{G}_K, \mathbb{Z}/d\mathbb{Z})$. Via the Kummer isomorphism

$$K^\times/(K^\times)^d \times K^\times/(K^\times)^d \xrightarrow{\sim} H^1(\mathcal{G}_K, \mu_d \times \mu_d)$$

and the above identification the pair (a, b) corresponds to a surjection $\phi : \mathcal{G}_K \to \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$ such that $\bar{K}^{ker\phi} = K(\sqrt[d]{a}, \sqrt[d]{b})$. Let L be a subextension of $K(\sqrt[d]{a}, \sqrt[d]{b})|K$. Then there is a quotient Q of $\mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$ such that the image ϕ_Q of ϕ under the map

$$Hom(\mathcal{G}_K, \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}) \to Hom(\mathcal{G}_K, Q)$$

satisfies $\bar{K}^{ker\phi_Q} = L$. By Lemma 4.4 below Q takes the form

$$\begin{array}{ccc} \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z} & \longrightarrow & \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \\ (x, y) & \mapsto & (\bar{x}, \overline{rx + sy}) \end{array}$$

with $m, n|d$, $0 \leqslant r, s \leqslant m - 1$ and $gcd(r, s, m) = 1$. Via Kummer theory the homomorphism ϕ_Q thus corresponds to $(a, a^r b^s) \in K^\times/(K^\times)^n \times K^\times/(K^\times)^m$. This proves the result. $\square$

Lemma 4.4: *Every quotient of $\mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$ takes the form*

$$\begin{array}{ccc} \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z} & \longrightarrow & \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \\ (x, y) & \mapsto & (\bar{x}, \overline{rx + sy}) \end{array}$$

with $m, n|d$, $0 \leqslant r, s \leqslant m - 1$ and $gcd(r, s, m) = 1$.

Proof: Let Q be a quotient of $\mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$. Consider the homomorphism

$$\varphi : \mathbb{Z}/d\mathbb{Z} \xrightarrow{id \times 0} \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z} \xrightarrow{\pi} Q.$$

The image of φ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ with $n|d$ and generated by $\pi(1, 0)$. The homomorphism from $\mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$ to the cokernel of φ factors through the second projection $\mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z} \to \mathbb{Z}/d\mathbb{Z}$. Hence, its cokernel is isomorphic to $\mathbb{Z}/m\mathbb{Z}$ with $m|d$ and generated by $\pi(r, s)$ with $0 \leqslant r, s \leqslant m - 1$ and $gcd(r, s, m) = 1$. $\square$

As a direct corollary of the generalized Abhyankar lemma (Theorem 4.1) and Lemma 4.3 we obtain:

Corollary 4.5: *Let X/B be the strict henselization of an arithmetic surface at a geometric point and D a tidy divisor on X with two irreducible components $Z_1 = div\ g_1$ and $Z_2 = div\ g_2$. Let $(X_1, D_1) \to (X, D)$ be a connected tame covering. Then, X_1 is the normalization of X in a function field extension*

$$K(X)[\sqrt[m]{g_1}, \sqrt[n]{g_1^r g_2^s}]|K(X)$$

with m, n prime to the residue characteristic, $0 \leqslant r, s \leqslant n - 1$ and $gcd(r, s, n) = 1$.

Before concluding this section, we prove another result about desingularized tame covering. We do not need the explicit description from the rest of this section. As the assertion also concerns the shape of tame coverings, we state it here.

Lemma 4.6: *Let $Y' \to Y$ be a flat morphism of schemes which is locally of finite presentation. Let Z be a closed subscheme of Y and denote by Z' its preimage in Y'. Then every connected component of Z dominates a connected component of Z'.*

Proof: By [EGAIV.2], Théorème 2.4.6, the morphism $Y' \to Y$ is universally open. In particular, $Z' \to Z$ is open and thus $Spec\ \mathcal{O}_{Z',z'} \to Spec\ \mathcal{O}_{Z,z}$ is surjective for every point $z' \in Z'$ mapping to $z \in Z$. Suppose there is a connected component Z'_0 of Z' mapping nonsurjectively to a connected component Z_0 of Z. Then there are irreducible components Z_1 and Z_2 of Z_0 with nontrivial intersection such that Z_1 is contained in the image of Z_0 and Z_2 is not. Let $z \in Z_0$ be a point in the intersection of Z_1 and Z_2 and $z' \in Z'_0$ a preimage of z. This produces a contradiction as $Spec\ \mathcal{O}_{Z',z'} \to Spec\ \mathcal{O}_{Z,z}$ is not surjective. $\square$

Corollary 4.7: *Let X/B be an arithmetic surface and $D \subseteq X$ a tidy divisor. Let $(X_1, D_1) \to (X, D)$ be a tame covering. Then every connected component of D_1 dominates a connected component of D.*

4.2 Rational Singularities and the dual graph

We are interested in the type of singularities that arise in tame coverings $(X_1, D_1) \to (X, D)$, with an arithmetic surface X/B and a tidy divisor D. Since X_1 is normal, the singular locus is a finite set of closed points. In the following sections we examine the exceptional fibres of a tidy desingularization of (X_1, D_1). The exceptional fibres are curves over the residue fields of the respective singular points such that all irreducible components are regular and have normal crossings with all other irreducible components. They are determined by their respective irreducible components with multiplicity and their dual graph, which is defined as follows.

Definition 4.8: *Let C be a curve, i. e., a one-dimensional excellent scheme. The dual graph Γ_C of C is the graph defined as follows. Each vertex represents an irreducible component of C and the number of edges between two vertices is given by the number of intersection points of the corresponding irreducible components.*

Lemma 4.9: *Let Y be a normal surface, $Z \subset Y$ a proper closed subscheme, and $\phi : (X, D) \to (Y, Z)$ a tidy desingularization of (Y, Z). Let Γ_D denote the dual graph of D. Then $\pi_1(\Gamma_D)$ is independent of the chosen desingularization.*

Proof: Let $\phi' : (X', D') \to (Y, Z)$ be another desingularization with dual graph $\Gamma_{D'}$. There is a birational map $X' \dashrightarrow X$ over Y. Using elimination of indeterminacies, we find a regular scheme X'' over Y and birational morphisms $\psi : X'' \to X$ and $\psi' : X'' \to X'$ which are isomorphisms over D and D', respectively, such that the following diagram commutes

$$\begin{tikzcd} & X'' \arrow[dl, "\psi"'] \arrow[dr, "\psi'"] & \\ X \arrow[rr, dashed] \arrow[dr] & & X' \arrow[dl] \\ & Y. & \end{tikzcd}$$

The morphisms ψ and ψ' are consecutive blowups in closed points of D and D', respectively. It thus suffices to prove that blowing up X in a closed point x of D leaves $\pi_1(\Gamma_D)$ invariant. Assume first that x is a regular point of D. Blowing up in x a vertex is added to Γ_D and connected with the vertex corresponding to the irreducible component of D containing x. Now assume that x is a singular point of D represented by an edge of the dual graph connecting the vertices a and b, say. Then this edge is removed from the dual graph. A new vertex is added and connected with a and with b. In both cases $\pi_1(\Gamma_D)$ remains invariant. □

Definition 4.10: *A projective (not necessarily integral) curve C over a field k is called rational if $H^1(C, \mathcal{O}_C) = 0$. An arithmetic surface X/B has rational singularities if there is a desingularization $\phi : X_1 \to X$ such that $R^1\phi_*\mathcal{O}_C = 0$.*

Remark 4.11: *(i) By flat base change a curve is rational if and only if its base change to the algebraic closure of the base field is rational.*

(ii) A curve C over an algebraically closed field k is rational if and only if all its irreducible components are isomorphic to $\mathbb{P}^1_k$ and its dual graph is a tree (see [Deb], Definition 4.23).

(iii) If the geometric exceptional fibres of one desingularization of X are rational, the same is true for any desingularization. Indeed, the exceptional fibres of two desingularizations differ only by rational irreducible components and by Lemma 4.9, if the dual graph of one desingularization is a tree, the same holds for the other one.

A particularly simple example of a rational curve is a chain of $\mathbb{P}^1$'s which we define as follows.

Definition 4.12: *Let C be a curve with irreducible components $C_1, \dots, C_n$. We say that C is a chain of $\mathbb{P}^1$'s if $C_1, \dots, C_n$ are isomorphic to $\mathbb{P}^1_k$ for some field k, for $i = 1, \dots, n-1$ the curve C_i intersects C_{i+1} in exactly one point, which is moreover k-rational, and $C_i \cap C_j$ is empty for $|i - j| \geqslant 2$.*

If C is a closed subscheme of another curve C_0, we say that C is a bridge of $\mathbb{P}^1$'s in C_0 if C is a chain of $\mathbb{P}^1$'s and C intersects exactly two of the remaining irreducible components of C_0 and this intersection takes place in two k-rational points $p_1 \in C_1$ and $p_n \in C_n$.

In particular, if C is chain of $\mathbb{P}^1$'s, it is a rational curve over some field such that the dual graph is of the form

.

If moreover C is a bridge of $\mathbb{P}^1$'s in C_0, the dual graph of C_0 near C is of the form

,

where Z_1 and Z_n denote the irreducible components of C_0 intersecting C but not contained in C.

4.3 Explicit desingularizations

For this section we fix a henselian discrete valuation ring $\mathcal{O}$ with algebraically closed residue field k and uniformizer π. We denote the closed point of $Spec\ \mathcal{O}$ by s and the generic point by η. Furthermore, let $Z = Spec\ R/\mathcal{O}$ be smooth of finite type of relative dimension 3 and fix a closed point z of Z lying over the closed point s of $Spec\ \mathcal{O}$. We assume there are $u, v, w \in R$ such that $div(\pi uvw)$ is an snc-divisor such that the intersection of an arbitrary subset of $\{div\ \pi, div\ u, div\ v, div\ w\}$ is irreducible and (π, u, v, w) is the maximal ideal corresponding to z. In particular, (π, u, v, w) is a system of parameters of $\mathcal{O}_{Z,z}$. Let j, k, l, m, r, s be non-negative integers such that $j + k + l > 0$, $m > 0$, $r + s > 0$, and $gcd(m, r, s) = 1$. Let α be a unit of R. We define

$$A = R/(u^j v^k w^l - \alpha\pi, w^m - u^r v^s).$$

This is an integral domain as $gcd(m, r, s) = 1$ and it is of relative dimension 1 over $\mathcal{O}$. We denote its quotient field by K. The special fibre of $Spec\ A$ has two irreducible components corresponding to the prime ideals (u, w) and (v, w) except if $k = l = 0$ or $j = l = 0$. If $k = l = 0$, the special fibre has one component corresponding to the prime ideal (u, w) and if $j = l = 0$, it has one component corresponding to (v, w).

We want to desingularize $Spec\ A$. In particular, we are interested in the exceptional fibres of a desingularization of $Spec\ A$. In general, $Spec\ A$ is not even normal but if either r or s is zero, the normalization of $Spec\ A$ is already regular as the next lemma shows. Note that by relabeling variables we may interchange r and s. Therefore, it suffices to treat the case $s = 0$.

Lemma 4.13: *Assume $s = 0$, i. e., $A = R/(u^j v^k w^l - \alpha\pi, w^m - u^r)$. Consider the ring homomorphism*

$$R \to R' = R[u']/(u'^m - u, u'^r - w).$$

Define

$$A' = R'/(u'^{mj+rl} v^k - \alpha\pi).$$

Then the above homomorphism induces an integral ring extension

$$\phi : A \to A',$$

which is the normalization homomorphism of A. In particular, the normalization of A is regular. The support of the divisor div $w \subset$ Spec A has only one irreducible component and the same holds for div $w \subset$ Spec A'. In particular, the dual graph of div w in Spec A' coincides with the one of div w in Spec A. If the support of div $w \subset$ Spec A is vertical and isomorphic to $\mathbb{A}^1_k$, the same holds for the support of div w in Spec A'. Moreover, div $w \subseteq$ Spec A' is tidy.

Proof: Write $am + br = 1$ with coprime integers a, b. In K the equation

$$X^m - u$$

has the solution $u^a w^b$. It follows that u', being the image of $u^a w^b$, is integral over A and thus the ring homomorphism

$$A \to A'$$

constitutes an integral extension of A. Furthermore, A' is a regular ring so the normalization of A is regular. The induced map of the support of *div w* in *Spec A'* to the support of *div w* in *Spec A* is given by the ring homomorphism

$$R/rad(u^j v^k w^l - \alpha\pi, u^r, w) \to R'/rad(u'^{mj+rl} v^k - \alpha\pi, w)$$

with

$$rad(u^j v^k w^l - \alpha\pi, u^r, w) = \begin{cases} (v^k - \alpha\pi, u, w) & \text{if } j = l = 0, \\ (\pi, u, w) & \text{else} \end{cases}$$

and

$$rad(u'^{mj+rl} v^k - \alpha\pi, w) \begin{cases} (v^k - \alpha\pi, u') & \text{if } j = l = 0, \\ (\pi, u') & \text{else.} \end{cases}$$

In both cases the support of *div w* in *Spec A* and in *Spec A'* has only one irreducible component. If the support of *div w* in *Spec A* is isomorphic to $\mathbb{A}^1_k$, it is not possible that $j = l = 0$ (in which case *div w* is horizontal). We thus have

$$R/rad(u^j v^k w^l - \alpha\pi, u^r, w) = R/(\pi, u, w) \cong k[T]$$

and

$$R'/rad(u'^{mj+rl} v^k - \alpha\pi, w) = R'/(\pi, u') = R[u']/(\pi, u, w, u') \cong k[T, u']/(u') \cong k[T].$$

□

In general, if r and s are both positive, it is quite complicated to write down the normalization of *Spec A* explicitly. However, we can take one step towards the normalization of *Spec A*:

Lemma 4.14: *Assume that r and s are positive. Set*

$$g_1 = gcd(m, s), \qquad g_2 = gcd(m, r),$$

and

$$m' = \frac{m}{g_1 g_2}.$$

Write

$$\frac{r}{g_2} = am' + r'', \qquad \frac{s}{g_1} = bm' + s''$$

with non-negative integers a, b, r'', s'' *such that* $r'', s'' < m'$. *If* $m' = 1$, *set* $g_3 = 1$ *and otherwise* $g_3 = gcd(r'', s'')$ *and define*

$$r' = \frac{r''}{g_3}, \qquad s' = \frac{s''}{g_3},$$

$$j' = g_1 j + al, \qquad k' = g_2 k + bl, \qquad l' = g_3 l.$$

Consider the extension

$$R \to R' = R[u', v', w']/(u'^{g_1} - u, v'^{g_2} - v, u'^a v'^b w'^{g_3} - w)$$

and define

$$A' = R'/(u'^{j'} v'^{k'} w'^{l'} - \alpha\pi, w'^{m'} - u'^{r'} v'^{s'}).$$

Then the above homomorphism $R \to R'$ *induces an integral ring extension*

$$\phi : A \to A'$$

and the dual graphs of div w *in Spec* A *and div* w' *in Spec* A' *coincide. Furthermore, if a vertical irreducible component of div* $w \subset$ *Spec* A *is isomorphic to* $\mathbb{A}^1_k$, *the same holds for the corresponding component of div* $w' \subset$ *Spec* A'. *If* $m' = 1$, A' *is regular and* ϕ *is a normalization homomorphism of* A. *In particular, in this case the normalization of* A *is regular and div* $w \subseteq$ *Spec* A' *is tidy.*

Proof: One checks that ϕ is well defined and injective. There is exactly one point z' of *Spec* A' lying above $z \in$ *Spec* A. The induced map of function fields

$$\phi_K : K \to K'$$

is an isomorphism. Indeed, write

$$1 = c_1 r + d_1 g_1, \qquad 1 = c_2 s + d_2 g_2, \qquad 1 = c_3 m + d_3 g_3$$

with integers c_i and d_i. Identifying K with its image in K' we have

$$u' = u^{d_1} v^{-c_1(bm' + g_3 s')} w^{c_1 g_2 m'}, \qquad v' = u^{-c_2(am' + g_3 r')} v^{d_2} w^{c_2 g_1 m'},$$
$$w' = u'^{-d_3 a} v'^{-d_3 b} u^{c_3 g_2 r'} v^{c_3 g_1 s'} w^{d_3},$$

and thus $K = K'$. Moreover, u', v' and w' are integral over A being roots of the normalized polynomials in $A[X]$

$$X^{g_1} - u, \qquad X^{g_2} - v, \qquad \text{and} \qquad X^m - u^{g_2 r'} v^{g_1 s'},$$

respectively.

Let us show the assertion concerning the dual graph. One checks that the support of *div* w' in *Spec* A' coincides with the support of *div* w in *Spec* A'. In order to see that the dual graphs of *div* $w \subset$ *Spec* A and *div* $w \subset$ *Spec* A' coincide we examine the induced morphism of the underlying reduced subschemes. It is given by

$$R/rad(u^j v^k w^l - \alpha\pi, u^r v^s, w) \to R'/\mathrm{rad}(u'^{j'} v'^{k'} w'^{l} - \alpha\pi, w'^{m'} - u'^{r'} v'^{s'}, u'^a v'^b w'^{g_3})$$

with

$$rad(u^j v^k w^l - \alpha\pi, u^r v^s, w) = \begin{cases} (u^j - \alpha\pi, uv, w) & \text{case 1: } k = l = 0, \\ (v^k - \alpha\pi, uv, w) & \text{case 2: } j = l = 0, \\ (\pi, uv, w) & \text{case 3: else,} \end{cases}$$

$$rad(u'^{j'}v'^{k'}w'^{l}-\alpha\pi, w'^{m'}-u'^{r'}v'^{s'}, u'^{a}v'^{b}w'^{g_3}) = \begin{cases} (u'^{g_1 j}-\alpha\pi, u'v', w') & \text{case 1 and } m'>1,\\ (v'^{g_2 k}-\alpha\pi, u'v', w') & \text{case 2 and } m'>1,\\ (\pi, u'v', w') & \text{case 3 and } m'>1,\\ \\ (u'^{g_1 j}-\alpha\pi, u'v', w'-1) & \text{case 1 and } m'=1,\\ (v'^{g_2 k}-\alpha\pi, u'v', w'-1) & \text{case 2 and } m'=1,\\ (\pi, u'v', w'-1) & \text{case 3 and } m'=1. \end{cases}$$

Assume first that $m'>1$. In case 1 the above morphism becomes

$$R/(u^j-\alpha\pi, uv, w) \to R'/(u'^{g_1 j}-\alpha\pi, u'v', w') = (R/(u^j-\alpha\pi, uv, w))[u'v']/(u'^{g_1}-u, v'^{g_2}-v).$$

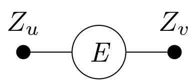
By our assumptions on $div(\pi uvw)$ the spectrum of $R/(u^j-\alpha\pi, uv, w)$ has two irreducible components, one irreducible, smooth curve over k with parameter u at z and one regular horizontal component. The same holds for $(R/(u^j-\alpha\pi, uv, w))[v']/(v'^{g_1}-v)$, where the vertical prime divisor has parameter u' at z'. We conclude that the dual graphs coincide. If the vertical irreducible component of *Spec* $R/(u^j-\alpha\pi, uv, w)$ is isomorphic to $\mathbb{A}^1_k$, i. e., $R/(\pi, u, w) \cong k[v]$, we have

$$(R/(\pi, u, w))[v']/(v'^{g_1}-v) \cong k[v, v']/(v'^{g_1}-v) \cong k[v']$$

and thus also the vertical component of $(R/(u^j-\alpha\pi, uv, w))[v']/(u'^{g_1}-u, v'^{g_2}-v$ is isomorphic to $\mathbb{A}^1_k$. Cases 2 and 3 are analogous, as well as the case where $m'=1$. □

After the construction in Lemma 4.14 we are now ready to examine a desingularization of *Spec A*:

Lemma 4.15: *Assume that $m>r, s>0$ and that m, r and s are pairwise coprime. Let A' be the normalization of A. Then the dual graphs of the special fibre of Spec A and of Spec A' coincide and if an irreducible component of $(Spec\ A)_s$ is isomorphic to $\mathbb{A}^1_k$, the same holds for the corresponding component of $(Spec\ A')_s$. Furthermore, there is a desingularization $X \to Spec\ A'$ such that div $w \subseteq X$ is tidy and the exceptional fibre E of $X \to Spec\ A'$ is a bridge of $\mathbb{P}^1$'s in div $w \subseteq X$ and the dual graph of div w has the form*



.

Proof: We use induction on $(m, r+s)$ given the lexicographical ordering (note that $2 \leqslant r+s \leqslant 2m-2$). The ring A is a noetherian ring of dimension 2 and it is singular at the maximal ideal generated by u, v, and w.

Let us first examine $div\ w \subseteq Spec\ A$. It is the subscheme of *Spec A* determined by the ideal (w):

$$A/(w) = R/(u^j v^k w^l - \alpha\pi, u^r v^s, w).$$

The underlying reduced scheme is given by

$$rad(u^j v^k w^l - \alpha\pi, u^r v^s, w) = \begin{cases} (u^j-\alpha\pi, uv, w) & \text{case 1: } k=l=0,\\ (v^k-\alpha\pi, uv, w) & \text{case 2: } j=l=0,\\ (\pi, uv, w) & \text{case 3: else.} \end{cases}$$

In case 1 $div\ w$ has one horizontal component Z_u corresponding to the prime ideal $(u^j-\alpha\pi, v, w)$ and one vertical component Z_v corresponding to (π, u, w). In case 2 it has one horizontal component Z_v corresponding to $(v^k-\alpha\pi, u, w)$ and one vertical component Z_u corresponding to (π, v, w). In case 3 it has two vertical components Z_u and Z_v corresponding to (π, v, w) and (π, u, w).

Consider the blowup Y of $Spec\ A$ in z. Let Y' denote the normalization of Y. The morphism $Y' \to Y$ factors through $Y \times_A Spec\ A' \to Y$ as the latter is finite birational. We thus have the following diagram:

$$\begin{tikzcd}
Spec\ A & Y \arrow[l] & \\
Spec\ A' \arrow[u] & Y \times_A Spec\ A' \arrow[l] \arrow[u] & \\
 & & Y'. \arrow[ull, bend left] \arrow[ul] \arrow[uul, bend right]
\end{tikzcd}$$

The morphism $Y' \to Spec\ A'$ is a birational morphism of normal surfaces with exceptional locus $z' \in Spec\ A'$. Let us calculate Y. By [Liu], Lemma 8.1.2(e), Y can be covered by 3 affine open subschemes $Spec\ B_u$, $Spec\ B_v$, and $Spec\ B_w$, where

$$B_u = A[\frac{v}{u}, \frac{w}{u}], \qquad B_v = A[\frac{u}{v}, \frac{w}{v}], \qquad B_w = A[\frac{u}{w}, \frac{v}{w}]$$

considered as subrings of the quotient field of A. More precisely, define

$$\begin{aligned}
R_u &= R[v_u, w_u]/(v_u u - v, w_u u - w),\\
R_v &= R[u_v, w_v]/(u_v v - u, w_v v - w),\\
R_w &= R[u_w, v_w]/(u_w w - u, v_w w - v).
\end{aligned}$$

Then $Spec\ R_u$, $Spec\ R_v$, and $Spec\ R_w$ are smooth and of finite type over $\mathcal{O}$ and

$$\begin{aligned}
B_u &= R_u/I_u,\\
I_u &= \begin{cases} (u^{j+k+l}v_u^k w_u^l - \alpha\pi, w_u^m u^{m-r-s} - v_u^s) & \text{if } m > r+s,\\ (u^{j+k+l}v_u^k w_u^l - \alpha\pi, w_u^{r+s} - v_u^s) & \text{if } m = r+s,\\ (u^{j+k+l}v_u^k w_u^l - \alpha\pi, w_u^m - u^{r+s-m}v_u^s) & \text{if } m < r+s,\end{cases}
\end{aligned}$$

$$\begin{aligned}
B_v &= R_v/I_v,\\
I_v &= \begin{cases} (u_v^j v^{j+k+l} w_v^l - \alpha\pi, w_v^m v^{m-r-s} - u_v^r) & \text{if } m > r+s,\\ (u_v^j v^{j+k+l} w_v^l - \alpha\pi, w_v^{r+s} - u_v^r) & \text{if } m = r+s,\\ (u_v^j v^{j+k+l} w_v^l - \alpha\pi, w_v^m - u_v^r v^{r+s-m}) & \text{if } m < r+s,\end{cases}
\end{aligned}$$

$$\begin{aligned}
B_w &= R_w/I_w,\\
I_w &= \begin{cases} (u_w^j v_w^k w^{j+k+l} - \alpha\pi, w^{m-r-s} - u_w^r v_w^s) & \text{if } m > r+s,\\ (u_w^j v_w^k w^{j+k+l} - \alpha\pi, 1 - u_w^r v_w^s) & \text{if } m = r+s,\\ (u_w^j v_w^k w^{j+k+l} - \alpha\pi, 1 - u_w^r v_w^s w^{r+s-m}) & \text{if } m < r+s.\end{cases}
\end{aligned}$$

Let us first treat the case where $m = r+s$. Since $u_w, v_w \in B_w$ are invertible, B_u and B_v are subrings of B_w and thus $Spec\ B_w \subseteq Spec\ B_u \cap Spec\ B_v$. We can therefore omit B_w. By Lemma 4.13 the morphisms

$$\begin{aligned}
B_u &\to B_u' := R[x]/(u^{j+k+l}x^{k(r+s)+ls} - \alpha\pi, x^{r+s}u - v, x^s u - w),\\
v_u &\mapsto x^{r+s}\\
w_u &\mapsto x^s
\end{aligned}$$

$$\begin{aligned}
B_v &\to B_v' := R[y]/(v^{j+k+l}y^{j(r+s)+lr} - \alpha\pi, y^{r+s}v - u, x^r v - w)\\
u_v &\mapsto y^{r+s}\\
w_v &\mapsto y^r
\end{aligned}$$

constitute a normalization Y' of Y, where in the quotient field K of A we have $x = y^{-1}$. Since Y' is regular, it is already a desingularization of *Spec* A'. The divisor *div* $w \subseteq Y'$ is given by

$$\begin{aligned} div\ w \cap Spec\ B'_u &= Spec\ B'_u/(w) = Spec\ R[x]/(u^{j+k+l}x^{k(r+s)+ls} - \alpha\pi, x^s u, v, w),\\ div\ w \cap Spec\ B'_v &= Spec\ B'_v/(w) = Spec\ R[y]/(v^{j+k+l}y^{j(r+s)+lr} - \alpha\pi, y^r v, u, w),\end{aligned}$$

and its underlying reduced subscheme by

$$(div\ w \cap Spec\ B'_u)_{red} = \begin{cases} Spec\ R[x]/(u^{j+k+l} - \alpha\pi, ux, v, w) & \text{if } k = l = 0,\\ Spec\ R[x]/(\pi, ux, v, w) & \text{else,}\end{cases}$$

$$(div\ w \cap Spec\ B'_v)_{red} = \begin{cases} Spec\ R[y]/(v^{j+k+l} - \alpha\pi, vy, u, w) & \text{if } j = l = 0,\\ Spec\ R[y]/(\pi, vy, u, w) & \text{else.}\end{cases}$$

The divisor *div* $w \cap$ *Spec* B'_u has two irreducible components. The first one is given by the prime ideals (π, u, v, w) of $R[x]$ and the second one by (π, v, w, x) or $(u^{j+k+l} - \alpha\pi, v, w, x)$, respectively. The component corresponding to (π, u, v, w) is isomorphic to $\mathbb{A}^1_k$ and the component corresponding to (π, v, w, x), respectively $(u^{j+k+l} - \alpha\pi, ux, v, w)$, is isomorphic to the irreducible component Z_u of $(Spec\ A)_s$. Analogously, *div* $w \cap$ *Spec* B'_v has two irreducible components, one of them isomorphic to $\mathbb{A}^1_k$ and the other one isomorphic to the irreducible component Z_v of $(Spec\ A)_s$. We readily check that the irreducible components of *div* w intersect transversally implying that *div* w is tidy. The components of *div* $w \cap$ *Spec* B'_u and *div* $w \cap$ *Spec* B'_v that are isomorphic to $\mathbb{A}^1_k$ patch together to form the exceptional fibre $E \cong \mathbb{P}^1_k$ of $Y' \to$ *Spec* A'. Furthermore, we read off that the dual graph of the special fibre looks as follows:

$$\begin{tikzcd} \overset{Z_u}{\bullet} \arrow[r, no head] & \overset{E}{\bullet} \arrow[r, no head] & \overset{Z_v}{\bullet} \end{tikzcd}.$$

Assume next that $r + s > m$. By the same reason as before we can omit B_w. Let us determine *div* w.

$$\begin{aligned} div\ w \cap Spec\ B_u &= Spec\ B_u/(w)\\ &= Spec\ R[v_u, w_u]/(u^{j+k+l}v_u^k w_u^l - \alpha\pi, w_u^m - u^{r+s-m}v_u^s, v_u u - v, w_u u, w),\\ div\ w \cap Spec\ B_v &= Spec\ B_v/(w)\\ &= Spec\ R[u_v, w_v]/(u_v^j v^{j+k+l} w_u^l - \alpha\pi, w_v^m - u_v^r v^{r+s-m}, u_v v - u, w_v v, w).\end{aligned}$$

The underlying reduced scheme is given by

$$(div\ w \cap Spec\ B_u)_{red} = \begin{cases} Spec\ R[v_u]/(u^{j+k+l} - \alpha\pi, uv_u, v, w) & \text{if } k = l = 0,\\ Spec\ R[v_u]/(\pi, uv_u, v, w) & \text{else,}\end{cases}$$

$$(div\ w \cap Spec\ B_v)_{red} = \begin{cases} Spec\ R[u_v]/(v^{j+k+l} - \alpha\pi, u_v v, u, w) & \text{if } j = l = 0,\\ Spec\ R[u_v]/(\pi, u_v v, u, w) & \text{else.}\end{cases}$$

Again, the exceptional fibre E is isomorphic to $\mathbb{P}^1_k$ and we read off the dual graph of the special fibre:

$$\begin{tikzcd} \overset{Z_u}{\bullet} \arrow[r, no head] & \overset{E}{\bullet} \arrow[r, no head] & \overset{Z_v}{\bullet} \end{tikzcd}.$$

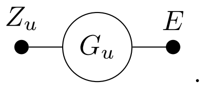
By Lemma 4.14 we can replace B_u by

$$B'_u = R'_u/(x'^{j'}y'^{k'}z'^{l'} - \alpha\pi, z'^{m'} - x'^{r'}y'^{s'})$$

with

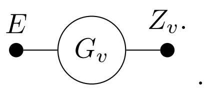
$$R'_u = R_u[x', y', z']/(x'^{g_1} - u, y'^{g_2} - v_u, x'^a y'^b z'^{g_3} - w_u),$$

where we use $r+s-m$ instead of r and $j+k+l$ instead of j and similarly for B_v. The integers $m', j', k', l', r', s', g_1, g_2$ and g_3 are defined as in Lemma 4.14. Note that *Spec* R'_u is smooth and of finite type over $\mathcal{O}$. Since s and m are coprime, we have that $m' > 1$ and $r', s' > 0$. Furthermore, $(m', r'+s') \leqslant (m, (r+s-m)+s) < (m, r+s)$ in the lexicographical ordering as $m > s$. Hence, the induction hypothesis states that there is a desingularization X_u of the normalization Y'_u of *Spec* B'_u (which coincides with the preimage of *Spec* B_u in Y') such that *div* z' is tidy, the exceptional fibre G_u of $X_u \to Y'_u$ is a bridge of $\mathbb{P}^1$'s in *div* z' and the dual graph of *div* z' has the form

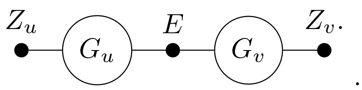


We check that on *Spec* B'_u the divisors *div* w and *div* z' have the same support implying that their dual graphs coincide.

Analogously, there is a desingularization X_v of the normalization Y'_v of *Spec* B'_v such that *div* $w \subseteq X_v$ is tidy, the exceptional fibre G_v of $X_v \to Y'_v$ is a bridge of $\mathbb{P}^1$'s in *div* w and the dual graph of *div* $w \subset X_v$ has the form



Since Y' has only isolated singularities, the desingularizations $X_u \to$ *Spec* B_u and $X_v \to$ *Spec* B_v can be patched together to a desingularization $X \to Y'$. Combining the dual graphs of *div* w in *Spec* B_u and *Spec* B_v we obtain



Since G_u and G_v are bridges of $\mathbb{P}^1$'s by induction, the whole exceptional fibre is a bridge of $\mathbb{P}^1$'s and the assertion is proven for $m < r+s$.

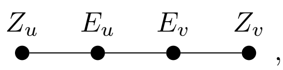
Let us finally treat the case where $m > r+s$. We have

$$(div\ w \cap Spec\ B_u)_{red} = \begin{cases} Spec\ R[w_u]/(u^{j+k+l} - \alpha\pi, uw_u, v, w) & \text{if } k = l = 0, \\ Spec\ R[w_u]/(\pi, uw_u, v, w) & \text{else,} \end{cases}$$

$$(div\ w \cap Spec\ B_v)_{red} = \begin{cases} Spec\ R[w_v]/(v^{j+k+l} - \alpha\pi, w_v v, u, w) & \text{if } j = l = 0, \\ Spec\ R[w_v]/(\pi, w_v v, u, w) & \text{else,} \end{cases}$$

$$(div\ w \cap Spec\ B_w)_{red} = Spec\ R[u_w, v_w]/(\pi, u_w v_w, u, v, w).$$

We read off the dual graph of the special fibre:



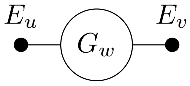
where E_u and E_v are rational. By Lemma 4.14 we can replace B_w by

$$B'_w = R'_w/(x'^{j'} y'^{k'} z'^{l'} - \alpha\pi, z'^{m'} - x'^{r'} y'^{s'})$$

with

$$R'_w = R_w[x', y', z']/(x'^{g_1} - u_w, y'^{g_2} - v_w, x'^{a} y'^{b} z'^{g_3} - w)$$

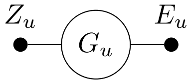
using $j+k+l$ instead of l and $m-r-s$ instead of m. If $m' = 1$, which happens if $m-r-s$ divides rs, B'_w is regular and *div* $w \subseteq$ *Spec* B'_w is tidy. Otherwise, since $(m', r'+s') < (m, r+s)$, the induction hypothesis states that there is a desingularization $X_w \to Y'_v$ such that *div* z' is tidy, the exceptional fibre G_w of $X_w \to Y'_v$ is a bridge of $\mathbb{P}^1$'s in *div* z', and the dual graph of *div* z' has the form



.

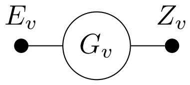
Again, the dual graphs of *div* z' and of *div* w coincide.

By analogous considerations for B_u and B_v we obtain: The normalization Y'_u of *Spec* B_u is regular if s divides $m - r - s$. Otherwise, there is a desingularization $X_u \to Y'_u$ such that *div* $w \subseteq X_u$ is tidy, the exceptional fibre G_u is a bridge of $\mathbb{P}^1$'s in *div* w and the dual graph of *div* $w \cap X_u$ has the form



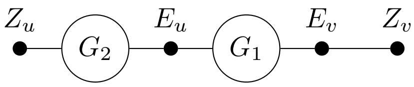
.

Similarly, the normalization Y'_v of *Spec* B_v is regular if r divides $m - r - s$. Otherwise, there is a desingularization $X_v \to Y'_v$ such that *div* $w \subseteq X_v$ is tidy, the exceptional fibre G_v is a bridge of $\mathbb{P}^1$'s in *div* w and the dual graph of *div* $w \cap X_v$ has the form



.

As in the case $r + s > m$, the desingularizations X_u, X_v and X_w patch together to a desingularization $X \to Y'$. Putting the pieces of the dual graphs together, we check that in any case ($s|m - r - s$ or not etc.) the exceptional fibre of $X \to$ *Spec* A' is a bridge of $\mathbb{P}^1$'s in *div* w and the dual graph of *div* w is of the form as stated in the lemma. For instance, if $r|m - r - s$ we have



.

This completes the proof in the case $m > r + s$. □

We just proved that there exists a desingularization of *Spec* A' such that the exceptional fibres are bridges of $\mathbb{P}^1$'s in *div* w. In order to show that this is true for any tidy desingularization of (*Spec* A', *div* w) we need the following lemma.

Lemma 4.16: *Let X/B be an arithmetic surface and $D \subseteq X$ a tidy divisor. Let E be an irreducible component of D which is a -1-curve with field of definition k (i. e., a vertical prime divisor with self-intersection $-[k : k(b)]$, where b is the image point of E in B). Suppose that E intersects each vertical divisor and each irreducible component of D in at most one point such that all intersection points are k-rational and the total number of intersection points is at most* 2. *Then the push-forward D' of D to the contraction $\pi : X \to X'$ of E is a tidy divisor.*

Proof: The contraction $X \to X'$ exists and is regular at the image point p of E by Castelnuovo's criterion as E is a -1-curve (see [Liu], Chapter 9, Theorem 3.8). We have to show that D' is tidy at p. Let W' be either an irreducible component of D' or a vertical prime divisor passing through p. Denote by W its strict transform in X. Then W is either an irreducible component of D or a vertical prime divisor and intersects E transversally at a k-rational point, i. e., $E \cdot W = [k : k(b)]$. Since X' is regular at p, we have

$$\pi^*(W') = W + m_p(W') \cdot E,$$

where $m_p(W')$ denotes the multiplicity of W' in p, i. e., the maximal power of the maximal ideal corresponding to p which contains the ideal sheaf at p corresponding to W'. By the projection formula

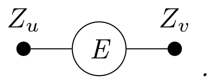
$$0 = E \cdot \pi^*(W') = E \cdot (W + m_p(W') \cdot E) = [k : k(b)](1 - m_p(W')),$$

and thus $m_p(W') = 1$, i. e., W' is regular at p. If W' is the only prime divisor passing through p which is either vertical or contained in D, we are done. If not, there is exactly one other such prime divisor Z', which is regular at p by the same reason as W'. We have to show that Z' intersects W' transversally at p. Since the problem is local on X', we may assume that p is the only intersection point of Z' and W'. Hence, Z and W do not intersect and

$$Z' \cdot W' = \pi^*(Z') \cdot \pi^*(W') = (Z+E) \cdot (W+E) = [k : k(b)],$$

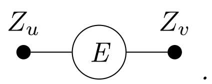
which is equivalent to saying that Z' and W' intersect transversally at p. $\square$

Corollary 4.17: *In the situation of Lemma 4.15 let $X' \to Spec\ A'$ be the minimal desingularization of $Spec\ A'$. Then $div\ w \subseteq X'$ is tidy, the exceptional fibre is a bridge of $\mathbb{P}^1$'s in $div\ w$ and the dual graph of $div\ w$ has the form*



Proof: If the desingularization $X \to Spec\ A'$ constructed in Lemma 4.15 is not the minimal desingularization of $Spec\ A'$, there is an irreducible component E of the exceptional fibre which is a -1-curve. Using that the exceptional fibre of $X \to Spec\ A'$ is a bridge of $\mathbb{P}^1$'s in $div\ w$ we verify that the assumptions on E in Lemma 4.16 are satisfied. Contracting E we obtain another desingularization of $Spec\ A'$ with the required properties and with one irreducible component less. Having contracted all -1-curves we arrive at the minimal desingularization of $Spec\ A'$, which is thus of the desired form. $\square$

Corollary 4.18: *In the situation of Lemma 4.15, for* any *tidy desingularization of $Spec\ A'$ the exceptional fibre is a bridge of $\mathbb{P}^1$'s in $div\ w$ and the dual graph of $div\ w$ has the form*



Proof: By Lemma 4.17 the minimal desingularization $X \to Spec\ A'$ has the asserted properties. Any other tidy desingularization evolves from X by successively blowing up special points of D. In our situation the special points coincide with the singular points of D. After blowing up a singular point of D the exceptional fibre still has the above described form. $\square$

4.4 The generalized exceptional fibre of a desingularized tame covering

In this section we describe the singularities arising in a tame covering $(X_1, D_1) \to (X, D)$ of an arithmetic surface X/B with tidy divisor D. It turns out that the singularities of X_1 are locally of the form $Spec\ A$, where A is the ring defined in the previous section. More precisely, we have the following result.

Proposition 4.19: *Let X/B be an arithmetic surface and $D \subseteq X$ a tidy divisor. Let $(X_1, D_1) \to (X, D)$ be a tame covering of (X, D) and $(X'_{min}, D'_{min}) \to (X_1, D_1)$ the minimal desingularization of (X_1, D_1). Then D'_{min} is a tidy divisor and the exceptional fibres of $X'_{min} \to X_1$ are bridges of $\mathbb{P}^1$'s in D'_{min}. In particular, $(X'_{min}, D'_{min}) \to (X, D)$ is a tidy desingularization of (X, D). Moreover, for any other desingularized tame covering $(X', D') \to (X, D)$ the generalized exceptional fibres are bridges of $\mathbb{P}^1$'s in D', as well.*

Proof: The assertions of the proposition are étale local. Indeed, let $(Y, Z) \to (X, D)$ be an étale cover. We obtain a cartesian diagram

$$\begin{array}{ccccc} (Y',Z') & \longrightarrow & (Y_1,Z_1) & \longrightarrow & (Y,Z) \\ \downarrow & & \downarrow & & \downarrow \\ (X'_{min},D'_{min}) & \longrightarrow & (X_1,D_1) & \longrightarrow & (X,D). \end{array}$$

The upper row defines a desingularized $\mathfrak{c}$-covering of (Y, Z). The exceptional fibre E_{y_1} of a closed point $y_1 \in Y_1$ is the base change of the exceptional fibre E_{x_1} of the image point $x_1 \in X_1$ to the residue field of y_1. If E_{x_1} is a bridge of $\mathbb{P}^1$'s, the same holds for its base change to $k(y_1)$. Suppose that E_{y_1} is a bridge of $\mathbb{P}^1$'s with dual graph

$$\overset{A_1}{\bullet}\!-\!\!-\!\!\overset{E_1}{\bullet}\!-\!\!-\!\!\overset{E_2}{\bullet}\!-\!\!-\;\cdots\cdots\;-\!\!-\!\!\overset{E_{n-1}}{\bullet}\!-\!\!-\!\!\overset{E_n}{\bullet}\!-\!\!-\!\!\overset{A_n}{\bullet}\ ,$$

where A_1 and A_n denote the irreducible components of Z_1 passing through y_1. The images of A_1 and A_n in D_1 are two distinct irreducible components B_1 and B_n of D_1 intersecting in x_1. The exceptional fibre E_{x_1} connects the strict transforms of B_1 and B_n. Each irreducible component of E_{x_1} has at most two intersection points because otherwise there would be irreducible components of E_{y_1} with more than two intersection points. With these restrictions the dual graph of E_{x_1} has to be of the form

$$\overset{B_1}{\bullet}\!-\!\!-\!\!\overset{F_1}{\bullet}\!-\!\!-\!\!\overset{F_2}{\bullet}\!-\!\!-\;\cdots\cdots\;-\!\!-\!\!\overset{F_{m-1}}{\bullet}\!-\!\!-\!\!\overset{F_m}{\bullet}\!-\!\!-\!\!\overset{B_n}{\bullet}\ ,$$

with $m \leqslant n$. Furthermore, the intersection point of the strict transform of B_n with F_m is $k(x_1)$-rational. Hence, the field of definition of F_m is $k(x_1)$ and thus F_m is isomorphic to $\mathbb{P}^1_{k(x_1)}$ and E_n equals the base change of F_m to $k(y_1)$. The intersection point of F_m with F_{m-1} also has to be $k(x_1)$-rational because otherwise E_n would have more than two intersection points. Continuing this process, we obtain by induction on n that E_{x_1} is a bridge of $\mathbb{P}^1$'s. By the same reasoning as in proposition 4.17 we obtain the statement about the minimal desingularization of (Y_1, Z_1) and as in Corollary 4.18 we conclude that the generalized exceptional fibres are bridges of $\mathbb{P}^1$'s for any tidy desingularization of (Y_1, D_1).

Since X_1 is normal, it has only isolated singularities. Let x be a singular point of X lying over a closed point $b \in B$. By Lemma 2.11 there is a smooth, connected scheme of finite type Z over B of relative dimension 2, a closed point $z \in Z_b$, and a surjective homomorphism $\mathcal{O}_{Z,z} \to \mathcal{O}_{X,x}$ whose kernel is generated by an element $F \in \mathcal{O}_{Z,z}$ which takes the following form: There is a system of parameters (π, u, v) of $\mathcal{O}_{Z,z}$, a unit $\alpha \in \mathcal{O}_{Z,z}^\times$, and non-negative integers j, k with $j + k > 0$ such that

$$F = u^j v^k - \alpha\pi.$$

Furthermore, if we denote by $\bar{u}$ and $\bar{v}$ the image of u and v in $Spec\ \mathcal{O}_{X,x}$, we have either

$$(D_{\mathcal{O}_{X,x}})_{red} = div\ \bar{u} + div\ \bar{v} \quad \text{or} \quad (D_{\mathcal{O}_{X,x}})_{red} = div\ \bar{u}.$$

Shrinking Z we may assume that $Z = Spec\ R$ is affine and that u and v are contained in R such that $div\ (\pi uv)$ is an snc divisor and the intersection of an arbitrary subset of $\{div\ \pi, div\ u, div\ v\}$ is irreducible and (π, u, v) is the maximal ideal corresponding to z. We may then replace X by $Spec\ R/F$ and D by $div(uv)$ or $div(u)$, respectively. By Lemma 4.2 it suffices to treat the case where D is singular, i. e., $D = div(uv)$.

Let $x_1 \in X_1$ be a point in the preimage of x. By Corollary 4.5 the strict henselization $(X_1)^{sh}_{x_1}$ of X_1 at x_1 is the normalization of X^{sh}_x in a function field extension

$$K(X^{sh}_x)[\sqrt[n]{u},\ \sqrt[m]{u^r v^s}]|K(X^{sh}_x)$$

with m, n prime to the residue characteristic, $0 \leqslant r, s \leqslant m - 1$ and $gcd(r, s, m) = 1$. Since the assertions of the proposition are étale local by the first paragraph of the proof, we may assume that μ_m and μ_n are constant sheaves on X. Then, étale locally at x, the tame covering $X_1 \to X$ is the normalization of X in the function field extension

$$K(X)[\sqrt[n]{u},\ \sqrt[m]{u^r v^s}]|K(X).$$

The normalization X_2 of X in $K(X)[\sqrt[n]{u}]$ ramifies only in *div* u, which is regular. Lemma 4.2 thus implies that X_2 is regular. So without loss of generality we may replace X with X_2 and assume that X_1 is the normalization of X in $K(X)[\sqrt[m]{u^r v^s}]$. Hence, X_1 is isomorphic to the normalization of

$$Spec\ R[w]/(u^j v^k - \alpha\pi, w^m - u^r v^s).$$

Replacing X by an open subscheme we may assume that the assumptions on $R[w]$ from the beginning of section 4.3 are satisfied: The divisor $div(\pi uvw)$ of $Spec\ R[w]$ is an snc-divisor such that the intersection of an arbitrary subset of $\{div\ \pi, div\ u, div\ v, div\ w\}$ is irreducible. Moreover, by construction, (π, u, v, w) is the maximal ideal corresponding to the image point of x_1 in $Spec\ R[w]$. The assertions now follow from Corollary 4.17 and Corollary 4.18. □

Corollary 4.20: *Let X/B be an arithmetic surface and $D \subseteq X$ a tidy divisor. Let $(X_1, D_1) \to (X, D)$ be a tame covering of (X, D) and $(X', D') \to (X_1, D_1)$ a desingularization of (X_1, D_1). Assume that every irreducible component of an exceptional fibre of $(X', D') \to (X_1, D_1)$ intersects the other irreducible components of D' in at least two points. Then $(X', D') \to (X_1, D_1)$ is a tidy desingularization.*

Proof: We can factor $(X', D') \to (X_1, D_1)$ as

$$(X', D') := (X'_0, D'_0) \to (X'_1, D'_1) \to \ldots \to (X'_n, D'_n) \to (X_1, D_1),$$

where $(X'_n, D'_n) \to (X_1, D_1)$ is the minimal desingularization of (X_1, D_1) and for $i = 1, \ldots, n$ the morphism $(X'_{i-1}, D'_{i-1}) \to (X'_i, D'_i)$ is the blowup of X'_i in a closed point p_i of D'_i. By proposition 4.19 the minimal desingularization $(X'_n, D'_n) \to (X_1, D_1)$ is a tidy desingularization. Moreover, blowing up in closed points does not destroy the tidiness of a divisor. Hence, D'_i is a tidy divisor of X'_i for all $i = 0, \ldots n$. Suppose that $(X', D') \to (X_1, D_1)$ is not a tidy desingularization. Then there is an index i such that p_i is not a special point of D'_i, i. e., p_i is a regular point of D_i. Let i_0 be the smallest such index. Then the exceptional fibre of $(X'_{i_0-1}, D'_{i_0-1}) \to (X'_{i_0}, D'_{i_0})$ has only one intersection point with the other irreducible components of D'_{i_0-1}. This does not change by blowing up D'_{i_0-1} in special points. We thus obtain a contradiction. □

Our knowledge of the local structure of desingularized tame coverings now puts us in the position of proving Lemma 2.21 from the previous chapter. For the convenience of the reader we restate it here:

Lemma: *The following assertions hold:*

(i) If $(X', D') \to (X, D)$ and $(X'', D'') \to (X', D')$ are both desingularized $\mathfrak{c}$-coverings, the composite $(X'', D'') \to (X, D)$ is again a desingularized $\mathfrak{c}$-covering.

(ii) If $(X', D') \to (X, D)$ and $(X'', D'') \to (X, D)$ are desingularized $\mathfrak{c}$-coverings, there is a commutative diagram of desingularized $\mathfrak{c}$-coverings

$$\begin{tikzcd} & (X', D') \arrow[dr] & \\ (X''', D''') \arrow[ur] \arrow[dr] & & (X, D). \\ & (X'', D'') \arrow[ur] & \end{tikzcd}$$

(iii) Let X'/B' be another arithmetic surface with tidy divisor $D' \subseteq X'$. Let $\bar{x} \to (X' - D')$ be a geometric point. There is at most one desingularized tame covering $(X', D') \to (X, D)$ such that $\bar{x} \to (X' - D') \to (X - D)$ coincides with the fixed geometric point $\bar{x} \to (X - D)$.

Proof: (i). Let X_1 be the normalization of X in $K(X')$ and X_2 its normalization in $K(X'')$. Furthermore, denote by X_1' the normalization of X' in $K(X'')$. We obtain a cartesian diagram

$$\begin{tikzcd} D'' \arrow[r] \arrow[d, hook] & D_2 \arrow[r] \arrow[d, hook] & D \arrow[d, hook] \\ X'' \arrow[r] & X_2 \arrow[r] & X. \end{tikzcd}$$

Since $U' = X' - D'$ is the normalization of $U = X - D$ in $K(X')$ and $U'' = X'' - D''$ is the normalization of U' in $K(X'')$, U'' is also the normalization of U in $K(X'')$. It is thus an open subscheme of X_2 and $U'' \to U$ is a finite étale $\mathfrak{c}$-covering as finite étale $\mathfrak{c}$-coverings are stable under composition. Hence, $X'' \to X_2$ is birational and an isomorphism on U''. Moreover, $D'' \subseteq X''$ is a tidy divisor. The only remaining question is whether $X'' \to X_2$ is obtained from the minimal desingularization of (X_2, D_2) by successively blowing up in special points. By Corollary 4.20 it suffices to show that every irreducible component of an exceptional fibre of $X'' \to X_2$ meets the other irreducible components of D'' in at least two points. The morphisms $X'' \to X'$ and $X' \to X$ factor as

$$\begin{aligned} (X'', D'') &= (Y_0, Z_0) \to \dots \to (Y_n, Z_n) = (X_1', D_1') \to (X', D'), \\ (X', D') &= (Y_{n+1}, Z_{n+1}) \to \dots \to (Y_m, Z_m) = (X_1, D_1) \to (X, D), \end{aligned}$$

where $(Y_n, Z_n) \to (X_1', D_1')$ and $(Y_m, Z_m) \to (X_1, D_1)$ represent the minimal desingularizations of (X_1', D_1') and (X_1, D_1), respectively, and for $i = 1, \dots, n$ and $i = n+2, \dots, m$ the morphism $(Y_{i-1}, Z_{i-1}) \to (Y_i, Z_i)$ is the blowup of Y_i in a special point p_i of Z_i. Let E be an irreducible component of an exceptional fibre of $X'' \to X_2$. There is $i \in \{1, \dots, n\} \cup \{n+2, \dots m\}$ such that the image of E in Y_{i-1} is one-dimensional and its image in Y_i is a closed point. This closed point is precisely the point p_i and we obtain a finite morphism from E to the exceptional fibre of $Y_{i-1} \to Y_i$ in X''. Since $X_{i-1} \to X_i$ is the blowup of X_i in p_i and p_i is a special point, its exceptional fibre intersects the other irreducible components of Z_{i-1} in two points. The intersection points of E contain the preimages of these two points and thus there are at least two intersection points.

(ii). Let K''' be the compositum of $K(X')$ and $K(X'')$ and X_3 the normalization of X in K'''. This defines a $\mathfrak{c}$-covering $(X_3, D_3) \to (X, D)$. We obtain rational maps $X_3 \dashrightarrow X'$ and $X_3 \dashrightarrow X''$, which, restricted to $U_3 = X_3 - D_3$, are finite étale $\mathfrak{c}$-coverings of $U' = X' - D'$ and $U'' = X'' - D''$, respectively. Using elimination of indeterminacies and the existence of tidy desingularizations we find a desingularization $(X''', D''') \to (X_3, D_3)$ dominating (X', D') and (X'', D'') such that D''' is tidy. Suppose there is an irreducible component E of an exceptional fibre of X''' with only one intersection point with the other irreducible components of D'''. By similar arguments as in the proof of part (i) the image of E in X' as well as in X'' is a point. Let us write

$$(X''', D''') = (X_0''', D_0''') \to \dots \to (X_n''', D_n''') \to (X_3, D_3),$$

where $(X_n''', D_n''') \to (X_3, D_3)$ is the minimal desingularization of (X_3, D_3) and for $i = 1, \dots, n$ the morphism $X_{i-1}''' \to X_i'''$ is the blowup of X_i''' in a closed point $p_i \in D_i'''$. There is $i \in \{1, \dots, n\}$ such that the image of E is the point p_i and the image of E in X_{i-1}''' is the exceptional fibre E_i of $X_{i-1}''' \to X_i'''$. Since E has only one intersection point, the same holds for E_i. Furthermore, the blowup points p_k for $k = 1, \dots, i-1$ must not lie above E_i except possibly above the intersection point q_i of E_i with the other irreducible components. One checks that after blowing up in q_i the strict transform of E_i is still a -1-curve. Therefore, we can contract E. Moreover, by similar arguments as in the proof of part (i) the image of E in X' as well as in X'' is a point. Hence, the contraction still factors through $X' \to X$ and $X'' \to X$. After finitely many contractions we may assume that all irreducible components of exceptional fibres of $X''' \to X_3$ have at least two intersection points. Then the same holds for the generalized exceptional fibres of $X''' \to X'$ and of $X''' \to X''$ as these are contained in the exceptional fibres of $X''' \to X_3$. The assertion now follows from Lemma 4.20.

(iii). Set $U = X - D$ and $U' = X' - D'$. Denote by x' the image of the geometric point $\bar{x}$ in X'. There is at most one étale covering $U' \to U$ which induces $x' \to x$ and a morphism $X' \to X$ is completely determined by its restriction to a dense open subset. $\square$

4.5 Multiplicities of the exceptional divisors

Given an arithmetic surface X/B, a tidy divisor $D \subseteq X$ on X, and a desingularized tame covering $(X', D') \to (X, D)$, we have determined in the preceding sections the structure of the support of D'. However, we have not yet dealt with the multiplicities in D' of the irreducible components of the generalized exceptional fibres. More precisely, we are interested in the pullback of an irreducible component of D to X'.

Definition 4.21: *Let X/B be an arithmetic surface and $D \subseteq X$ a Cartier divisor. Let $f : (X', D') \to (X, D)$ be a morphism such that pullback of Cartier divisors is defined (e. g. birational or flat). Let $x' \in D'$ be a closed point and denote by $x \in D$ the image of x' in X. Let us call $D_1, \ldots, D_n$ the irreducible components of D passing through x and $D'_1, \ldots, D'_m$ the irreducible components of D' passing through x'. Restricting f to a suitable neighborhood of x', the pullback of Cartier divisors via f induces a homomorphism*

$$\mathbb{Q} \cdot D_1 \oplus \ldots \oplus \mathbb{Q} \cdot D_n \to \mathbb{Q} \cdot D'_1 \oplus \ldots \oplus \mathbb{Q} \cdot D'_m.$$

We call this morphism multiplicity homomorphism at x' and its transformation matrix with respect to the above bases multiplicity matrix at x'.

In particular, the multiplicity homomorphisms are defined for a tidy divisor D and a desingularized tame covering $(X', D') \to (X, D)$. Moreover, multiplicity homomorphisms are compatible with composition. If $(X'', D'') \to (X', D')$ is another morphism as above and x'' a closed point of D'' mapping to $x' \in D'$, the multiplicity homomorphism of $(X'', D'') \to (X', D')$ at x'' is the composition of the multiplicity homomorphism of $(X'', D'') \to (X', D')$ at x'' and the multiplicity homomorphism of $(X', D') \to (X, D)$ at x'.

Lemma 4.22: *Let X/B be an arithmetic surface and $D \subseteq X$ a tidy divisor. Let $(X', D') \to (X, D)$ be the blowup of X in a singular point p of D. Then all multiplicity homomorphisms are surjective.*

Proof: Denote by D_1 and D_2 the irreducible components of D passing through p and by D'_1 and D'_2 their strict transforms in X'. Furthermore, let E denote the singular fibre of $X' \to X$. On $E \subseteq D'$ there are two points p'_1 and p'_2 where D' is singular, namely the respective intersection points with D'_1 and D'_2. The pullback of D_i is given by $D'_i + E$. Hence, the intersection matrix at p'_1 as well as at p'_2 (with respect to the bases $\{(D_1, D_2), (D'_1, E)\}$ and $\{(D_1, D_2), (E, D'_2)\}$, respectively) is

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

which is invertible. If $p' \in E$ is a nonsingular point of D', its multiplicity matrix is

$$\begin{pmatrix} 1 & 1 \end{pmatrix},$$

which is nonzero and thus its multiplicity homomorphism is surjective. The intersection homomorphism at any other closed point of D' is the identity. $\square$

Lemma 4.23: *Let X/B be the localization of an arithmetic surface at a closed point x of codimension 2 and $D \subseteq X$ a tidy divisor. Let $\pi : (X', D') \to (X_1, D_1) \to (X, D)$ be a desingularized tame covering such that (X_1, D_1) is local with closed point x_1 and $(X', D') \to (X_1, D_1)$ is the minimal desingularization of (X_1, D_1). Suppose that X_1 is singular, i. e., that $X' \to X_1$ is not the identity. Let Z be an irreducible component of D. Then, there is exactly one irreducible component Z_1 of D_1 lying above Z and the exceptional fibre E of $X' \to X_1$ is a chain of $\mathbb{P}^1$'s with dual graph*

Moreover, E intersects the strict transform Z' of Z_1 in one point $p \in E_1$ and the pullback π^*Z of Z to X' takes the form

$$a_0 Z' + a_1 E_1 + \ldots a_n E_n$$

with $a_0 > a_1 > \ldots > a_n > 0$.

Proof: All assertions except the one about the coefficients $a_0, \ldots, a_n$ follow from Corollary 4.17. Denote by b the image of x in B. In order to simplify notation, we set $E_0 := Z'$. By the projection formula we have

$$0 = \pi^*Z \cdot E_n = (a_0 E_0 + a_1 E_1 + \ldots a_n E_n) \cdot E_n = [k(x) : k(b)](a_{n-1} + a_n E_n^2).$$

Since the desingularization $X' \to X_1$ is minimal, E_n cannot be a -1-curve and thus $E_n^2 < -1$. (The self-intersection of E_n has to be negative by [Liu], chapter 9, Theorem 1.27.) Hence,

$$a_{n-1} = -a_n E_n^2 > a_n.$$

By induction we may assume that $a_{i+1} < a_i$ for $0 < k \leqslant i < n$. Again by the projection formula we obtain

$$0 = \pi^*Z \cdot E_k = [k(x) : k(b)](a_{k-1} + a_k E_k^2 + a_{k+1}).$$

By induction and using $E_k^2 \leqslant -2$ we conclude that

$$a_{k-1} = -a_{k+1} - a_k E_k^2 > -a_{k+1} + 2a_k > a_k.$$

□

Lemma 4.24: *Let X/B be an arithmetic surface and $D \subseteq X$ a tidy divisor. Let $\pi : (X', D') \to (X_1, D_1) \to (X, D)$ be a desingularized tame covering. Then all multiplicity homomorphisms are surjective.*

Proof: By Lemma 4.22 we may assume that $X' \to X_1$ is the minimal desingularization of X_1. Let $x' \in D'$ be a closed point and denote by x_1 and x the image of x' in X_1 and X, respectively. Without loss of generality we may replace X by its localization at x and X_1 by its localization at x_1 and X' by its base change to the localization of X_1 at x_1. If x' is a regular point of D', there is only one irreducible component of D' passing through x'. Hence, the multiplicity homomorphism at x' is surjective if and only if it is nonzero, which is clear by taking the pullback of any irreducible component of D passing through x.

Suppose that x' is a singular point of D'. Then also x_1 and x are singular points of D_1 and D, respectively. There are two irreducible components Z_1 and W_1 of D_1 passing through x_1 mapping to the irreducible components W and Z of D passing through x. Assume first that x_1 is a regular point of X_1. Then $X_1 = X'$ and

$$\pi^*Z = aZ_1 \qquad \text{and} \qquad \pi^*W = bW_1$$

with positive integers a and b. Hence, the multiplicity matrix is

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$

which is invertible. If x_1 is a singular point of X_1, we are in the situation of Lemma 4.23. Using the notation of this lemma we have

$$\pi^* Z = a_0 Z' + a_1 E_1 + \dots a_n E_n$$

with $a_0 > a_1 > \dots a_n > 0$ and

$$\pi^* W = b_1 E_1 + \dots b_n E_n + b_{n+1} W'$$

with $b_1 < \dots < b_n < b_{n+1}$ and where W' denotes the strict transform of W_1 in X'. Setting $E_0 := Z'$ and $E_{n+1} := W'$ we know that there is an integer i with $0 \leqslant i \leqslant n$ such that x' is the intersection point of E_i with E_{i+1}. The intersection matrix at x' is

$$\begin{pmatrix} a_i & b_i \\ a_{i+1} & b_{i+1} \end{pmatrix}$$

and

$$det \begin{pmatrix} a_i & b_i \\ a_{i+1} & b_{i+1} \end{pmatrix} = a_i b_{i+1} - a_{i+1} b_i > a_i b_i - a_i b_i = 0$$

as $a_{i+1} < a_i$ and $b_{i+1} > b_i$. Therefore, also in this case the multiplicity homomorphism is surjective. $\square$

Chapter 5

Cohomology with support

Given an arithmetic surface X/B, a tidy divisor $D \subseteq X$ and a full class of finite groups $\mathfrak{c}$ we want to lift, in the limit over all desingularized $\mathfrak{c}$-coverings, cohomology classes on U to cohomology classes on X. Via the excision sequence associated with $D \hookrightarrow X$ this amounts to showing the vanishing in the limit over $\mathfrak{I}_{X,D,\bar{x}}$ of the cohomology groups with support

$$H^i_D(X, \mathbb{Z}/m\mathbb{Z})$$

for $m \in \mathbb{N}(\mathfrak{c})$. Assuming all integers in $\mathbb{N}(\mathfrak{c})$ are invertible on X, we can apply absolute cohomological purity to compute these cohomology groups. Since in general, D is not regular (it is singular in the special points S), we have to divide this task in two steps making use of the excision sequences associated with $D - S \subseteq X - S$ and $S \subseteq X$.

5.1 Absolute cohomological purity

Definition 5.1: *Let c be a non-negative integer. A regular pair of codimension c is a pair (X, Z) where X is a noetherian regular scheme and Z is a closed subscheme of X of pure codimension c whose underlying reduced scheme is regular. A morphism of regular pairs $(X', Z') \to (X, Z)$ is a cartesian diagram*

$$\begin{tikzcd} Z' \arrow[r, hook] \arrow[d] & X' \arrow[d] \\ Z \arrow[r, hook] & X. \end{tikzcd}$$

Remark 5.2: *At first sight it seems more natural to require Z in the definition of a regular pair (X, Z) to be regular, not only its underlying reduced subscheme. However, we want tame coverings of a regular pair (X, Z) to be morphisms of regular pairs. But if a tame covering $(X', Z') \to (X, Z)$ ramifies in Z, Z' cannot be reduced.*

For non-negative integers n and r we write $\mathbb{Z}/n\mathbb{Z}(r)$ for the r-fold tensor product of μ_n with itself and define $\mathbb{Z}/n\mathbb{Z}(-r) = \mathscr{H}om(\mathbb{Z}/n\mathbb{Z}(r), \mathbb{Z}/n\mathbb{Z})$. The following theorem by Gabber (see [Fuj]) is known as absolute cohomological purity.

Theorem 5.3: *Let (X, Z) be a regular pair of codimension c and n a positive integer invertible on X. Set $\Lambda = \mathbb{Z}/n\mathbb{Z}$. Then*

$$\underline{H}^q_Z(\Lambda) \cong \begin{cases} 0 & \text{for } q \neq 2c \\ \Lambda_Z(-c) & \text{for } q = 2c. \end{cases}$$

Remark 5.4: *In [Fuj] it is required that Z be regular. But the étale sites of Z and Z_{red} are equivalent and hence, the statement also holds if only Z_{red} is regular.*

Proposition 5.5: *Let $f : (X', Z') \to (X, Z)$ be a morphism of regular pairs of codimension c. Suppose that Z and Z' are irreducible and as cycles we have $f^*Z_{red} = e \cdot Z'_{red}$ with a positive integer e (the ramification index). Then, for any $n \in \mathbb{N}$ invertible on X the following diagram commutes*

$$\begin{tikzcd}
H^i_Z(X, \mathbb{Z}/n\mathbb{Z}) \arrow[ddd] & H^{i-2c}(Z, \mathbb{Z}/n\mathbb{Z}(-c)) \arrow[l, "\sim"', "purity"] \arrow[d] \\
& H^{i-2c}(Z', \mathbb{Z}/n\mathbb{Z}(-c)) \arrow[dd, "\cdot e"] \\
& \\
H^i_{Z'}(X', \mathbb{Z}/n\mathbb{Z}) & H^{i-2c}(Z', \mathbb{Z}/n\mathbb{Z}(-c)). \arrow[l, "\sim", "purity"']
\end{tikzcd}$$

Proof: Set $\Lambda := \mathbb{Z}/n\mathbb{Z}$ and for any scheme Y denote by Λ_Y the constant sheaf on Y with stalks Λ. Consider the following diagram of sheaves on Z

$$\begin{tikzcd}
f^*\underline{H}^{2c}_Z(\Lambda_X(c)) \arrow[ddd] & f^*\Lambda_Z \arrow[l, "\sim"'] \arrow[d] \\
& \Lambda_{Z'} \arrow[dd, "\cdot e"] \\
& \\
\underline{H}^{2c}_{Z'}(\Lambda_{X'}(c)) & \Lambda_{Z'}. \arrow[l, "\sim"']
\end{tikzcd}$$

The horizontal maps are induced by the cycle maps which map $1 \in \Lambda$ to the fundamental class $s_{Z_{red}/X}$ and $s_{Z'_{red}/X'}$, respectively. We want to show that the diagram commutes. It suffices to do so for global sections as all sheaves involved are constant. Under the composition

$$f^*\Lambda_Z \to \Lambda_{Z'} \xrightarrow{\cdot e} \Lambda_{Z'} \to \underline{H}^{2c}_{Z'}(X', \Lambda_{X'}(c))$$

the element $1 \in \Lambda$ is mapped to $e \cdot s_{Z'_{red}/X'}$ and under the composition

$$f^*\Lambda_Z \to f^*\underline{H}^2_Z(X, \Lambda_X(c)) \to \underline{H}^{2c}_{Z'}(X', \Lambda_{X'}(c))$$

it is mapped to $f^*s_{Z_{red}/X}$, which equals $e \cdot s_{Z'_{red}/X'}$ because $f^*Z_{red} = e \cdot Z'_{red}$. Twisting by $(-c)$ and taking cohomology we obtain the commutative diagram

$$\begin{tikzcd}
H^{i-2c}(Z, \underline{H}^{2c}_Z(X, \Lambda)) \arrow[ddd] & H^{i-2c}(Z, \Lambda(-c)) \arrow[l, "\sim"'] \arrow[d] \\
& H^{i-2c}(Z', \Lambda(-c)) \arrow[dd, "\cdot e"] \\
& \\
H^{i-2c}(Z', \underline{H}^{2c}_{Z'}(X', \Lambda)) & H^{i-2c}(Z', \Lambda(-c)). \arrow[l, "\sim"']
\end{tikzcd}$$

By Theorem 5.3 the cohomology groups on the left are canonically isomorphic to $H^i_Z(X, \Lambda)$ and $H^i_{Z'}(X', \Lambda)$ respectively. This proves the result. □

Corollary 5.6: *Let X be a noetherian, regular scheme and $f : X' \to X$ a tamely ramified covering such that the branch locus $D \subseteq X$ is regular. Let Z be an irreducible component of D and let Z' denote its preimage in X'. Then, for any integer n dividing the ramification index of each irreducible component of Z', the canonical map*

$$H^i_Z(X,\Lambda) \to H^i_{Z'}(X',\Lambda)$$

is the zero map for all $i \in \mathbb{N}$.

Proof: By Lemma 4.2, X' and the underlying reduced subscheme of Z' are regular because the branch locus D is regular. Denote by Z'_k, $k = 1,\dots,r$ the irreducible components of Z'. For each k we can now apply proposition 5.5 to the morphism

$$X' - \bigcup_{i\neq k} Z'_i \to X$$

to conclude that

$$H^i_Z(X,\Lambda) \to H^i_{Z'_k}(X' - \bigcup_{i\neq k} Z'_i,\Lambda)$$

is the zero map. But

$$H^i_{Z'}(X',\Lambda) = \bigoplus_k H^i_{Z'_k}(X' - \bigcup_{i\neq k} Z'_i,\Lambda),$$

and the corollary follows. □

Lemma 5.7: *Let (X,Z) be a regular pair of codimension c and set $U = X - Z$. Let $\pi : X \to Y$ be a proper morphism such that Z is flat over Y. Set $\Lambda = \mathbb{Z}/n\mathbb{Z}$ for an integer n prime to the residue characteristics of X. Then for any closed $y \in Y$ and any integer d the base change morphisms*

$$(R^q(\pi_U)_*\Lambda(d))_y \to H^q(U_y,\Lambda(d))$$

are isomorphisms for any $q \geqslant 0$.

Proof: Without loss of generality we may assume Y is the spectrum of a strictly henselian local ring with closed point y. Then, $\mu_n \cong \mathbb{Z}/n\mathbb{Z}$ on X and it suffices to prove the lemma for $d = 0$. We may further assume that Z is reduced. We need to show that

$$H^q(U,\Lambda) \to H^q(U_y,\Lambda)$$

is an isomorphism. Consider the following diagram of excision sequences

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & H^q_Z(X,\Lambda) & \longrightarrow & H^q(X,\Lambda) & \longrightarrow & H^q(U,\Lambda) & \longrightarrow & \dots. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & H^q_{Z_y}(X_y,\Lambda) & \longrightarrow & H^q(X_y,\Lambda) & \longrightarrow & H^q(U_y,\Lambda) & \longrightarrow & \dots
\end{array}$$

The homomorphisms $H^q(X,\Lambda) \to H^q(X_y,\Lambda)$ are isomorphisms due to proper base change. By flatness the morphism $(X_y, Z_y) \to (X,Z)$ is a morphism of regular pairs of codimension c yielding a commutative diagram

$$\begin{array}{ccc}
H^q_Z(X,\Lambda) & \xrightarrow{\sim} & H^{q-2c}(Z,\Lambda(-d)). \\
\downarrow & & \downarrow \\
H^q_{Z_y}(X_y,\Lambda) & \xrightarrow{\sim} & H^{q-2c}(Z_y,\Lambda(-d))
\end{array}$$

The horizontal maps are purity isomorphisms and the vertical map on the right is an isomorphism by proper base change. Hence, the vertical map on the left is an isomorphism and the lemma follows by applying the five lemma to the above diagram of exact sequences. □

5.2 Cohomology and dual graphs

Let X/B be an arithmetic surface and $D \subseteq X$ a tidy divisor In this section we relate the homology of the dual graph of D with the cohomology of X with support in D. Later, we will apply this in the situation where D is an exceptional fibre of a desingularized tame covering.

Proposition 5.8: *Let C be a projective curve over an algebraically closed field k with only ordinary double points and let Γ_C denote its dual graph. Let $S \subset C$ be a finite set of closed points containing the set C_{sing} of singular points of C. Define $C_N := \coprod_i C_i$, where C_i are the normalizations of the irreducible components of C. Set $S_N = S \times_C C_N$. For $n \in \mathbb{N}$ prime to the characteristic of k consider the homomorphisms of cohomology groups with coefficients in $\mathbb{Z}/n\mathbb{Z}$*

$$\begin{tikzcd} H^1(C-S) = H^1(C_N - S_N) \arrow[r, "\alpha"] \arrow[rrr, bend right, "\beta"'] & H^2_{S_N}(C_N) & H^0(S_N)(-1) \arrow[l, "\sim"', "purity"] \arrow[r, "norm"'] & H^0(S)(-1), \end{tikzcd}$$

where α is the connecting homomorphism of the excision sequence associated to (C_N, S_N). Then

$$\frac{ker(\beta)}{ker(\alpha)} \cong H_1(\Gamma_C, \mathbb{Z}/n\mathbb{Z}), \qquad coker(\beta) \cong H_0(\Gamma_C, \mathbb{Z}/n\mathbb{Z}),$$

where $H_i(\Gamma_C, \mathbb{Z}/n\mathbb{Z})$ denotes singular homology with coefficients in $\mathbb{Z}/n\mathbb{Z}$ and such that for each $s \in S$ the canonical map

$$H^0(s)(-1) \to H^0(S)(-1) \to coker(\beta)$$

is identified with the inclusion of the direct summand of $H_0(\Gamma_C, \mathbb{Z}/n\mathbb{Z})$ corresponding to the connected component of Γ_C containing s.

Proof: The group $H_1(\Gamma_C, \mathbb{Z}/n\mathbb{Z})$ can be calculated using a cellular chain complex. The zero-skeleton $(\Gamma_C)_0$ consists of the nodes of the graph which correspond to the irreducible components C_i and the one-skeleton $(\Gamma_C)_1$ is all of Γ_C. Thus, the one-cells are the edges of the graph, which correspond to the singular points in C_{sing}. We give each edge s a direction by choosing an initial node $C_1(s)$ and an end node $C_2(s)$. Then $H_i(\Gamma_C, \mathbb{Z}/n\mathbb{Z})$ is the i^{th} homology of the sequence

$$0 \to H_1((\Gamma_C)_1, (\Gamma_C)_0, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{d} H_0((\Gamma_C)_0, \mathbb{Z}/n\mathbb{Z}) \to 0$$

and the map d can be identified with

$$\begin{aligned} \bigoplus_{s \in C_{sing}} \mathbb{Z}/n\mathbb{Z} \cdot s &\to \bigoplus_i \mathbb{Z}/n\mathbb{Z} \cdot C_i. \\ s &\mapsto C_2(s) - C_1(s) \end{aligned}$$

We conclude that

$$\begin{aligned} H_1(\Gamma_C, \mathbb{Z}/n\mathbb{Z}) &\cong ker(\bigoplus_{s \in C_{sing}} \mathbb{Z}/n\mathbb{Z} \cdot s \to \bigoplus_i \mathbb{Z}/n\mathbb{Z} \cdot C_i) \\ H_0(\Gamma_C, \mathbb{Z}/n\mathbb{Z}) &\cong coker(\bigoplus_{s \in C_{sing}} \mathbb{Z}/n\mathbb{Z} \cdot s \to \bigoplus_i \mathbb{Z}/n\mathbb{Z} \cdot C_i). \end{aligned}$$

Let us first compute $ker(\beta)/ker(\alpha)$.

$$\begin{aligned} \frac{ker(\beta)}{ker(\alpha)} &= ker\left(\frac{H^1(C-S)}{ker(\alpha)} \to H^0(S)(-1)\right) \\ &= ker(Im(\alpha) \to H^0(S)(-1)) \\ &= ker(ker(H^2_{S_N}(C_N) \to H^2(C_N)) \to H^0(S)(-1)) \\ &= ker(H^2_{S_N}(C_N) \to H^2(C_N)) \cap ker(H^2_{S_N}(C_N) \to H^0(S)(-1)). \end{aligned}$$

Via the purity isomorphism $H^0(S_N)(-1) \to H^2_{S_N}(C_N)$, the cohomology group $H^2_{S_N}(C_N)$ is identified with

$$\bigoplus_{s_N \in S_N} \mathbb{Z}/n\mathbb{Z} \cdot s_N$$

by mapping $s_N \in \bigoplus_{s_{N'} \in S_{N'}} \mathbb{Z}/n\mathbb{Z} \cdot s_{N'}$ to the fundamental class s_{S_N/C_N}. Let us first examine $(H^2_{S_N}(C_N) \to H^2(C_N))$. We identify $H^2(C_N)$ with

$$\bigoplus_i \mathbb{Z}/n\mathbb{Z} \cdot C_i$$

via the degree maps $H^2(C_i) \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$ of the components C_i. For each component C_i of C_N and each $s_N \in C_i \cap S_N$ the images of the fundamental classes s_{S_N/C_N} in $H^2(C_i)$ are the same. With these identifications the map $H^2_{S_N}(C_N) \to H^2(C_N)$ is identified with

$$\bigoplus_{s_N \in S_N} \mathbb{Z}/n\mathbb{Z} \cdot s_N \to \bigoplus_i \mathbb{Z}/n\mathbb{Z} \cdot C_i,$$
$$s_N \mapsto C(s_N)$$

where $C(s_N)$ is the component of C_N which contains s_N. Next we consider the map $H^2_{S_N}(C_N) \to H^0(S)(-1)$, which is induced by the norm map $H^0(S_N) \to H^0(S)$. Via the above identifications it takes the form

$$\bigoplus_{s_N \in S_N} \mathbb{Z}/n\mathbb{Z} \cdot s_N \to \bigoplus_{s \in S} \mathbb{Z}/n\mathbb{Z} \cdot s, \tag{5.1}$$
$$s_N \to s(s_N) \tag{5.2}$$

where $s(s_N)$ is the image of s_N in S. The kernel of this map is generated by $(s_N)_2(s) - (s_N)_1(s)$ where $(s_N)_j(s) \in C_j(s)$ are the two preimages in S_N of a point $s \in C_{sing}$. We thus get an isomorphism

$$\bigoplus_{s \in C_{sing}} \mathbb{Z}/n\mathbb{Z} \cdot s \to ker(\bigoplus_{s_N \in S_N} \mathbb{Z}/n\mathbb{Z} \cdot s_N \to \bigoplus_{s \in S} \mathbb{Z}/n\mathbb{Z} \cdot s).$$
$$s \mapsto (s_N)_2(s) - (s_N)_1(s)$$

Therefore, $ker(\beta)/ker(\alpha)$ is isomorphic to the kernel of the composition

$$\bigoplus_{s \in C_{sing}} \mathbb{Z}/n\mathbb{Z} \cdot s \to \bigoplus_{s_N \in S_N} \mathbb{Z}/n\mathbb{Z} \cdot s_N \to \bigoplus_i \mathbb{Z}/n\mathbb{Z} \cdot C_i,$$

which maps $s \in C_{sing}$ to $C_2(s) - C_1(s)$. Comparing with the calculation of $H_1(\Gamma_C, \mathbb{Z}/n\mathbb{Z})$ at the beginning of the proof we see that

$$\frac{ker(\beta)}{ker(\alpha)} \cong H_1(\Gamma_C, \mathbb{Z}/n\mathbb{Z}).$$

Next we compute $coker(\beta)$. In the above notation the image of α is given by

$$\{\sum a_{s_N} s_N \in \bigoplus_{s_N \in S_N} \mathbb{Z}/n\mathbb{Z} \cdot s_N | \sum_{s_N \in C_i} a_{s_N} = 0 \ \forall i\}.$$

It is generated by all elements of the form $(s_N)_1 - (s_N)_2$ with $(s_N)_1$ and $(s_N)_2$ lying on the same component of C_N. The image of β is the image of this set under the map (5.1), i. e., it is the subgroup of $\bigoplus_{s \in S} \mathbb{Z}/n\mathbb{Z} \cdot s$ generated by $s_1 - s_2$ with s_1 and s_2 on the same irreducible component of C. This subgroup coincides with the subgroup generated by $s_1 - s_2$ with s_1 and s_2 on the same *connected* component of C, which equals

$$\{\sum a_s s \in \bigoplus_{s \in S} \mathbb{Z}/n\mathbb{Z} \cdot s | \sum_{s \in Z} a_s = 0 \ \forall Z \subseteq C \text{ connected component}\}.$$

Hence, $coker(\beta)$ is the direct sum of a copy of $\mathbb{Z}/n\mathbb{Z}$ for each connected component and the result follows. $\square$

Proposition 5.9: *Let X/B be an arithmetic surface and $D \subset X$ an snc-divisor. Let $S \subset D$ be a set of closed points containing the set D_{sing} of singular points of D. Denote by D_N the normalization of D and set $S_N = S \times_D D_N$. Then the following diagram of cohomology groups with coefficients in $\Lambda = \mathbb{Z}/n\mathbb{Z}$ (n prime to the residue characteristics of X) commutes*

$$\begin{tikzcd}
H^3_{D-S}(X-S,\Lambda) \arrow[r] \arrow[rr, bend left, "\delta"] & H^3(X-S,\Lambda) \arrow[r, "\delta"] & H^4_S(X,\Lambda) \\
H^1(D-S,\Lambda(-1)) \arrow[u, "purity", "\sim"'] & & H^0(S,\Lambda(-2)) \arrow[u, "purity", "\sim"'] \\
H^1(D_N-S_N,\Lambda(-1)) \arrow[u, equal] \arrow[r, "\delta"] & H^2_{S_N}(D_N,\Lambda(-1)) & H^0(S_N,\Lambda(-2)). \arrow[l, "\sim"', "purity"] \arrow[u, "norm"]
\end{tikzcd}$$

All maps δ denote connecting homomorphisms of excision sequences.

Proof: Denote by D_i, $i = 1, \dots r$ the irreducible components of D. Since

$$H^3_{D-S}(X-S,\Lambda) = \bigoplus_i H^3_{D_i-S}(X-S,\Lambda),$$

it suffices to prove the proposition for each component D_i separately. We may thus assume without loss of generality that D is a regular irreducible curve. In this case the above diagram reduces to

$$\begin{tikzcd}
H^3_{D-S}(X-S,\Lambda) \arrow[r] \arrow[rr, bend left, "\delta"] & H^3(X-S,\Lambda) \arrow[r, "\delta"] & H^4_S(X,\Lambda) \\
H^1(D-S,\Lambda(-1)) \arrow[u, "purity", "\sim"'] \arrow[r, "\delta"] & H^2_S(D,\Lambda(-1)) & H^0(S,\Lambda(-2)). \arrow[l, "\sim"', "purity"] \arrow[u, "purity", "\sim"']
\end{tikzcd}$$

Consider the commutative diagram

$$\begin{tikzcd}
H^3_{D-S}(X-S,\Lambda) \arrow[r, "\delta"] & H^4_S(X,\Lambda) \\
H^1(D-S,\underline{H}^2_{D-S}(X-S,\Lambda)) \arrow[u, "\sim"] \arrow[r, "\delta"] \arrow[d, "\sim"'] & H^2_S(D,\underline{H}^2_D(X,\Lambda)) \arrow[u, "\sim"] \arrow[d, "\sim"'] \\
H^1(D-S,\Lambda(-1)) \otimes H^2_{D-S}(X-S,\Lambda(1)) \arrow[r, "\delta\otimes res^{-1}"] & H^2_S(D,\Lambda(-1)) \otimes H^2_D(X,\Lambda(1)) \\
H^1(D-S,\Lambda(-1)) \arrow[u, "\sim", "\otimes s_{D-S/X-S}"'] \arrow[r, "\delta"] & H^2_S(D,\Lambda(-1)). \arrow[u, "\sim", "\otimes s_{D/X}"']
\end{tikzcd}$$

The restriction

$$res : H^2_D(X,\Lambda(1)) \to H^2_{D-S}(X-S,\Lambda(1))$$

is an isomorphism which maps the fundamental class $s_{D/X}$ to $s_{D-S/X-S}$. For this reason, the homomorphism $\delta \otimes res^{-1}$ in the third line of the diagram is well defined and the lowermost square commutes. Commutativity of the middle square follows because $\underline{H}_D(X)$ is a free sheaf which restricts to $\underline{H}_{D-S}(X-S)$ on $D-S$. The upper square commutes due to compatibility of the spectral sequences

$$
\begin{aligned}
H^i_S(D, \underline{H}^j_D(X,\Lambda)) &\Rightarrow H^{i+j}_S(X,\Lambda),\\
H^i(D-S, \underline{H}^j_{D-S}(X-S,\Lambda)) &\Rightarrow H^{i+j}_{D-S}(X-S,\Lambda).
\end{aligned}
$$

Furthermore, by [Fuj], Proposition 1.2.1 the following diagram commutes

$$
\begin{tikzcd}
H^4_S(X,\Lambda) & \\
H^2_S(D, \underline{H}^2_D(X,\Lambda)) \arrow[u, "\sim"] \arrow[d, "\sim"'] & \\
H^2_S(D,\Lambda(-1)) \otimes H^2_D(X,\Lambda(1)) & \\
H^2_S(D,\Lambda(-1)) \arrow[u, "\sim", "\otimes s_{D/X}"'] & H^0(S,\Lambda(-2)). \arrow[l, "\sim", "\text{purity}"'] \arrow[uuul, "\text{purity}", "\sim"']
\end{tikzcd}
$$

Putting the two diagrams together, the assertion of the proposition follows. □

5.3 Killing cohomology with support

For this section fix a proper arithmetic surface $\bar{X}/B$ with geometric point $\bar{x} \to \bar{X}$ lying over a closed point $x \in \bar{X}$. We assume that the residue fields of B are either finite or algebraic closures of finite fields. Let $\bar{D} \subseteq \bar{X}$ be a tidy divisor whose support does not contain x. Let $\bar{D}_h$ be the maximal subdivisor of $\bar{D}$ with support on the isolated horizontal components of $\bar{D}$, i. e., on the horizontal components which do not intersect any other component. Set $X = \bar{X} - \bar{D}_h$ and $U = \bar{X} - \bar{D}$ and denote by $D \subseteq X$ the restriction of $\bar{D}$ to X. We write D_v for the maximal vertical subdivisor of D and D_h for the maximal horizontal subdivisor, such that $D = D_v + D_h$. Notice that by construction D_v is also the maximal vertical subdivisor of $\bar{D}$ and the maximal horizontal subdivisor of $\bar{D}$ is given by $\bar{D}_h + D_h$. Let W denote the union of all vertical prime divisors which are contained in a singular fibre of $\bar{X} \to B$ but not contained in $\bar{D}$. Put differently, W is the Zariski closure of the union of all reduced fibres $(U_b)_{red}$ such that $\bar{X}_b$ is singular. Denote by S the finite set of special points of $\bar{D}$, i. e., the set of singular points of $\bar{D}_{red}$.

Furthermore, we fix a full class of finite groups $\mathfrak{c}$ such that all elements of $\mathbb{N}(\mathfrak{c})$ are invertible on $\bar{X}$ and for all prime numbers $l \in \mathbb{N}(\mathfrak{c})$ we have $\mu_l \cong \mathbb{Z}/l\mathbb{Z}$ on X. We choose an integer $n \in \mathbb{N}(\mathfrak{c})$ and set $\Lambda = \mathbb{Z}/n\mathbb{Z}$.

We denote by $\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}$ the category of all pointed desingularized $\mathfrak{c}$-coverings of $(\bar{X},\bar{D})$ as defined in Section 2.3. Viewing $\bar{x}$ as geometric point of B we write $\mathfrak{I}_{B,\bar{x}}$ for the category of pointed finite étale $\mathfrak{c}$-coverings of B. By

$$(B' \to B) \mapsto ((\bar{X} \times_B B', \bar{D} \times_B B') \to (\bar{X},\bar{D}))$$

$\mathfrak{I}_{B,\bar{x}}$ becomes a subcategory of $\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}$.

For $(\bar{X}',\bar{D}') \to (\bar{X},\bar{D})$ in $\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}$ let

$$\bar{X}' \to B' \to B$$

be the Stein factorization of $\bar{X}' \to \bar{X} \to B$. Then $\bar{X}'$ is an arithmetic surface over B'. We use analogous notation for $(\bar{X}',\bar{D}')$ as for $(\bar{X},\bar{D})$: We write U' for $\bar{X}' - \bar{D}$, $\bar{D}'_h$ for the maximal subdivisor of $\bar{D}'$ with support on the isolated horizontal components of $\bar{D}$ and so on. Moreover, we write E' for the generalized exceptional divisor of $(\bar{X}',\bar{D}') \to (\bar{X},\bar{D})$.

By Lemma 4.2 the preimage of $\bar{D}_h$ under a desingularized $\mathfrak{c}$-covering $(\bar{X}', \bar{D}') \to (\bar{X}, \bar{D})$ is $\bar{D}'_h$ and thus the preimage of X is X'. Furthermore, the preimage of D_v is D'_v. Note that the preimage of D_h is the sum of D'_h and a divisor with support in E'.

We want to investigate whether U has the $K(\pi, 1)$ property with respect to $\mathfrak{c}$, i. e., whether

$$\varinjlim_{(\bar{X}', \bar{D}') \in \mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}} H^i(U', \Lambda) = 0.$$

for all $\Lambda = \mathbb{Z}/n\mathbb{Z}$ with $n \in \mathbb{N}(\mathfrak{c})$. In this section we show that

$$\varinjlim_{\mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}} H^i(X', \Lambda) \to \varinjlim_{\mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}} H^i(U', \Lambda)$$

is surjective for $i \geqslant 2$. To this end we examine the vanishing of the cohomology groups with support $H^i_D(X, \Lambda)$ in the limit over $\mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}$.

Lemma 5.10: *Let $\mathfrak{c}$ be a full class of finite groups and $M \in \mathfrak{c}$ a finite abelian group with an operation of $\hat{\mathbb{Z}}$. Then $H^q(\hat{\mathbb{Z}}, M) = 0$ for $q \geqslant 2$ and there is an open subgroup $H \subseteq \hat{\mathbb{Z}}$ of index in $\mathbb{N}(\mathfrak{c})$ such that the restriction map*

$$H^1(\hat{\mathbb{Z}}, M) \to H^1(H, M)$$

is the zero map.

Proof: By the example after Corollary 3.3.4 in [NSW] the cohomological dimension of $\hat{\mathbb{Z}}$ is one. In order to show the statement concerning the first cohomology group, we write $\hat{\mathbb{Z}} = G_1 \times G_2$ with

$$G_1 = \prod_{l \in \mathbb{N}(\mathfrak{c})} \mathbb{Z}_l \qquad \text{and} \qquad G_2 = \prod_{l \notin \mathbb{N}(\mathfrak{c})} \mathbb{Z}_l.$$

By [NSW], Proposition 1.6.2 the cohomology groups $H^i(G_2, M)$ vanish for $i \geqslant 1$ and thus by the Hochschild-Serre spectral sequence associated to $G_2 \subseteq \hat{\mathbb{Z}}$ we have

$$H^1(\hat{\mathbb{Z}}, M) = H^1(G_1, M^{G_2}).$$

There is an open subgroup H_1 of G_1 such that the restriction map $H^1(G_1, M^{G_2}) \to H^1(H_1, M^{G_2})$ is the zero map and the lemma follows by setting $H = H_1 \times G_2$. □

Remark 5.11: *The statement about the first cohomology of $\hat{\mathbb{Z}}$ is a consequence of the fact that $\hat{\mathbb{Z}}$ is a good group with respect to $\mathfrak{c}$. Here, a profinite group G is said to be good with respect to a full class of finite groups $\mathfrak{c}$ if for all G-modules $M \in \mathfrak{c}$ and all $q \geqslant 0$ the inflation*

$$H^q(G(\mathfrak{c}), M^{ker(G \to G(\mathfrak{c}))}) \to H^q(G, M)$$

is an isomorphism.

Corollary 5.12: *Let B be a Dedekind scheme with finite residue fields at closed points, $\bar{b} \to B$ a geometric point above a closed point b and $\mathfrak{c}$ a full class of finite groups. Suppose that*

$$\pi_1(b, \bar{b})(\mathfrak{c}) \to \pi_1(B, \bar{b})(\mathfrak{c})$$

is injective. Then for any finite $Gal_{k(b)}$-module M in $\mathfrak{c}$ the cohomology groups $H^q(k(b), M)$ vanish for $q \geqslant 2$ and there is a finite étale $\mathfrak{c}$-covering $B' \to B$ such that

$$H^1(b, M) \to H^1(b \times_B B', M)$$

is the zero map.

Proof: Since $\mathrm{Gal}_{k(b)} \cong \hat{\mathbb{Z}}$, the corollary follows from Lemma 5.10 noting that by assumption any finite Galois $\mathfrak{c}$-extension $k'|k(b)$ is globally realized by a finite Galois covering $B' \to B$. $\square$

Proposition 5.13: *Let $Z \leqslant D_v$ be a rational subdivisor (i. e., Z is a rational curve) and let V be an open subscheme of X. Suppose that for every geometric point $\bar{b}$ above a closed point $b \in B$ the natural map*

$$\pi_1(b, \bar{b})(\mathfrak{c}) \to \pi_1(B, \bar{b})(\mathfrak{c})$$

is injective. Then the cokernel of

$$H^2(V - S, \Lambda) \to H^2(V - Z, \Lambda)$$

vanishes in the limit over $\mathfrak{I}_{B,\bar{x}}$ for all $m \in \mathbb{N}(\mathfrak{c})$. (Remember that S is the set of special points of $\bar{D}$.)

Proof: Without loss of generality we may assume that $V \cap Z$ is dense in Z. Otherwise, Z has irreducible components in the complement of V, which we can remove without changing the above cohomology groups. Denote by T the union of S with the finite set of closed points $Z - V$. Then $V - S = V - T$. By proposition 5.9 we have the following commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & H^2(V - Z, \Lambda) \arrow[r] & H^3_{Z-T}(V - T, \Lambda) \arrow[r] & H^3(V - T, \Lambda) \arrow[r] \arrow[d] & \dots \\
 & & H^1(Z - T, \Lambda(-1)) \arrow[u, "\sim"] \arrow[r, "\beta(-1)"] & H^0(T, \Lambda(-2)), &
\end{tikzcd}$$

where $\beta(-1)$ is the (-1)-twist of the map β defined in proposition 5.8. It thus suffices to show that the kernel of β vanishes in the limit over $\mathfrak{I}_{B,\bar{x}}$. Without loss of generality we may assume that Z is contained in a single closed fibre of $X \to B$ over some point $b \in B$ with residue field $k(b)$. Let $\overline{k(b)}$ be an algebraic closure of $k(b)$ and denote by $\bar{Z}$ and $\bar{T}$ the base change of Z and T, respectively, to $\overline{k(b)}$. Moreover, write Z_N for the normalization of Z and $\bar{Z}_N$ for its base change to $\overline{k(b)}$. Consider the diagram of cohomology groups with coefficients in Λ

$$\begin{tikzcd}
 & 0 & 0 & \\
0 \arrow[r] & H^1(\bar{Z}_N)^{G_{k(b)}} \arrow[u] \arrow[r] & H^1(\bar{Z} - \bar{T})^{G_{k(b)}} \arrow[u] \arrow[r, "\bar{\beta}"] & H^0(\bar{T})(-1)^{G_{k(b)}} \\
0 \arrow[r] & H^1(Z_N) \arrow[u] \arrow[r] & H^1(Z - T) \arrow[u] \arrow[r, "\beta"] & H^0(T)(-1), \arrow[u] \\
 & H^1(k(b))^d \arrow[u] \arrow[r, "="] & H^1(k(b))^d \arrow[u] & \\
 & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where d is the number of components of Z_N. The vertical sequences are exact sequences induced by the Hochschild-Serre spectral sequences

$$H^i(k(b), H^j(\bar{Z}_N), \Lambda) \Rightarrow H^{i+j}(Z_N, \Lambda),$$
$$H^i(k(b), H^j(\bar{Z}_N - \bar{T}_N, \Lambda)) \Rightarrow H^{i+j}(Z_N - T_N, \Lambda).$$

The upper horizontal sequence is exact by the following reason: According to proposition 5.8, the first homology group $H_1(\Gamma_Z, \mathbb{Z}/n\mathbb{Z})$ of the dual graph Γ_Z of $\bar{Z}$ is isomorphic to $ker(\bar{\beta})/ker(\bar{\alpha})$, where $\bar{\alpha}$ denotes the connecting homomorphism of the excision sequence associated to $\bar{T}_N \hookrightarrow \bar{Z}_N$. Since we assumed Z to be rational, Γ_Z is a tree, and thus its first homology group vanishes. It follows that the kernel of $\bar{\beta}$ equals the image of the map

$$\gamma : H^1(\bar{Z}_N, \Lambda) \hookrightarrow H^1(\bar{Z}_N - \bar{T}_N, \Lambda) = H^1(\bar{Z} - \bar{T}, \Lambda).$$

Taking $G_{k(b)}$-invariants we obtain the upper sequence of the above diagram which is therefore exact. A diagram chase now shows the exactness of the lower horizontal sequence.

Again by the rationality assumption on Z, the cohomology group $H^1(\bar{Z}_N)$ vanishes. The above diagram shows that the kernel of β equals $H^1(k(b))^d$. By Corollary 5.12 this group vanishes in the limit over $\mathfrak{I}_{B,\bar{x}}$. $\square$

Definition 5.14: *Let Y/B be an arithmetic surface and $Z \subset Y$ a tidy divisor. We say that (Y, Z) has enough tame coverings at a closed point p of Z if for every irreducible component C of Z passing through p there is $f \in K(Y)^\times$ with support in Z such that $deg_C(f) > 0$ and $deg_W(f) = 0$ for any other prime divisor W passing through p. We say that (Y, Z) has enough tame coverings if it has enough tame coverings at every closed point of Z.*

The following lemma sheds some light on this definition.

Lemma 5.15: *Assume that $(\bar{X}, \bar{D})$ has enough tame coverings. Let $\bar{y}$ be a geometric point of $\bar{X}$. Denote by $\bar{X}^{sh}_{\bar{y}}$ the strict henselization of $\bar{X}$ at $\bar{y}$ and by $\bar{D}^{sh}_{\bar{y}}$ the restriction of $\bar{D}$ to $\bar{X}^{sh}_{\bar{y}}$. Let $(\bar{X}'_0, \bar{D}'_0) \to (\bar{X}^{sh}_{\bar{y}}, \bar{D}^{sh}_{\bar{y}})$ be a tame covering such that $\bar{X}'_0$ is local. Then there is a tame covering $(\bar{X}', \bar{D}') \to (\bar{X}, \bar{D})$ whose strict henselization at any lift $\bar{y}'$ of $\bar{y}$ coincides with $(\bar{X}'_0, \bar{D}'_0) \to (\bar{X}^{sh}_{\bar{y}}, \bar{D}^{sh}_{\bar{y}})$.*

Proof: If $\bar{y}$ does not lie above a point of $\bar{D}$, the restriction $\bar{D}^{sh}_{\bar{y}}$ is empty. In this case there are no nontrivial tame coverings of $(\bar{X}^{sh}_{\bar{y}}, \bar{D}^{sh}_{\bar{y}})$ and we can take $\bar{X}' \to \bar{X}$ to be the identity. Suppose now that $\bar{y}$ lies above a singular point y of $\bar{D}$, i. e., there are two irreducible components D_1 and D_2 of D intersecting each other at y. Since $(\bar{X}, \bar{D})$ has enough tame coverings, there are f_1 and f_2 in $K(\bar{X})^\times$ with support in D such that

$$\begin{aligned} &deg_{D_1}(f_1) = m_1 > 0, \qquad && deg_{D_2}(f_2) = 0, \\ &deg_{D_1}(f_2) = 0, && deg_{D_2}(f_2) = m_2 > 0. \end{aligned}$$

Let h_1 and h_2 be functions in $K(\bar{X}^{sh}_{\bar{y}})^\times$ such that $div\ h_i = (D_i)_{\bar{X}^{sh}_{\bar{y}}}$ for $i = 1, 2$ on $\bar{X}^{sh}_{\bar{y}}$. Then on $\bar{X}^{sh}_{\bar{y}}$ we have

$$div\ f_i = m_i \cdot div\ h_i.$$

By Corollary 4.5 the strictly henselian arithmetic surface $\bar{X}'_0$ is the normalization of $\bar{X}^{sh}_{\bar{y}}$ in a function field extension of the form

$$K(\bar{X}^{sh}_{\bar{y}})[\sqrt[n]{h_1},\ \sqrt[m]{h_1^r h_2^s}]|K(\bar{X}^{sh}_{\bar{y}})$$

with m, n prime to the residue characteristic, $0 \leqslant r, s \leqslant m-1$ and $gcd(r, s, m) = 1$. For $i = 1, 2$ write

$$a_i m_i + b_i m = g_i = gcd(m_i, m)$$

with integers a_i, b_i and similarly,

$$am_1 + bn = g = gcd(m_1, n)$$

with integers a, b. Note that in particular, g_i and g are prime to the residue characteristic of $\bar{y}$. Then, on $\bar{X}^{sh}_{\bar{y}}$ we have the equalities

$$\begin{aligned} g_i \cdot div\ h_i &= a_i \cdot div\ f_i + b_i m \cdot div\ h_i. \\ g \cdot div\ h_1 &= a \cdot div\ f_1 + bn \cdot div\ h_1 \end{aligned}$$

Therefore,

$$K(\bar{X}^{sh}_{\bar{y}})[\sqrt[n]{h_1},\ \sqrt[m]{h_1^r h_2^s}] = K(\bar{X}^{sh}_{\bar{y}})[\sqrt[ng]{f_1^a},\ \sqrt[mg_1g_2]{f_1^{rg_2a_1} f_2^{sg_1a_2}}].$$

Hence, we can take $\bar{X}'$ to be the normalization of $\bar{X}$ in the function field extension

$$K(\bar{X})[\sqrt[ng]{f_1^a}, \sqrt[mg_1g_2]{f_1^{rg_2a_1} f_2^{sg_1a_2}}]|K(\bar{X}).$$

□

Lemma 5.16: *Let Y be a regular, noetherian scheme such that all elements in $\mathbb{N}(\mathfrak{c})$ are invertible on Y. Choose $\alpha \in K(Y)$ and set $Z = supp\ \alpha$. For an integer $d \in \mathbb{N}(\mathfrak{c})$ consider the extension of function fields $L_d = K(Y)(\sqrt[d]{\alpha}, \mu_d)|K(Y)$. Let Y_d denote the normalization of Y in L_d and define Z_d to be the preimage of Z in Y_d. Then, for any $n \in \mathbb{N}(\mathfrak{c})$, there is $M \in \mathbb{N}(\mathfrak{c})$ such that for all $d \in \mathbb{N}(\mathfrak{c})$ with $M|d$ the ramification index of each irreducible component of Z_d is divisible by n.*

Proof: The morphism $Y_d \to Y$ is tamely ramified as d is prime to the residue characteristics of Y. It is at most ramified in Z. Write $div\ \alpha = \sum_i a_i Z_i$ with $a_i \neq 0$. Then $Z = \bigcup_i Z_i$ and the ramification index in Z_i is

$$e_i = \frac{d}{gcd(a_i, d)}.$$

We set $M' = n \cdot \prod_i a_i$ and define M to be the maximal factor of M' lying in $\mathbb{N}(\mathfrak{c})$. If $M|d$, we claim that $n|e_i$ for all i. It suffices to check this for $d = M$ because the ramification indices for $M|d$ are multiples of the ramification indices for $d = M$. Writing a_i' for the maximal factor of a_i contained in $\mathbb{N}(\mathfrak{c})$ we have $M = n \cdot \prod_i a_i'$ and

$$e_i = \frac{M}{gcd(a_i, M)} = n \cdot \prod_{j \neq i} a_i',$$

which is divisible by n. □

The lemma shows that if (Y, Z) has enough tame coverings, for all $n \in \mathbb{N}(\mathfrak{c})$ we can find a tame $\mathfrak{c}$-covering of (Y, Z) such that n divides all ramification indices.

Proposition 5.17: *Suppose that $(\bar{X}, \bar{D})$ has enough tame coverings. Suppose that for every geometric point $\bar{b}$ above a closed point $b \in B$ the natural map*

$$\pi_1(b, \bar{b})(\mathfrak{c}) \to \pi_1(B, \bar{b})(\mathfrak{c})$$

is injective. Then the cokernel of the restriction

$$H^2(X, \Lambda) \to H^2(U, \Lambda)$$

vanishes in the limit over $\mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}$.

Proof: Since $(\bar{X}, \bar{D})$ has enough tame coverings, we can use Lemma 5.16 in order to find a desingularized $\mathfrak{c}$-covering

$$(X', D') \to (X_1, D_1) \to (X, D),$$

such that m divides the ramification index of each irreducible component of D_1. We have the following cartesian diagram

$$\begin{tikzcd}
X' - D' = U' \arrow[r, hook] \arrow[d] & X' \arrow[d] & D' \arrow[l, hook'] \arrow[d] & S' \cup E' \arrow[l, hook'] \arrow[d] \\
X_1 - D_1 = U_1 \arrow[r, hook] \arrow[d] & X_1 \arrow[d] & D_1 \arrow[l, hook'] \arrow[d] & S_1 \arrow[l, hook'] \arrow[d] \\
X - D = U \arrow[r, hook] & X & D \arrow[l, hook'] & S. \arrow[l, hook']
\end{tikzcd}$$

It induces the following commutative diagram of excision sequences with coefficients in $\Lambda = \mathbb{Z}/n\mathbb{Z}$:

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & H^2(X-S) & \longrightarrow & H^2(U) & \longrightarrow & H^3_{D-S}(X-S) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & H^2(X'-(S'\cup E')) & \longrightarrow & H^2(U') & \longrightarrow & H^3_{D'-S'}(X'-(S'\cup E')) & \longrightarrow & \dots .
\end{array}$$

Let ϕ be an element of $H^2(U,\Lambda)$ and let ϕ' be its image in $H^2(U',\Lambda)$. By proposition 5.6 ϕ' is mapped to zero in $H^3_{Z'-S'}(X'-(S'\cup E'),\Lambda)$. Hence, there is $\phi'_1 \in H^2(X'-(S'\cup E'),\Lambda)$ mapping to ϕ'. Since E' is rational by Proposition 4.19, we can apply Proposition 5.13 with $V = X'-S'$ and $Z = E'$ to obtain a finite étale $\mathfrak{c}$-covering $B'' \to B'$ and thus via base change a finite étale $\mathfrak{c}$-covering $\bar{X}'' \to \bar{X}'$ such that the image of ϕ'_1 in $H^2(X''-(S''\cup E'),\Lambda)$ lies in the image of

$$H^2(X''-S'',\Lambda) \to H^2(X''-S''\cup E'',\Lambda).$$

and thus can be lifted to an element $\phi''_2 \in H^2(X''-S'',\Lambda)$. Taking into account that $H^3_{S''}(X'')$ vanishes by purity (see 5.3), the excision sequence associated to (X'',S'') shows that the restriction map

$$H^2(X'',\Lambda) \to H^2(X''-S'',\Lambda)$$

is surjective. Hence, ϕ''_2 lifts to $H^2(X'',\Lambda)$. We have thus constructed a lift to $H^2(X'',\Lambda)$ of the image of ϕ in $H^2(U'',\Lambda)$. □

Lemma 5.18: *Let C be an integral projective curve over an algebraically closed field k. Let $f : C' \to C$ be a (possibly ramified) covering of degree d. Then for any integer n prime to the residue characteristic of k the following diagram commutes.*

$$\begin{array}{ccc}
H^2(C,\mu_n) & \longrightarrow & H^2(C',\mu_n) \\
{\scriptstyle tr}\downarrow{\scriptstyle \sim} & & {\scriptstyle tr}\downarrow{\scriptstyle \sim} \\
\mathbb{Z}/n\mathbb{Z} & \xrightarrow{\cdot d} & \mathbb{Z}/n\mathbb{Z}.
\end{array}$$

Proof: The trace map is induced by the degree map

$$Pic(C) \to \mathbb{Z}$$

via the surjection

$$Pic(C) \cong H^1(C,\mathcal{O}^\times) \longrightarrow H^2(C,\mu_n).$$

For a Cartier divisor D on C representing a class in $Pic(C)$ we have

$$deg(f^*D) = d \cdot deg(D).$$

This accounts for the factor d occurring in the statement of the lemma. □

Proposition 5.19: *Suppose that the following conditions are satisfied:*

(i) $(\bar{X},\bar{D})$ has enough tame coverings.

(ii) Every connected component of D has at least one horizontal component.

(iii) For every geometric point $\bar{b}$ above a closed point $b \in B$ the natural map

$$\pi_1(b,\bar{b})(\mathfrak{c}) \to \pi_1(B,\bar{b})(\mathfrak{c})$$

is injective.

Then the cokernel of

$$H^3(X,\Lambda) \to H^3(U,\Lambda)$$

vanishes in the limit over $\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}$.

Proof: Since $(\bar{X},\bar{D})$ has enough tame coverings, we can use Lemma 5.16 to find a desingularized $\mathfrak{c}$-covering $(\bar{X}',\bar{D}') \to (\bar{X},\bar{D})$ such that m divides the ramification indices of all irreducible components of D_h. Denoting by $E' \subseteq X$ the union of all generalized exceptional fibres of $\bar{X}' \to \bar{X}$, we obtain the following diagram of excision sequences with coefficients in Λ

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & H^3(X-D_v) & \longrightarrow & H^3(U) & \longrightarrow & H^4_{D_h-S}(X-D_v) & \longrightarrow & \ldots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \ldots & \longrightarrow & H^3(X'-D'_v) & \longrightarrow & H^3(U') & \longrightarrow & H^4_{D'_h-S'}(X'-D'_v) & \longrightarrow & \ldots. \end{array}$$

Let ϕ be an element of $H^3(U,\Lambda)$ and denote by ϕ' its image in $H^3(U',\Lambda)$. By Corollary 5.6 ϕ' maps to 0 in $H^4_{D'_h-S'}(X'-D'_v,\Lambda)$ and thus can be lifted to an element $\phi'_1 \in H^3(X'-D'_v,\Lambda)$. Since $\bar{X}'$ satisfies the same conditions as $\bar{X}$ (see Lemma 7.2), we can replace $\bar{X}$ with $\bar{X}'$ and assume ϕ lifts to $\phi_1 \in H^3(X-D_v,\Lambda)$.

Next consider the excision sequence

$$\ldots \to H^3(X-S,\Lambda) \to H^3(X-D_v,\Lambda) \to H^4_{D_v-S}(X-S,\Lambda) \to \ldots.$$

By purity we have

$$H^4_{D_v-S}(X-S,\Lambda) \cong H^2(D_v-S,\Lambda(-1)).$$

For each component Z_i of D_v lying over a closed point $b_i \in B$ with geometric point $\bar{b}_i$ consider the Hochschild-Serre spectral sequence

$$H^r(b_i, H^s(Z_{i,\bar{b}_i} - S_{\bar{b}_i},\Lambda)) \Rightarrow H^{r+s}(Z_i - S,\Lambda).$$

Since $H^j(Z_{i,\bar{b}_i} - S_{\bar{b}_i},\Lambda) = 0$ for $j \geqslant 2$ as $Z_{i,\bar{b}_i} - S_{\bar{b}_i}$ is an affine curve over an algebraically closed field, we conclude that

$$H^1(b_i, H^1(Z_{i,\bar{b}_i} - S_{\bar{b}_i},\Lambda)) \cong H^2(Z_i - S,\Lambda).$$

By Corollary 5.12 there is a finite étale $\mathfrak{c}$-covering $B'_i \to B$ and thus via base change a finite étale $\mathfrak{c}$-covering $\bar{X}'_i \to \bar{X}$ such that

$$H^1(b_i, H^1(Z_{i,\bar{b}_i} - S_{\bar{b}_i},\Lambda)) \to H^1(b_i \times_B B'_i, H^1(Z_{i,\bar{b}_i} - S_{\bar{b}_i},\Lambda))$$

is the zero map. Let B' be the compositum of all extensions B'_i. By compatibility with the Hochschild-Serre spectral sequence and the purity isomorphism we conclude that ϕ_1 maps to 0 in $H^4_{D'_v-S'}(X'-S',\Lambda)$. As before we replace $\bar{X}$ by $\bar{X}'$ and may assume that ϕ_1 maps to 0 in $H^4_{D_v-S'}(X-S,\Lambda)$. Hence, ϕ_1 lifts to $\phi_2 \in H^3(X-S,\Lambda)$.

Now consider the following excision sequence:

$$\ldots \longrightarrow H^3(X,\Lambda) \longrightarrow H^3(X-S,\Lambda) \longrightarrow H^4_S(X,\Lambda) \longrightarrow \ldots.$$

The cohomology group $H^4_S(X,\Lambda)$ is the direct sum over the finitely many elements $s \in S$ (which are closed points of X) of the cohomology groups $H^4_s(X,\Lambda)$. For $s \in S$ choose an irreducible component D_s of D passing through s. Since $(\bar{X},\bar{D})$ has enough tame coverings, we find a desingularized $\mathfrak{c}$-covering $(\bar{X}',\bar{D}') \to (\bar{X}_1,\bar{D}_1) \to (\bar{X},\bar{D})$ such that m divides the ramification indices of all irreducible components of $\bar{D}_1$ lying over D_s and is unramified in all other prime divisors passing through s. By Lemma 4.2 the scheme $\bar{X}_1$ is regular at all preimage points $s'_1,\ldots,s'_r$

of s. Hence, we may assume that $\bar{X}' \to \bar{X}_1$ is an isomorphism in a neighborhood of $s'_1, \ldots, s'_r$. Therefore, by Proposition 5.5, the homomorphism

$$H^4_s(X, \Lambda) \to \bigoplus_i H^4_{s'_i}(X', \Lambda)$$

is the zero map. Take a desingularized $\mathfrak{c}$-covering $(\bar{X}'', \bar{D}'') \to (\bar{X}, \bar{D})$ dominating the coverings $(\bar{X}', \bar{D}') \to (\bar{X}, \bar{D})$ constructed for each $s \in S$. We obtain a diagram of excision sequences

$$\begin{tikzcd}
\ldots \arrow[r] & H^3(X, \Lambda) \arrow[r] \arrow[d] & H^3(X - S, \Lambda) \arrow[r] \arrow[d] & H^4_S(X, \Lambda) \arrow[r] \arrow[d] & \ldots \\
\ldots \arrow[r] & H^3(X'', \Lambda) \arrow[r] & H^3(X'' - (S'' \cup E''), \Lambda) \arrow[r] & H^4_{S'' \cup E''}(X'', \Lambda) \arrow[r] & \ldots,
\end{tikzcd}$$

where E'' denotes the union of all generalized exceptional fibres of $\bar{X}'' \to \bar{X}$. The homomorphism

$$H^4_S(X, \Lambda) \to H^4_{S'' \cup E''}(X'', \Lambda)$$

is the zero map and thus $\phi_2 \in H^3(X - S, \Lambda)$ lifts to $H^3(X'', \Lambda)$. In total we obtain a lift of $\phi \in H^3(U, \Lambda)$ to $H^3(X'', \Lambda)$ and we are done. □

Proposition 5.20: *Assume that for every geometric point $\bar{b}$ above a closed point $b \in B$ the natural map*

$$\pi_1(b, \bar{b})(\mathfrak{c}) \to \pi_1(B, \bar{b})(\mathfrak{c})$$

is injective and that $(\bar{X}, \bar{D})$ has enough tame coverings. Then for $q \geqslant 4$ the cokernel of

$$H^q(X, \Lambda) \to H^q(U, \Lambda)$$

vanishes in the limit over $\mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}$.

Proof: Let $q \geqslant 4$ and take an element ϕ of $H^q(U, \Lambda)$. By the same reasoning as in the beginning of the proof of 5.19, we may assume that ϕ lifts to $\phi_1 \in H^q(X - D_v, \Lambda)$. Consider the excision sequence

$$\ldots \longrightarrow H^q(X - S, \Lambda) \longrightarrow H^q(X - D_v, \Lambda) \longrightarrow H^{q+1}_{D_v - S}(X - S, \Lambda) \longrightarrow \ldots.$$

By purity and since $D_v - S$ is a union of affine curves over a finite field (which has cohomological dimension 1), we have

$$H^{q+1}_{D_v - S}(X - S, \Lambda) \cong H^{q-1}(D_v - S, \Lambda(-1)) = 0.$$

We conclude that

$$H^q(X - S, \Lambda) \to H^q(X - D_v, \Lambda)$$

is surjective and thus ϕ_1 lifts to $\phi_2 \in H^q(X - S, \Lambda)$. Next consider the excision sequence

$$\ldots \longrightarrow H^q(X, \Lambda) \longrightarrow H^q(X - S, \Lambda) \longrightarrow H^{q+1}_S(X, \Lambda) \longrightarrow \ldots.$$

By purity we have

$$H^{q+1}_S(X, \Lambda) \cong H^{q-3}(S, \Lambda(-2)).$$

The finite set of closed points S has cohomological dimension 1 implying that

$$H^{q+1}_S(X, \Lambda) = 0$$

for $q \geqslant 5$. We conclude that in this case the restriction

$$H^q(X, \Lambda) \to H^q(X - S, \Lambda)$$

is surjective and thus ϕ_2 lifts to $H^q(X, \Lambda)$. Assume that $q = 4$. By Corollary 5.12 there is a finite étale $\mathfrak{c}$-covering $B' \to B$ such that

$$H^5_S(X, \Lambda) \to H^5_S(X \times_B B', \Lambda)$$

is the zero map and the result follows. □

Chapter 6

Higher direct Images

We stick to the notation of paragraph 5.3: $\bar{X}/B$ is a proper arithmetic surface with geometric point $\bar{x} \to \bar{X}$ lying over a closed point $x \in \bar{X}$. We assume that $\bar{X} - x$ is regular. Furthermore, we fix a tidy divisor $\bar{D} \subseteq \bar{X}$ not containing x. We define X, U, $\bar{D}_h$, D_h, D_v, W and S as in paragraph 5.3.

We further take a full class of finite groups $\mathfrak{c}$ such that all integers in $\mathbb{N}(\mathfrak{c})$ are invertible on $\bar{X}$ and $\mu_l \cong \mathbb{Z}/l\mathbb{Z}$ on $\bar{X}$ for all prime numbers $l \in \mathfrak{c}$. Fix a positive integer $n \in \mathbb{N}(\mathfrak{c})$. We set $\Lambda = \mathbb{Z}/n\mathbb{Z}$ and also denote the corresponding constant sheaf on any scheme by Λ.

We add one more piece of notation to the general setup: Namely, we denote by η the generic point of B and choose a geometric point $\bar{\eta}$ above η. The absolute Galois group $\mathrm{Gal}(\bar{\eta}|\eta)$ of η will be denoted $\mathcal{G}_\eta$. For a geometric point $\bar{b}$ of B we write $I_{\bar{b}} \subseteq \mathcal{G}_\eta$ for the inertia group of $\mathcal{G}_\eta$ at $\bar{b}$. It can also be interpreted as the fundamental group $\pi_1(\eta_{\bar{b}}^{sh}, \bar{\eta})$ of the generic point $\eta_{\bar{b}}^{sh}$ of the strict henselization of B at $\bar{b}$.

Our final goal is to show that the limit over $\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}$ of $H^q(U', \Lambda)$ vanishes for $q \geqslant 2$ (for $q = 1$ this is automatically true). In section 5.3 we showed that, under certain hypotheses on X, for $q \geqslant 2$ the cokernel of

$$H^q(X, \Lambda) \to H^q(U, \Lambda)$$

vanishes in the limit over the category $\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}$. This breaks down to the existence of a desingularized $\mathfrak{c}$-covering $(\bar{X}', D') \to (\bar{X}, D)$ such that the image of each cohomology class $\phi \in H^q(U, \Lambda)$ in $H^q(U', \Lambda)$ lifts to $H^q(X', \Lambda)$. It remains to prove that $H^q(X', \Lambda)$ vanishes in the limit. In case $q = 2$ we will only show that the cokernel of

$$H^2_{D'}(X', \Lambda) \to H^2(X', \Lambda)$$

vanishes in the limit, which is sufficient. Indeed, the excision sequence

$$\ldots \to H^2_{D'}(X', \Lambda) \to H^2(X', \Lambda) \to H^2(U', \Lambda) \to \ldots$$

induces an injection

$$\mathit{coker}(H^2_{D'}(X', \Lambda) \to H^2(X', \Lambda)) \hookrightarrow H^2(U', \Lambda),$$

and we have to show the vanishing in the limit of precisely the cohomology classes lying in the image.

Since the assumptions made so far on $\bar{X}$ are also satisfied by $\bar{X}'$, we can change notation and henceforth seek to prove the vanishing in the limit over $\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}$ of $H^q(X, \Lambda)$ for $q \geqslant 3$ and of $\mathit{coker}(H^2_D(X, \Lambda) \to H^2(X, \Lambda))$.

6.1 The Leray Spectral Sequence

Let $\mathcal{S}_m$ denote the category whose objects are first quadrant E_m-spectral sequences

$$E_m^{p,q} \Rightarrow E^{p+q}$$

and whose morphisms

$$\phi : (E_m^{p,q} \Rightarrow E^{p+q}) \to (\bar{E}_m^{p,q} \Rightarrow \bar{E}^{p+q})$$

are collections $(\phi_r^{p,q}, \phi^n)$ where $\phi_r^{p,q} : E_r^{p,q} \to \bar{E}_r^{p,q}$ constitute a map of spectral sequences and $\phi^n : E^n \to \bar{E}^n$ are homomorphisms which are compatible with $\phi_r^{p,q}$ in the sense that

$$\phi^n(F^p E^n) \subseteq \bar{F}^p \bar{E}^n$$

and that the induced maps of associated graded groups

$$gr\ E^n \to gr\ \bar{E}^n$$

are induced from the maps $\phi_r^{p,q}$.

Lemma 6.1: *Let $s \leqslant n$ and m be positive integers. Let $\mathcal{C}$ be a cofiltered category and $\Phi : \mathcal{C} \to \mathcal{S}_m$ a contravariant functor. Suppose that*

$$\varinjlim_{\mathcal{C}} E_m^{p,q} = 0$$

for all integers p, q such that $p + q = n$ and $p \geqslant s$. Then

$$\varinjlim_{\mathcal{C}} F^s E^n = 0.$$

Proof: Let C be an object of $\mathcal{C}$ and let $E_m^{p,q} \Rightarrow E^{p+q}$ be the spectral sequence corresponding to $\Phi(C)$. Let $x \in F^s E^n$ be nontrivial and denote by $p \geqslant s$ the natural number such that $x \in F^p E^n$ but $x \notin F^{p+1} E^n$. It follows that x lifts to some $y \in E_m^{p,q}$ where $q = n - p$. By assumption, there is a morphism $\bar{C} \to C$ in $\mathcal{C}$ with associated map of spectral sequences

$$(\phi_r^{p,q}, \phi^n) : (E_m^{p,q} \Rightarrow E^{p+q}) \to (\bar{E}_m^{p,q} \Rightarrow \bar{E}^{p,q}),$$

such that $\phi_m^{p+q}(y) = 0$. By compatibility of ϕ^n with $\phi_r^{p,q}$ it follows that $\phi^n(x)$ is contained in $\bar{F}^{p+1}\bar{E}^n$. The assertion now follows by induction on p noting that $F^p E^n = 0$ for $p > n$ for all spectral sequences in $\mathcal{S}_m$. □

Corollary 6.2: *Suppose that*

$$\varinjlim_{\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} H^i(B', R^j \pi'_* \Lambda) = 0$$

for all integers i, j such that $i + j \geqslant 3$ and for all integers i, j such that $i + j = 2$ and $i \geqslant 1$. Then

$$\varinjlim_{\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} H^n(X', \Lambda) = 0$$

for $n \geqslant 3$ and

$$\varinjlim_{\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} ker(H^2(X', \Lambda) \stackrel{edge}{\to} H^0(B', R^2 \pi'_* \Lambda)) = 0.$$

Proof: We would like to apply Lemma 6.1 to the category $\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}$ of desingularized $\mathfrak{c}$-coverings of $(\bar{X}, \bar{D})$, which is cofiltered by Lemma 2.21. With each arithmetic surface $\bar{\pi}' : \bar{X}' \to B'$ in $\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}$, we associate the Leray spectral sequence

$$H^i(B', R^j \pi'_* \Lambda) \Rightarrow H^{i+j}(X', \Lambda).$$

A morphism $(\bar{X}'', \bar{D}'') \to (\bar{X}', \bar{D}')$ in $\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}$ induces a morphism $X'' \to X'$ and thus a map of spectral sequences

$$H^i(B'', R^j\pi''_*\Lambda) \to H^i(B', R^j\pi'_*\Lambda).$$

This map of spectral sequences is compatible with the pullback map $H^n(X'', \Lambda) \to H^n(X', \Lambda)$. We conclude that the above construction defines a functor

$$\Phi : \mathfrak{I}_{\bar{X},\bar{D},\bar{x}} \to \mathcal{S}_2.$$

Setting $s = 0$ for $n \geqslant 3$ and $s = 1$ for $n = 2$ we obtain from Lemma 6.1 that

$$\varinjlim_{\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} F^s H^n(X', \Lambda) = 0.$$

But

$$F^0 H^n(X', \Lambda) = H^n(X', \Lambda) \qquad \text{and} \qquad F^1 H^2(X', \Lambda) = \mathit{ker}(H^2(X, \Lambda) \overset{edge}{\to} H^0(B, R^2\pi'_*\Lambda)),$$

and hence the assertion of the corollary holds. □

6.2 Killing the Cohomology of higher direct images

Corollary 6.2 of the previous section provides us with a method to prove the vanishing of the cohomology groups $H^n(X', \Lambda)$ in the limit over $\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}$, namely by showing that the limit of $H^i(B', R^j\pi'_*\Lambda)$ for $i + j = n$ vanishes. In this section we show the vanishing in the limit of $H^i(B', R^j\pi'_*\Lambda)$ for most combinations of (i, j). The two most complicated cases – $(i, j) = (0, 2)$ and $(i, j) = (1, 1)$ – are treated in separate sections.

Lemma 6.3: *Suppose that either $j \geqslant 3$ or $j = 2$, $i \geqslant 2$ and $\bar{D}_h$ is not empty. Then*

$$H^i(B, R^j\pi_*\Lambda) = 0.$$

If in addition, for every geometric point $\bar{b}$ above a closed point $b \in B$ the natural map

$$\pi_1(b, \bar{b})(\mathfrak{c}) \to \pi_1(B, \bar{b})(\mathfrak{c})$$

is injective, the group

$$H^1(B, R^2\pi_*\Lambda)$$

vanishes in the limit over $\mathfrak{I}_{B,\bar{x}}$.

Proof: Let $\bar{b}$ be a geometric point of B. By Lemma 5.7 the stalk of $R^j\pi_*\Lambda$ at $\bar{b}$ is isomorphic to $H^j(X_{\bar{b}}, \Lambda)$. Since $X_{\bar{b}}$ is a curve over an algebraically closed field, it has cohomological dimension less or equal to 2. Therefore, $R^j\pi_*\Lambda = 0$ for $j \geqslant 3$. Assume now that $\bar{D}_h$ is nonempty and hence, for all geometric points $\bar{b}$ of B the curve $X_{\bar{b}}$ is not complete. If $X_{\bar{b}}$ is regular, it is irreducible and thus affine. Since affine curves over algebraically closed fields have cohomological dimension less than or equal to 1, $H^2(X_{\bar{b}}, \Lambda)$ is nontrivial only if $X_{\bar{b}}$ is singular. We conclude that $R^2\pi_*\Lambda$ is a skyscraper sheaf and

$$H^i(B, R^2\pi_*\Lambda) = 0$$

for $i \geqslant 2$ as the residue fields at closed points of B have cohomological dimension 1. Furthermore, applying Corollary 5.12 to every closed point b of B in the support of $R^2\pi_*\Lambda$ we find a $\mathfrak{c}$-extension $B' \to B$ such that

$$H^1(B, R^2\pi_*\Lambda) \to H^1(B', R^2\pi'_*\Lambda)$$

is the zero map. □

In case $\bar{X}/B$ is of local type we can even say a bit more:

Lemma 6.4: *Suppose that $\bar{X}$ is of local type. Then*

$$H^i(B, R^j\pi_*\Lambda) = 0$$

if either $i \geqslant 2$ or $j \geqslant 3$. Moreover, there is an étale $\mathfrak{c}$-covering $B' \to B$ such that

$$H^1(B, R^j(\pi_Y)_*\Lambda) \to H^1(B', R^j(\pi_Y)_*\Lambda)$$

is the zero map for all $q \geqslant 0$.

Proof: By Lemma 6.3 the cohomology groups $H^i(B, R^j\pi_*\Lambda)$ vanish for $j \geqslant 3$. Since

$$H^i(B, R^j\pi_*\Lambda) \cong H^i(b, R^j\pi_*\Lambda) \cong H^i(k(b), H^j(X_{\bar{b}}, \Lambda)),$$

by Lemma 5.7, the remaining assertions follow from Corollary 5.12. □

Lemma 6.4 shows that in the local case the hypotheses of Corollary 6.2 are satisfied. Therefore,

$$\varinjlim_{\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} H^n(X', \Lambda) = 0$$

for $n \geqslant 3$ and

$$\varinjlim_{\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} ker(H^2(X', \Lambda) \overset{edge}{\to} H^0(B', R^2\pi'_*\Lambda)) = 0.$$

In case $n = 2$ we still have to do some work in order to deduce that

$$\varinjlim_{\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} H^n(U', \Lambda) = 0.$$

This will be done in section 6.5 (for both the local and the global case).

In the remaining part of this section we continue investigating the global case. The following lemma and its corollary are valid in the global as well as in the local case. However, in the local case they are trivially true (compare with Lemma 6.4) so their relevance lies in the global case.

Lemma 6.5: *Let B be a Dedekind scheme with generic point η and $\mathcal{F}$ a $\mathfrak{c}$-constructible sheaf on $B_{ét}$. Suppose that for every geometric point $\bar{b}$ above a closed point $b \in B$ the natural map*

$$\pi_1(b, \bar{b})(\mathfrak{c}) \to \pi_1(B, \bar{b})(\mathfrak{c})$$

is injective. Choose a finite set of closed points $T \subseteq B$ such that $\mathcal{F}$ is locally constant on $B - T$. Assume that the action of $\pi_1(B-T, \bar{\eta})$ on $\mathcal{F}_{\bar{\eta}}$ factors through $\pi_1(B-T, \bar{\eta})(\mathfrak{c})$. Denote by $\tilde{B}(\mathfrak{c})$ the universal $\mathfrak{c}$-covering of B and by $\tilde{T}(\mathfrak{c})$ the preimage of T in $\tilde{B}(\mathfrak{c})$. Suppose that $B - T$ is $K(\pi, 1)$ with respect to $\mathfrak{c}$ and

$$\pi_1(\tilde{B}(\mathfrak{c}) - \tilde{T}(\mathfrak{c}), \bar{\eta}))(\mathfrak{c}) \cong \mathop{\ast}_{x \in \tilde{T}(\mathfrak{c})} \pi_1(\eta_x^{sh}, \bar{\eta})(\mathfrak{c}) = \mathop{\ast}_{x \in \tilde{T}(\mathfrak{c})} I_x(\mathfrak{c}),$$

where η_x^{sh} denotes the generic point of the strict henselization of B at the point x and I_x the inertia group of x in the absolute Galois group of η. Then for all $j \geqslant 2$ there is a finite étale $\mathfrak{c}$-covering $B' \to B$ such that

$$H^i(B, \mathcal{F}) \to H^i(B', \mathcal{F})$$

is the zero map.

Proof: Denote by $\iota : \eta \to B$ the inclusion of the generic point. We will first reduce to the case where
$$\mathcal{F} \to \iota_* \iota^* \mathcal{F}$$
is an isomorphism. We have an exact sequence
$$0 \to \mathcal{G}_1 \to \mathcal{F} \to \iota_* \iota^* \mathcal{F} \to \mathcal{G}_2 \to 0$$
with skyscraper sheaves $\mathcal{G}_1$ and $\mathcal{G}_2$ with support in T. Since the points in T have finite residue field,
$$H^j(B, \mathcal{G}_i) = 0$$
for $j \geqslant 2$ and there is a finite étale $\mathfrak{c}$-covering $T' \to T$ such that
$$H^1(B, \mathcal{G}_i) = H^1(T, \mathcal{G}_i) \to H^1(T', \mathcal{G}_i)$$
is the zero map. By hypothesis all finite étale $\mathfrak{c}$-coverings of T are globally realized so there is a finite étale $\mathfrak{c}$-covering $B' \to B$ such that
$$H^1(B, \mathcal{G}_i) \to H^1(B', \mathcal{G}_i)$$
is the zero map. By splitting the above exact sequence in two short exact sequences
$$0 \to \mathcal{G}_1 \to \mathcal{F} \to \mathcal{H} \to 0, \qquad 0 \to \mathcal{H} \to \iota_* \iota^* \mathcal{F} \to \mathcal{G}_2 \to 0,$$
and using the associated long exact cohomology sequences we obtain isomorphisms
$$H^j(B, \mathcal{F}) \xrightarrow{\sim} H^j(B, \iota_* \iota^* \mathcal{F})$$
for $j \geqslant 3$ and a short exact sequence
$$0 \to H^1(B, \mathcal{G}_2) \to H^2(B, \mathcal{F}) \to H^2(B, \iota_* \iota^* \mathcal{F}) \to 0.$$
Noting that the hypotheses are stable under base change by a finite étale $\mathfrak{c}$-extension $B' \to B$ we conclude that it suffices to prove the statement in case $\mathcal{F} \cong \iota_* \iota^* \mathcal{F}$.

In this case $\mathcal{F}$ corresponds to a finite $\mathfrak{c}$-torsion $\pi_1(B - T)$-module $\mathcal{M}$. Consider the excision sequence
$$\ldots \to H^j_T(B, \mathcal{F}) \to H^j(B, \mathcal{F}) \to H^j(B - T, \mathcal{F}) \to \ldots.$$
By assumption we have
$$\varinjlim_{B' \to B} H^j(B' - T', \mathcal{F}) \cong H^j(\tilde{B}(\mathfrak{c}) - \tilde{T}(\mathfrak{c}), \mathcal{F}) \cong H^j(\pi_1(\tilde{B}(\mathfrak{c}) - \tilde{T}(\mathfrak{c}))(\mathfrak{c}), \mathcal{M}) \cong \bigoplus_{x \in \tilde{T}(\mathfrak{c})} {}' H^j(I_x(\mathfrak{c}), \mathcal{M}),$$
where the limit is taken over all finite étale $\mathfrak{c}$-extensions $B' \to B$ and T' denotes the preimage of T in B'. Since $cd_{\mathfrak{c}}\ I_x \leqslant 1$ for all closed points $x \in B$ by [NSW], Theorem 7.1.8, we conclude that
$$\varinjlim_{B' \to B} H^j(B' - T', \mathcal{F}) = 0$$
for $j \geqslant 2$ and thus there is a finite étale $\mathfrak{c}$-extension $B' \to B$ such that
$$H^j(B - T, \mathcal{F}) \to H^j(B' - T', \mathcal{F})$$
is the zero map. Taking into account that the hypotheses are stable under base change by finite étale $\mathfrak{c}$-coverings of B it remains to show that for $j \geqslant 2$ the image of $H^j_T(B, \mathcal{F}) \to H^j(B, \mathcal{F})$ vanishes in the limit over all finite étale $\mathfrak{c}$-coverings.

For a closed point $x \in B$ denote by B^h_x the henselization of B at x. Consider the excision sequence
$$\ldots \to \bigoplus_{x \in T} H^{j-1}(B^h_x - x, \mathcal{F}) \to H^j_T(B, \mathcal{F}) \to \bigoplus_{x \in T} H^j(B^h_x, \mathcal{F}) \to \ldots.$$

We have

$$H^j(B_x^h, \mathcal{F}) \cong H^j(x, \mathcal{F}_x) = 0$$

for $j \geqslant 2$ and

$$H^{j-1}(B_x^h - x, \mathcal{F}) \cong H^{j-1}(K_x^h, \mathcal{M})$$

where K_x^h denotes the fraction field of B_x^h. Let $\bar{x}$ be a geometric point over x. Then the cohomological $\mathfrak{c}$-dimension of $K_{\bar{x}}^{sh}$ is equal to one by [NSW], Theorem 7.1.8. Since all unramified extensions of K_x^h are globally realized, $H^{j-1}(B_x^h - x, \mathcal{F})$ vanishes in the limit over all finite étale $\mathfrak{c}$-extensions $B' \to B$ if $j \geqslant 3$. For $j = 2$ consider the commutative diagram of excision sequences

$$\begin{tikzcd} H^1(B-T,\mathcal{F}) \arrow[r] \arrow[d] & H^2_T(B,\mathcal{F}) \arrow[r] & H^2(B,\mathcal{F}) \\ \bigoplus_{x\in T} H^1(K_x^h,\mathcal{M}). \arrow[ur, two heads] & & \end{tikzcd}$$

Taking the limit over all finite étale $\mathfrak{c}$-coverings $B' \to B$ we obtain

$$\begin{tikzcd} H^1(\tilde{B}(\mathfrak{c}) - \tilde{T}(\mathfrak{c}),\mathcal{F}) \arrow[r] \arrow[d, "\sim"] & H^2_{\tilde{T}(\mathfrak{c})}(\tilde{B}(\mathfrak{c}),\mathcal{F}) \arrow[r] & H^2(\tilde{B}(\mathfrak{c}),\mathcal{F}) \\ \bigoplus_{x\in \tilde{T}(\mathfrak{c})} H^1(K_x^{sh},\mathcal{M}). \arrow[ur, two heads] & & \end{tikzcd}$$

We conclude that in the limit over all finite étale $\mathfrak{c}$-extensions of B the image of $H^2_T(B, \mathcal{F})$ in $H^2(B, \mathcal{F})$ vanishes. □

Corollary 6.6: *Suppose that B meets the hypotheses of Lemma 6.5 for the image T of S in B. Then for all $j \geqslant 0$ and all $i \geqslant 2$ the cohomology groups*

$$H^i(B, R^j\pi_*\Lambda)$$

vanish in the limit over $\mathfrak{I}_{B,\bar{x}}$.

Proof: By [SGA4], XIV, Théorème 1.1 the sheaves $R^j\pi_*\Lambda_{\bar{X}}$ are constructible and by [SGA4], XVI, Corollaire 2.2 their restriction to $B - S$ is locally constant. The corollary now follows from Lemma 6.5. □

Combining Lemma 6.3 and Corollary 6.6 we conclude that under certain hypotheses on $(\bar{X}, \bar{D})$

$$H^i(B, R^j\pi_*\Lambda)$$

vanishes in the limit over $\mathfrak{I}_{B,\bar{x}}$ if $i + j \geqslant 3$ or if $i + j = 2$ and $i \geqslant 2$. By Lemma 7.2, once these hypotheses are satisfied for $(\bar{X}, \bar{D})$ they are satisfied for all $(\bar{X}', \bar{D}')$ in $\mathfrak{I}_{B,\bar{x}}$. We then obtain that

$$\varinjlim_{\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} H^i(B', R^j\pi'_*\Lambda) = 0$$

if $i + j \geqslant 3$ or if $i + j = 2$ and $i \geqslant 2$. Therefore, in order to show that the hypotheses of Corollary 6.2 are satisfied there is only the case $(i, j) = (1, 1)$ left for examination. It will be treated in section 6.4.

6.3 The intersection matrix

In order to treat the cases $(i, j) = (1, 1)$ and $(i, j) = (0, 2)$ we have to deal with the cokernel of

$$H^2_D(X, \Lambda) \to H^2(X, \Lambda).$$

In this section we explain how to relate this homomorphism with the intersection matrix of the irreducible components of the singular fibres.

Lemma 6.7: *Suppose that B is strictly henselian with closed point s. Denote by ρ the intersection matrix of the components of the special fibre of $\bar{\pi} : \bar{X} \to B$. Then, for any integer c the following diagram commutes*

$$\begin{tikzcd}
H^2_{D_v}(X, \Lambda(c+1)) \arrow[r] & H^2(X, \Lambda(c+1)) \arrow[d, "\sim"', "base\ change"] \\
H^0(D_v, \Lambda(c)) \arrow[u, "purity", "\sim"'] & H^2(X_s, \Lambda(c+1)) \arrow[d, "\sim"', "deg"] \\
\bigoplus_{C \subseteq D_v} \Lambda(c) \cdot C \arrow[u, "\sim"'] \arrow[r, "\rho"] & \bigoplus_{C \subseteq \bar{X}_s, C \cap \bar{D}_h = \varnothing} \Lambda(c) \cdot C.
\end{tikzcd}$$

Here, C always denotes irreducible components.

Proof: It suffices to prove the lemma for $c = 0$. Since the residue field k is algebraically closed and thus contains μ_m, the general case follows by twisting by c. Consider the following commutative diagram

$$\begin{tikzcd}
H^2_{D_v}(X, \mu_m) \arrow[r] & H^2(X, \mu_m) \arrow[r, "\sim"] & H^2(X_s, \mu_m) \\
\bigoplus_{C \subseteq D_v} H^1_C(X, \mathbb{G}_m) \otimes \Lambda \arrow[u, "\sim"] \arrow[r] & H^1(X, \mathbb{G}_m) \otimes \Lambda \arrow[u] \arrow[r] & \bigoplus_{C \cap \bar{D}_h = \varnothing} H^1(C, \mathbb{G}_m) \otimes \Lambda \arrow[u, "\sim"] \\
\bigoplus_{C \subseteq D_v} \Lambda \cdot C \arrow[u, "\sim"] \arrow[r] \arrow[d, equal] & Pic(X) \otimes \Lambda \arrow[u, "\sim"] \arrow[r] & \bigoplus_{C \cap \bar{D}_h = \varnothing} Pic(C) \otimes \Lambda \arrow[u, "\sim"] \arrow[d, "\sim"', "(deg_C)_C"] \\
\bigoplus_{C \subseteq D_v} \Lambda \cdot C \arrow[rr] & & \bigoplus_{C \cap \bar{D}_h = \varnothing} \Lambda \cdot C.
\end{tikzcd}$$

Note that the direct sums on the right hand side run only over irreducible components of X_s with trivial intersection with $\bar{D}_h$. The reason is that these are precisely the components of X_s which are proper over B. The upper right horizontal isomorphism comes from Lemma 5.7. The upper vertical maps are connecting homomorphisms of the Kummer sequence. Note that the concatenation of the left hand side vertical arrows yields the purity isomorphism by definition as the latter is normalized this way in codimension one. Moreover, the degree map in the statement of the lemma is defined as the composition

$$H^2(X_s, \mu_m) \overset{\sim}{\leftarrow} \bigoplus_{C \cap \bar{D}_h = \varnothing} H^1(C, \mathbb{G}_m) \otimes \Lambda \overset{\sim}{\leftarrow} \bigoplus_{C \cap \bar{D}_h = \varnothing} Pic(C) \otimes \Lambda \overset{(deg_C)_C}{\to} \bigoplus_{C \cap \bar{D}_h = \varnothing} \Lambda.$$

The restrictions

$$Pic(X) \to Pic(C)$$

are given by $D \mapsto D \cdot C$ where $D \cdot C$ denotes the intersection product of the divisor D with the curve C. Composition with deg_C yields the intersection number $(D \cdot C)$. We conclude that the lower horizontal map is indeed given by the intersection matrix $\rho_{C_1,C_2} = (C_1 \cdot C_2)$. $\square$

We set

$$\mathbb{Z}_{\mathfrak{c}} = \varprojlim_{n \in \mathbb{N}(\mathfrak{c})} \mathbb{Z}/n\mathbb{Z} = \prod_{l \in \mathbb{N}(\mathfrak{c}) \text{ prime}} \mathbb{Z}_l.$$

Lemma 6.8: *Assume that $(\bar{X}, \bar{D})$ has enough tame coverings. Then, for every integer $m \in \mathbb{N}(\mathfrak{c})$ there is a desingularized $\mathfrak{c}$-covering $(\bar{X}', \bar{D}') \to (\bar{X}, \bar{D})$ such that the image of*

$$H^2_{\bar{D}}(\bar{X}, \mathbb{Z}_{\mathfrak{c}}(1)) \to H^2_{\bar{D}'}(\bar{X}', \mathbb{Z}_{\mathfrak{c}}(1))$$

is divisible by m.

Proof: As in Lemma 6.7 absolute cohomological purity provides us with a functorial isomorphism

$$\bigoplus_{C \subseteq \bar{D}} \Lambda \cdot C \xrightarrow{\sim} H^2_{\bar{D}}(\bar{X}, \mu_n)$$

for every integer $n \in \mathfrak{c}$. Taking the projective limit, we obtain

$$\bigoplus_{C \subseteq \bar{D}} \mathbb{Z}_{\mathfrak{c}} \cdot C \xrightarrow{\sim} H^2_{\bar{D}}(\bar{X}, \mathbb{Z}_{\mathfrak{c}}(1)).$$

Moreover, if $(\bar{X}', \bar{D}') \to (\bar{X}, \bar{D})$ is a desingularized $\mathfrak{c}$-covering, the induced map

$$\bigoplus_{C \subseteq \bar{D}} \mathbb{Z}_{\mathfrak{c}} \cdot C \to \bigoplus_{C' \subseteq \bar{D}'} \mathbb{Z}_{\mathfrak{c}} \cdot C'$$

is given by the pull-back of divisors. Let $C \subseteq \bar{D}$ be an irreducible component and $p \in C$ a closed point of $\bar{D}$. Since $(\bar{X}, \bar{D})$ has enough tame coverings, there is $f_p \in K(\bar{X})^\times$ such that $deg_C(f_p) = m_p > 0$ and $deg_Z(f_p) = 0$ for all other irreducible components Z of $\bar{D}$ passing through p. Hence, in a Zariski neighborhood U_p of p we have $div\ f_p = m_p C$. Denote by m'_p be the maximal factor of m_p contained in $\mathbb{N}(\mathfrak{c})$. Let $\phi_p : (\bar{X}_p, \bar{D}_p) \to (\bar{X}, \bar{D})$ be a desingularized $\mathfrak{c}$-covering with function field extension

$$K(\bar{X}_p) = K(\bar{X})[\sqrt[m'_p m]{f_p}] | K(\bar{X}).$$

Then $div\ f_p \subseteq \bar{X}_p$ is divisible by $m'_p m$. Thus, $\phi_p^*(C) \cap \phi_p^{-1}(U_p)$ is divisible by m, i. e., the coefficients of all irreducible components of $\phi_p^*(C)$ whose generic points lie over U_p are divisible by m. This property is conserved by further desingularized coverings.

There are finitely many closed points $p_1, \ldots, p_n \in C$ such that the open subschemes $U_{p_1}, \ldots, U_{p_n}$ cover C. Let $(\bar{X}', \bar{D}') \to (\bar{X}, \bar{D})$ be a desingularized $\mathfrak{c}$-covering dominating all coverings constructed above $(\bar{X}_{p_i}, \bar{D}_{p_i}) \to (\bar{X}, \bar{D})$. Then the pullback of C to $\bar{X}'$ is divisible by m. □

Remark 6.9: *Assume that $\bar{X}/B$ is the pullback of an arithmetic surface $\bar{X}_0/B_0$ to the strict henselization of B_0 in some geometric point. Moreover, assume that $\bar{D}$ is the pullback of a tidy divisor $\bar{D}_0 \subseteq \bar{X}_0$. If $(\bar{X}_0, \bar{D}_0)$ has enough tame coverings, the same holds for $(\bar{X}, \bar{D})$ and in the proof of Lemma 6.8 we can choose the functions $f_p \in K(\bar{X})^\times$ such that they are already contained in $K(\bar{X}_0)^\times$ and as such have support in $\bar{D}_0$. Hence, we may assume that the desingularized $\mathfrak{c}$-covering $(\bar{X}', \bar{D}') \to (\bar{X}, \bar{D})$ constructed in Lemma 6.8 is the pullback of a desingularized $\mathfrak{c}$-covering of $(\bar{X}_0, \bar{D}_0)$.*

Corollary 6.10: *Assume that B is strictly henselian and that $\bar{D}_h$ is nonempty and meets all irreducible components of W. If $(\bar{X}, \bar{D})$ has enough tame coverings, the cokernel of*

$$H^2_D(X, \hat{\mathbb{Z}}(\mathfrak{c})(1)) \to H^2(X, \hat{\mathbb{Z}}(\mathfrak{c})(1))$$

vanishes in the limit over $\mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}$.

Proof: It suffices to prove that the cokernel of

$$H^2_{D_v}(X, \hat{\mathbb{Z}}(\mathfrak{c})(1)) \to H^2(X, \hat{\mathbb{Z}}(\mathfrak{c})(1))$$

vanishes in the limit over $\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}$ as $H^2_{D_v}(X, \hat{\mathbb{Z}}(\mathfrak{c})(1))$ is a direct summand of $H^2_D(X, \hat{\mathbb{Z}}(\mathfrak{c})(1))$. Taking the inverse limit over all $\Lambda \cong \mathbb{Z}/n\mathbb{Z}$ with $n \in \mathbb{N}(\mathfrak{c})$ in the diagrams in Lemma 6.7, we obtain a commutative diagram

$$\begin{tikzcd}
H^2_{D_v}(X, \mathbb{Z}_{\mathfrak{c}}(c+1)) \arrow[r] & H^2(X, \mathbb{Z}_{\mathfrak{c}}(c+1)) \arrow[d, "\sim"', "\text{base change}"] \\
H^0(D_v, \mathbb{Z}_{\mathfrak{c}}(c)) \arrow[u, "\text{purity}", "\sim"'] & H^2(X_s, \mathbb{Z}_{\mathfrak{c}}(c+1)) \arrow[d, "\sim"', "\deg"] \\
\bigoplus_{C \subseteq D_v} \mathbb{Z}_{\mathfrak{c}}(c) \cdot C \arrow[u, "\sim"'] \arrow[r, "\rho"] & \bigoplus_{C \subseteq D_v} \mathbb{Z}_{\mathfrak{c}}(c) \cdot C.
\end{tikzcd}$$

Note that since we assumed that $\bar{D}_h$ meets all irreducible components of W, we have that $C \cap \bar{D}_h = \varnothing$ if and only if $C \subseteq D_v$. The map

$$\phi : H^2_{D_v}(X, \hat{\mathbb{Z}}(\mathfrak{c})(1)) \to H^2(X, \hat{\mathbb{Z}}(\mathfrak{c})(1))$$

is thus given by the intersection matrix ρ of the irreducible components of D_v. By [Liu], Theorem 9.1.23 the intersection matrix of the components of the special fibre is negative semidefinite and its radical is generated by the special fibre. Since we assumed that $\bar{D}_h$ is nonempty, the support of D does not comprise all irreducible components of the special fibre. Hence, the restriction of ρ to the components of D is negative definite. We conclude that

$$\phi \otimes \mathbb{Q} : H^2_D(X, \hat{\mathbb{Z}}(\mathfrak{c})(1)) \otimes \mathbb{Q} \to H^2(X, \hat{\mathbb{Z}}(\mathfrak{c})(1)) \otimes \mathbb{Q}$$

is an isomorphism and thus the cokernel of ϕ is $\mathfrak{c}$-torsion. Take $m \in \mathbb{N}(\mathfrak{c})$ such that $m \cdot coker\ \phi = 0$. By Lemma 6.8 there is a desingularized $\mathfrak{c}$-covering $(\bar{X}', \bar{D}') \to (\bar{X}, \bar{D})$ of $(\bar{X}, \bar{D})$ such that the image of

$$H^2_D(X, \hat{\mathbb{Z}}(\mathfrak{c})(1)) \to H^2_{D'}(X', \hat{\mathbb{Z}}(\mathfrak{c})(1))$$

is divisible by m. Taking into account that multiplication by m is injective on $H^2(X', \hat{\mathbb{Z}}(\mathfrak{c})(1))$ this proves the result. $\square$

Remark 6.11: *Assume that $\bar{D}_h$ is nonempty and meets all irreducible components of W. In case $(\bar{X}, \bar{D})$ has enough tame coverings, the base change of $\bar{X}$ to the strict henselization $B^{sh}_{\bar{b}}$ of B at any geometric point $\bar{b}$ meets the hypotheses of Corollary 6.10. Using in the proof of the corollary a desingularized $\mathfrak{c}$-covering of $(\bar{X}_{B^{sh}_{\bar{b}}}, \bar{D}_{B^{sh}_{\bar{b}}})$ which is the pullback of a desingularized $\mathfrak{c}$-covering of $(\bar{X}, \bar{D})$ (it exists by Remark 6.9), we see that the cokernel of*

$$H^2_{D_{B^{sh}_{\bar{b}}}}(X_{B^{sh}_{\bar{b}}}, \hat{\mathbb{Z}}(\mathfrak{c})(1)) \to H^2(X_{B^{sh}_{\bar{b}}}, \hat{\mathbb{Z}}(\mathfrak{c})(1))$$

vanishes not only in the limit over $\mathfrak{I}_{\bar{X}_{B^{sh}_{\bar{b}}}, \bar{D}_{B^{sh}_{\bar{b}}}, \bar{x}}$ but also in the limit over $\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}$.

6.4 The first higher direct image

In this section we treat the remaining case $(i, j) = (1, 1)$: We examine the vanishing of

$$\varinjlim_{\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} H^1(B', R^1\pi'_*\Lambda).$$

We will only be able to prove this if $\mathfrak{c}$ is the class $\mathfrak{c}(l)$ of finite l-groups for a prime number l. In this case it suffices to prove the assertion for $\Lambda = \mathbb{Z}/l\mathbb{Z}$ (see Corollary 2.5). Therefore, in this section $\mathfrak{c}$ is assumed to be $\mathfrak{c}(l)$ and $\Lambda = \mathbb{Z}/l\mathbb{Z}$. We assume throughout this section that μ_l is a constant sheaf on $\bar{X}$.

Let us recall the notation: The divisor $\bar{D}_h$ is the maximal subdivisor of $\bar{D}$ with support in the isolated horizontal components of $\bar{D}$, i. e., on the horizontal components which do not intersect any other component. Set $X = \bar{X} - \bar{D}_h$ and $U = \bar{X} - \bar{D}$ and denote by $D \subseteq X$ the restriction of $\bar{D}$ to X. We write $\bar{\pi} : \bar{X} \to B$ for the structure morphism of $\bar{X}$ and $\pi : X \to B$ for its restriction to X. The maximal vertical subdivisor of D is denoted D_v and the maximal horizontal subdivisor D_h, such that $D = D_v + D_h$. Let W denote the union of all vertical prime divisors which are contained in a singular fibre of $\bar{X} \to B$ but not contained in $\bar{D}$. Denote by S the finite set of special points of $\bar{D}$, i. e., the set of singular points of $\bar{D}_{red}$.

The sheaf $R^1\bar{\pi}_*\Lambda$ is closely related to the Jacobian of the geometric fibres of $\bar{X} \to B$. Indeed, the stalk of $R^1\bar{\pi}_*\Lambda$ at a geometric point $\bar{b}$ of B is

$$H^1(\bar{X}_{\bar{b}}, \Lambda) \cong H^1(\bar{X}_{\bar{b}}, \mu_l),$$

which parameterizes the l-division points of the Jacobian of $\bar{X}_{\bar{b}}$. We will not explicitly need the theory of the Jacobian variety. If we speak of l-division points of the Jacobian, it can just be thought of the cohomology group $H^1(\bar{X}_{\bar{b}}, \mu_l)$.

The diagram

$$\begin{tikzcd} \bar{X}_\eta \arrow[r, "\iota_{\bar{X}}"] \arrow[d, "\bar{\pi}_\eta"] & \bar{X} \arrow[d, "\bar{\pi}"] \\ \eta \arrow[r, "\iota"] & B \end{tikzcd}$$

induces the base change homomorphism

$$\iota^* R^1\bar{\pi}_*\mu_l \to R^1\bar{\pi}_{\eta *}(\iota^*_{\bar{X}}\mu_l) = R^1\bar{\pi}_{\eta *}\mu_l,$$

which by adjointness corresponds to a homomorphism

$$\phi : R^1\bar{\pi}_*\mu_l \to \iota_* R^1\bar{\pi}_{\eta *}\mu_l$$

of sheaves on B. We have the following

Lemma 6.12: *Assume that for every closed point b of B the prime l does not divide the greatest common divisor of the multiplicities of the irreducible components of $\bar{X}_b$. Then the above defined morphism ϕ is injective and its cokernel is a skyscraper sheaf whose stalk at a geometric point $\bar{b}$ over a closed point b of B is given by*

$$coker(H^2_{\bar{X}_{\bar{b}}}(\bar{X}^{sh}_{\bar{b}}, \mathbb{Z}_l) \to H^2(\bar{X}^{sh}_{\bar{b}}, \mathbb{Z}_l))[l],$$

where $\bar{X}^{sh}_{\bar{b}}$ denotes the base change of $\bar{X}$ to the strict henselization of B at $\bar{b}$.

Proof: The stalk $\phi_{\bar{b}}$ of this homomorphism at a geometric point $\bar{b}$ over a closed point b of B is given by the composition

$$H^1(\bar{X}_{\bar{b}}, \mu_l) \xleftarrow{\sim} H^1(\bar{X}^{sh}_{\bar{b}}, \mu_l) \hookrightarrow H^1(\bar{X}_{\eta_{\bar{b}}}, \mu_l) \longrightarrow H^1(\bar{X}_{\bar{\eta}}, \mu_l)^{I_{\bar{b}}},$$

where $\bar{X}_{\eta_{\bar{b}}}$ denotes the pullback of $\bar{X}$ to the generic point $\eta_{\bar{b}}$ of the strict henselization $B^{sh}_{\bar{b}}$ of B at $\bar{b}$ and $I_{\bar{b}}$ the inertia group of $\bar{b}$. We have a diagram

$$\begin{tikzcd}
 & & H^1(\bar{X}^{sh}_{\bar{b}}, \mu_l) \arrow[d, hook] \arrow[dr, "\phi_{\bar{b}}"] & & \\
0 \arrow[r] & H^1(I_{\bar{b}}, \mu_l) \arrow[r] \arrow[dr] & H^1(\bar{X}_{\eta_{\bar{b}}}, \mu_l) \arrow[r] \arrow[d] & H^1(\bar{X}_{\bar{\eta}}, \mu_l)^{I_{\bar{b}}} \arrow[r] & 0 \\
 & & H^2_{\bar{X}_{\bar{b}}}(\bar{X}^{sh}_{\bar{b}}, \mu_l) \arrow[d, "\bar{\rho}"] & & \\
 & & H^2(\bar{X}^{sh}_{\bar{b}}, \mu_l), & &
\end{tikzcd}$$

where the horizontal sequence comes from the Hochschild-Serre spectral sequence for $I_{\bar{b}}$ and the vertical sequence is the excision sequence associated to $\bar{X}_{\bar{b}} \hookrightarrow \bar{X}^{sh}_{\bar{b}}$. Let π be a uniformizer of $B^{sh}_{\bar{b}}$. The group $H^1(I_{\bar{b}}, \mu_l)$ is generated by the class σ given by

$$\begin{aligned} I_{\bar{b}} &\to \mu_l \\ g &\mapsto \frac{g\sqrt[l]{\pi}}{\sqrt[l]{\pi}}. \end{aligned}$$

Via the purity isomorphism the image of σ in $H^2_{\bar{X}_{\bar{b}}}(\bar{X}^{sh}_{\bar{b}}, \mu_l)$ corresponds to the divisor of $\bar{X}^{sh}_{\bar{b}}$ given by the special fibre $\bar{X}_{\bar{b}} = \sum a_i D_i$ (note that $\bar{X}$ is regular). Since we assumed that l does not divide $gcd(a_i)$, this image is not zero and thus the morphism

$$H^1(I_{\bar{b}}, \mu_l) \to H^2_{\bar{X}_{\bar{b}}}(\bar{X}^{sh}_{\bar{b}}, \mu_l)$$

is injective. This also proves the injectivity of $\phi_{\bar{b}}$.

In order to deal with the cokernel consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & H^2_{\bar{X}_{\bar{b}}}(\bar{X}^{sh}_{\bar{b}}, \mathbb{Z}_l(1)) \arrow[r, "\cdot l"] \arrow[d, "\rho"] & H^2_{\bar{X}_{\bar{b}}}(\bar{X}^{sh}_{\bar{b}}, \mathbb{Z}_l(1)) \arrow[r] \arrow[d, "\rho"] & H^2_{\bar{X}_{\bar{b}}}(\bar{X}^{sh}_{\bar{b}}, \mu_l) \arrow[r] \arrow[d, "\bar{\rho}"] & 0 \\
0 \arrow[r] & H^2(\bar{X}^{sh}_{\bar{b}}, \mathbb{Z}_l(1)) \arrow[r, "\cdot l"] & H^2(\bar{X}^{sh}_{\bar{b}}, \mathbb{Z}_l(1)) \arrow[r] & H^2(\bar{X}^{sh}_{\bar{b}}, \mu_l) \arrow[r] & 0.
\end{tikzcd}$$

The snake lemma implies the exact sequence

$$0 \to ker(\rho)/l \to ker(\bar{\rho}) \to coker(\rho)[l] \to 0.$$

The kernel of ρ is given via purity by the special fibre $\bar{X}_{\bar{b}} = \sum a_i D_i$. By the above considerations its image in $ker(\bar{\rho})$ thus coincides with the image of

$$H^1(I_{\bar{b}}, \mu_l) \to H^2_{\bar{X}_{\bar{b}}}(\bar{X}^{sh}_{\bar{b}}, \mu_l).$$

We conclude that

$$coker(\phi_{\bar{b}}) \cong coker(\rho)[l],$$

which proves the lemma. □

For the rest of this section we keep the assumption that for every closed point b of B the prime l does not divide the greatest common divisor of the multiplicities of the irreducible components of $\bar{X}_b$. According to Lemma 6.12 we have a short exact sequence

$$0 \to R^1\bar{\pi}_*\mu_l \to \iota_* R^1\bar{\pi}_{\eta *}\mu_l \to \mathcal{F} \to 0 \tag{6.1}$$

with a skyscraper sheaf $\mathcal{F}$ with stalks $\mathcal{F}_{\bar{b}} = coker(\rho)[l]$, where ρ is the intersection matrix at $\bar{b}$. Our first step will be to examine the sheaf $\mathcal{F}$. Naturally, this involves considerations concerning the intersection of vertical prime divisors.

Lemma 6.13: *Possibly after replacing $(\bar{X}, \bar{D})$ by a desingularized l-covering of $(\bar{X}, \bar{D})$, for each geometric point $\bar{b}$ such that $\bar{X}_{\bar{b}}$ is singular the following holds: For every vertical component $\bar{Z}$ of $\bar{D}_{\bar{b}}$ we have $\bar{Z} \cdot W_{\bar{b}} \leqslant 1$. (Intersection products are taken in the base change of $\bar{X}$ to the strict henselization of B at $\bar{b}$.)*

Proof: We first show that after replacing $(\bar{X}, \bar{D})$ by a desingularized l-covering of $(\bar{X}, \bar{D})$ the following holds for every closed point (not necessarily a geometric point) b of B: For every vertical component Z of $\bar{D}$ with nontrivial intersection with W there is exactly one intersection point x with W and Z is isomorphic to $\mathbb{P}^1_{k(x)}$.

Since there are only finitely many singular fibres, we can treat each fibre $\bar{X}_b$ for $b \in B$ separately. Let $(\bar{X}', \bar{D}') \to (\bar{X}, \bar{D})$ denote the blowup of $\bar{X}$ in $W \cap D_v$, which is a finite set of closed points $x_1, \ldots, x_n$. In particular, it is a desingularized l-covering. For the exceptional fibre E_i corresponding to x_i we have $[k(x_i) : k(b)] = (W' \cdot E_i)$, where W' denotes the strict transform of W in $\bar{X}'$ and there is exactly one intersection point y_i of W' with E_i. Moreover, E_i is isomorphic to $\mathbb{P}^1_{k(x)}$.

Now we prove that after the above construction the assertion of the lemma holds for the geometric singular fibres. Choose a geometric point $\bar{b}$ over a closed point $b \in B$ such that $\bar{X}_{\bar{b}}$ is singular, which is equivalent to $\bar{X}_b$ being singular. Let $\bar{Z}$ be an irreducible component of $\bar{D}_{\bar{b}}$ with nontrivial intersection with $W_{\bar{b}}$. Its image Z in $\bar{X}_b$ is an irreducible component of $\bar{D}_b$ with nontrivial intersection with W. By the preparations in the preceding paragraph of the proof, Z thus equals E_i for some i. Hence, there is exactly one intersection point $x \in Z$ and Z is isomorphic to $\mathbb{P}^1_{k(x)}$. The base change of Z to the strict henselization of B at $\bar{b}$ is the disjoint union of $[k(x) : k(b)]$ copies of $\mathbb{P}^1_{k(\bar{b})}$ each of which intersects $W_{\bar{b}}$ transversally in exactly one point. Thus, $\bar{Z}$ is one of these copies of $\mathbb{P}^1_{k(\bar{b})}$ and the proof is complete. □

If $(\bar{X}', \bar{D}') \to (\bar{X}, \bar{D})$ is a desingularized l-covering, Lemma 6.12 also holds on $\bar{X}'$. We denote the resulting skyscraper sheaf by $\mathscr{F}'$.

Lemma 6.14: *Assume that $(\bar{X}, \bar{D})$ has enough tame coverings and that every irreducible component of W has nontrivial intersection with D_v. Then, for every geometric point $\bar{b}$ of B we have*

$$\varinjlim_{\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} \mathscr{F}'_{\bar{b}} = 0.$$

Proof: We will prove in Lemma 7.2 that for any desingularized l-covering $(\bar{X}', \bar{D}') \to (\bar{X}, \bar{D})$ the pair $(\bar{X}', \bar{D}')$ has again enough tame coverings. Moreover, it is easy to see that the condition on singular fibres is also stable under desingularized l-coverings. It therefore suffices to show that $\mathscr{F}_{\bar{b}}$ vanishes in the limit over $\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}$.

Let $\bar{b}$ be a geometric point of B such that $\bar{X}_{\bar{b}}$ is singular (if $\bar{X}_{\bar{b}}$ is nonsingular, $\mathscr{F}_{\bar{b}} = 0$, anyway). An element of $\mathscr{F}_{\bar{b}} = coker(\rho_{\bar{b}})[l]$ is represented by a divisor $Z = \sum n_C C$ with support in $\bar{X}_{\bar{b}}$ such that there is a divisor A with support in $\bar{X}_{\bar{b}}$ such that

$$lZ = \rho_{\bar{b}}(A).$$

Write $A = A_W + A_D$ with *supp* $A_D \subseteq D_{\bar{b}}$ and *supp* $A_W \subseteq W_{\bar{b}}$. By Lemma 6.8 and Remark 6.9 there is a desingularized l-covering $(\bar{X}', \bar{D}') \to (\bar{X}, \bar{D})$ such that the image of A_D in $H^2_{\bar{X}'_{\bar{b}}}(\bar{X}'_{B'^{sh}_{\bar{b}}}, \mathbb{Z}_l(1))$ is divisible by l. On $\bar{X}'$ we have

$$\rho'_{\bar{b}}(A'_W) = l(Z' - \rho'_{\bar{b}}(A'_D)), \tag{6.2}$$

where A'_W is the pullback of A_W to $\bar{X}'_{\bar{b}}$, A'_D is the pullback of A_D to $\bar{X}'_{\bar{b}}$ divided by l and Z' is the image of Z in $H^2((\bar{X}')_{B'^{sh}_{\bar{b}}}, \mathbb{Z}_l(1))$. By hypothesis and by Lemma 6.13, we may assume that for each irreducible component C' of $W'_{\bar{b}}$ there is an irreducible component $D'_{C'}$ of $\bar{D}'_{\bar{b}}$ such that $(C' \cdot D'_{C'}) = 1$ and $(Z' \cdot D'_{C'}) = 0$ for all other components Z' of $W'_{\bar{b}}$. Write

$$A'_W = \sum a_{C'} C',$$

where the sum is over all irreducible components C' of $W'_{\bar{b}}$. The coefficient of $D'_{C'}$ in $\rho'_{\bar{b}}(A'_W)$ is $a_{C'} \cdot (D'_{C'} \cdot C') = a_{C'}$. By equation (6.2) it is divisible by l. In total we get $l | A'_W$, which concludes the proof of the lemma. $\square$

Corollary 6.15: *Under the assumptions of Lemma 6.14, we have*

$$\varinjlim_{\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} H^0(B', \mathcal{F}') = 0.$$

We need a slightly stronger version of Lemma 6.14, which states that the lemma also holds after base change by a possibly ramified l-extension $B_0 \to B$. Unfortunately, its proof is quite laborious and, even more annoyingly, it forces us to assume that W is regular. Let us set up notation. Suppose that W is regular. Let $\bar{b}_0$ be a geometric point above a closed point b_0 of B such that $\bar{X}_{\bar{b}_0}$ is singular. Let $B_0 \to B^{sh}_{\bar{b}_0}$ be a finite, local l-extension, i. e., B_0 is the normalization of B in some l-extension of the function field of $B^{sh}_{\bar{b}_0}$. Note that $B_0 \to B^{sh}_{\bar{b}_0}$ is purely tamely ramified. The base change of $\bar{X}$ to B_0 becomes singular at the special points of $\bar{X}_{\bar{b}_0}$. Since W is regular, these points are all contained in $\bar{D}_0$. Let $(\bar{X}_0, \bar{D}_0) \to (\bar{X} \times_B B_0, \bar{D} \times_B B_0)$ be a tidy desingularization. Define U_0, D_0, etc. as on $\bar{X}$.

The composite $(\bar{X}_0, \bar{D}_0) \to (\bar{X} \times_B B^{sh}_{\bar{b}_0}, \bar{D} \times_B B^{sh}_{\bar{b}_0})$ is not a desingularized l-covering as it is not étale over U. But by definition $(\bar{X}_0, Z_0) \to (\bar{X} \times_B B^{sh}_{\bar{b}_0}, \bar{X}_{\bar{b}_0})$ is a desingularized l-covering, where Z_0 is the preimage of $\bar{X}_{\bar{b}_0}$ in $\bar{X}_0$ (it is a multiple of the special fibre of $\bar{X}_0/B_0$).

Denote by K the union of the vertical irreducible components of $\bar{D}$ with nontrivial intersection with W and by K_0 its generalized strict transform in $\bar{X}_0$. Similarly, let W_0 be the generalized strict transform of W. By Proposition 4.19 the preimages in $\bar{X}_0$ of the intersection points of K with W are bridges of $\mathbb{P}^1$'s connecting K_0 with W_0. We denote their union by P_0. In other words, P_0 is the union of all exceptional fibres in $\bar{X}_0$ of intersection points of D_v with W.

Lemma 6.12 also holds on $\bar{X}_0$. We denote the resulting skyscraper sheaf by $\mathcal{F}_0$.

Lemma 6.16: *The sheaf $\mathcal{F}_0$ is independent of the choice of tidy desingularization $(\bar{X}_0, \bar{D}_0) \to (\bar{X} \times_B B_0, \bar{D} \times_B B_0)$.*

Proof: We have to show that blowing up $\bar{X}_0$ at a special point of $\bar{D}_0$ does not change $\mathcal{F}_0$. Consider the short exact sequence

$$0 \to R^1(\bar{\pi}_0)_* \mu_l \to \iota_* R^1(\bar{\pi}_0)_{\eta_0 *} \mu_l \to \mathcal{F}_0 \to 0$$

defining $\mathcal{F}_0$ (it is the analogue of sequence (6.1) on B_0). Since B_0 is strictly henselian, this sequence reads

$$0 \to H^1(\bar{X}_0, \mu_l) \to H^1((\bar{X}_0)_{\eta_0}, \mu_l)^{I_0} \to \mathcal{F}_0(B_0) \to 0,$$

where I_0 denotes the inertia group in the absolute Galois group of η_0. The group $H^1((\bar{X}_0)_{\eta_0}, \mu_l)^{I_0}$ is independent of the special fibre by definition. Furthermore, $H^1(\bar{X}_0, \mu_l)$ does not change with a blowup in a closed point as the exceptional fibre is rational. We conclude that also $\mathcal{F}_0$ is invariant under blowups in closed points. $\square$

If $(\bar{X}', \bar{D}') \to (\bar{X}, \bar{D})$ is a desingularized l-covering, we can construct a tidy desingularization $(\bar{X}'_0, \bar{D}'_0) \to (\bar{X}' \times_B B_0, \bar{D}' \times_B B_0)$ fitting into the diagram

$$\begin{array}{ccc}(\bar{X}'_0, \bar{D}'_0) & \longrightarrow & (\bar{X}', \bar{D}') \\ \downarrow & & \downarrow \\ (\bar{X}_0, \bar{D}_0) & \longrightarrow & (\bar{X}, \bar{D}).\end{array}$$

We obtain a morphism of sheaves

$$\mathscr{F}_0 \to \mathscr{F}'_0,$$

where $\mathscr{F}'_0$ is defined as $\mathscr{F}_0$ but with $(\bar{X}'_0, \bar{D}'_0)$ instead of $(\bar{X}_0, \bar{D}_0)$. By Lemma 6.16 the limit

$$\varinjlim_{\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} \mathscr{F}'_0$$

is well defined. Our goal is to prove that this limit vanishes.

If $(\bar{X}, \bar{D})$ has enough tame coverings, it is not clear whether the same holds for $(\bar{X}_0, \bar{D}_0)$ but we have the following partial result:

Lemma 6.17: *Let p_0 be a closed point of $\bar{D}_0$ which is not contained in P_0. Then, for every irreducible component Z_0 of $\bar{D}_0$ passing through p_0 there is $f \in K(\bar{X})^\times$ with support in $\bar{D}_0$ such that $deg_{Z_0}(f) > 0$ and $deg_{C_0}(f) = 0$ for any other prime divisor C_0 passing through p_0.*

Note that the function f is required to be contained not only in $K(\bar{X}_0)^\times$ but in $K(\bar{X})^\times$.

Proof: Let Z_0 be an irreducible component of $\bar{D}_0$ passing through p_0. Denote by $Z_1, \ldots, Z_n$ (for $n = 1$ or $n = 2$) the irreducible components of $\bar{D}$ passing through the image point $p \in \bar{D}$ of p_0. Since p is not contained in W, these are also the irreducible components of $W + \bar{D}$ passing through p. Since $(\bar{X}, \bar{D})$ has enough tame coverings, for $i = 1, \ldots, n$ there is $f_i \in K(\bar{X})^\times$ such that $deg_{Z_i}(f_i) > 0$ and $deg_{Z_j}(f_i) = 0$ for $i \neq j$. The projections of $div\ f_i$ to

$$\mathbb{Q} \cdot Z_1 \oplus \ldots \oplus \mathbb{Q} \cdot Z_n.$$

constitute a basis of this vector space. Let $Z_0 = Z_{0,1}, \ldots, Z_{0,m}$ denote the irreducible components of $\bar{D}_0$ passing through p_0. These are also the irreducible components of $W_0 + \bar{D}_0$ passing through p_0. As in section 4.5 we assign to $(\bar{X}_0, W_0 + \bar{D}_0) \to (\bar{X}, W + \bar{D})$ the multiplicity homomorphism

$$\phi_{p_0} : \mathbb{Q} \cdot Z_1 \oplus \ldots \oplus \mathbb{Q} \cdot Z_n \to \mathbb{Q} \cdot Z_{0,1} \oplus \ldots \oplus \mathbb{Q} \cdot Z_{0,m}$$

at p_0 induced by pullback. By Lemma 4.24 all multiplicity homomorphisms of a desingularized l-covering are surjective. Therefore, there is a linear combination of $div\ f_1, \ldots, div\ f_n$ with coefficients in $\mathbb{Q}$ mapping to $Z_{0,1} = Z_0$ under the multiplicity homomorphism. Clearing denominators we obtain

$$d \cdot Z_0 = \phi_{p_0}(k_1 div\ f_1 + \ldots k_n div\ f_n)$$

with integers $d, k_1, \ldots, k_n$ such that $d > 0$. In other words, setting $f = f_1^{k_1} \cdot \ldots f_n^{k_n}$ we have in a neighborhood of p_0

$$div\ f = d \cdot Z_0.$$

This is what we wanted to prove. $\square$

Corollary 6.18: *Let Z be a vertical divisor on $\bar{X}_0$ and $p_0 \in Z$ a closed point which is not contained in $P_0 + W_0$. There is a desingularized l-covering $(\bar{X}', \bar{D}') \to (\bar{X}, \bar{D})$ such that in the pullback of Z to $\bar{X}'_0$ the prime number l divides all coefficients of prime divisors whose image in $\bar{X}_0$ contain p_0.*

Proof: Using Lemma 6.17, we find $f \in K(\bar{X})^\times$ such that locally at p_0 we have

$$div\ f = dZ$$

with a positive integer d. Write $d = l^n d'$ such that l does not divide d'. The normalization of $\bar{X}$ in $K(\bar{X})[\sqrt[l^{n+1}]{f}]$ does the job. $\square$

Lemma 6.19: *Let Y/S be a proper, regular arithmetic surface over a strictly henselian Dedekind scheme S. Let $A \subseteq Y$ be a connected, tidy, vertical divisor with at least two irreducible components whose dual graph Γ_A is simply connected. Denote by $A_1, \dots, A_r$ (possibly $r = 0$) the irreducible components of A which have more than one intersection point with other irreducible components of A and by $A_{r+1}, \dots, A_s$ the remaining irreducible components of A. Then the intersection matrix*

$$\rho : A_1 \cdot \mathbb{F}_l \oplus \ldots \oplus A_{r+1} \cdot \mathbb{F}_l \to A_1 \cdot \mathbb{F}_l \oplus \ldots \oplus A_s \cdot \mathbb{F}_l$$

is injective.

Proof: If A has only two irreducible components A_1 and A_2, the intersection matrix is

$$\begin{pmatrix} A_1^2 \\ A_1 \cdot A_2 \end{pmatrix} = \begin{pmatrix} A_1^2 \\ 1 \end{pmatrix},$$

which is injective independently of the value of A_1^2. Assume now that $s > 2$. Let $a = \sum_{i=1}^{r+1} a_i A_i$ be an element in the kernel of ρ. Since Γ_A is simply connected, the number of components of A with only one intersection point is at least two, i. e., $s > r + 1$. Hence, A_s intersects only one other irreducible component A_i of A. We have $i \leqslant r$ because otherwise $s = 2$. Without loss of generality we may assume $i = 1$. The coefficient of A_s in $\rho(a)$ is a_1, whence $a_1 = 0$. Thus, a is in the kernel of the intersection matrix

$$\rho' : A_2 \cdot \mathbb{F}_l \oplus \ldots \oplus A_{r+1} \cdot \mathbb{F}_l \to A_2 \cdot \mathbb{F}_l \oplus \ldots \oplus A_s \cdot \mathbb{F}_l,$$

and the lemma follows by induction on s. □

Since $(\bar{X}_0, \bar{D}_0)$ does not necessarily have enough tame coverings in $p \in P_0$, it is slightly more complicated to kill elements of the cokernel of the intersection matrix. However, this is not the crucial point. The biggest problem is the following: Let $(\bar{X}_1, W_1) \to (\bar{X}_0, W_0)$ be a tidy desingularization. Then we cannot force the dual graph of W_1 to be simply connected except by requiring W to be regular. This is the reason why we assume in the next lemma that W is regular.

Lemma 6.20: *Assume W is regular and every irreducible component of W has nontrivial intersection with K. Suppose further that for every irreducible component K_i of $K_{\bar{b}_0}$ we have $K_i \cdot W_{\bar{b}_0} = 1$. Then*

$$\varinjlim_{\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} \mathscr{F}_0' = 0.$$

Proof: Let $(\bar{X}', \bar{D}') \to (\bar{X}, \bar{D})$ be a desingularized l-covering. Then also W' is regular and every irreducible component of W' has nontrivial intersection with K' by Lemma 7.2. Furthermore, by Lemma 6.13 we may assume that the statement concerning the intersection product also holds on $\bar{X}'$. Hence, it suffices to show that $\mathscr{F}_0$ vanishes in the limit over $\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}$.

As in the proof of Lemma 6.14 let $Z = \sum n_C C$ be a vertical divisor of $\bar{X}_0$ such that there is a vertical divisor A of $\bar{X}_0$ with

$$lZ = \rho_{\bar{b}_0}(A).$$

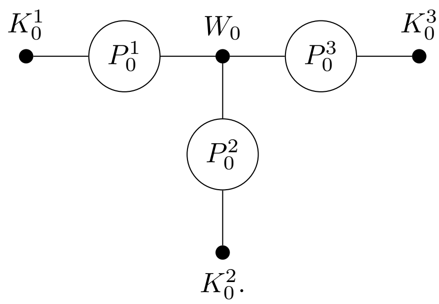
Write $A = A_K + A_W + A_P + A_R$ with *supp* $A_K \subseteq K_0$, *supp* $K_W \subseteq W_0$, *supp* $A_P \subseteq P_0$ and such that K_R has support in the union R_0 of the remaining irreducible components of the special fibre of $\bar{X}_0$. If $(\bar{X}', \bar{D}') \to (\bar{X}, \bar{D})$ is a desingularized l-covering, we write $A' = A'_K + A'_W + A'_P + A'_R$ for the analogous decomposition of the pullback of A to $\bar{X}'_0$. Let $T_0 \subseteq \bar{D}_0 \backslash P_0$ be a finite set of closed points containing all special points of $K_0 + R_0$ and such that each irreducible component of $K_0 + R_0$ contains at least one point in T_0. By Corollary 6.18, for each $p_0 \in T_0$ we find a desingularized l-covering $(\bar{X}', \bar{D}') \to (\bar{X}, \bar{D})$ such that l divides the coefficients in A' of all vertical prime divisors whose image in $\bar{X}$ contain p_0. Replacing $(\bar{X}, \bar{D})$ by a desingularized l-covering which dominates the just constructed coverings for all points $p_0 \in T_0$ we may assume that A_R and A_K are divisible by l. We can thus write

$$l(Z - A_K/l - A_R/l) = \rho_{\bar{b}_0}(A_W + A_P).$$

Changing notation we may assume that A has support in $W_0 + P_0$.

Blowing up in special points of K we may assume that K is regular. Let C be a connected component of $W_0 + P_0 + K_0$. There is precisely one irreducible component of C that is contained in W_0 as W and K are regular and $K_i \cdot W_{\bar{b}_0} = 1$ for every irreducible component K_i of $K_{\bar{b}_0}$. To this irreducible component in W_0 are attached several (at least one by assumption) bridges of $\mathbb{P}^1$'s connecting the said irreducible component of W_0 with an irreducible component of K_0. By our hypothesis on the intersection number of components of K and W, each component of K_0 is connected to W_0 by precisely one bridge of $\mathbb{P}^1$'s. The dual graph of C is thus of the form

C:



The vertex labeled W_0 denotes the irreducible component of C contained in W_0, K_0^1, K_0^2, K_0^3 are irreducible components of K_0, and P_0^1, P_0^2, P_0^3 denote bridges of $\mathbb{P}^1$'s in P_0. Of course, the number of bridges of $\mathbb{P}^1$'s in C does not have to be equal to three as in the above figure. In particular, the dual graph of C is simply connected. If there is more than one bridge of $\mathbb{P}^1$'s, the irreducible components of C with exactly one intersection point with the other irreducible components of C are precisely the irreducible components of K_0 in C. If there is only one bridge of $\mathbb{P}^1$'s, the components of C with only one intersection point are the irreducible component of W_0 and the irreducible component of K_0 contained in C. In both cases we obtain from Lemma 6.19 that the intersection matrix modulo l

$$\bar{\rho}_{\bar{b}_0} : \bigoplus_{C_i \subseteq C, C_i \nsubseteq K_0} \mathbb{F}_l \cdot C_i \longrightarrow \bigoplus_{C_i \subset C} \mathbb{F}_l \cdot C_i$$

is injective, where the direct sums run over irreducible components of C. We conclude that $A = 0$ *mod* l, i. e., A is divisible by l. Therefore, Z is contained in the image of the intersection matrix and defines a trivial element of $\mathcal{T}_0' = coker(\rho_{\bar{b}_0})[l]$. □

We will need Lemma 6.20 in order to prove that $H^1(B, \iota_* R^1 \bar{\pi}_{\eta *} \mu_l)$ vanishes in the limit over all desingularized l-coverings of $(\bar{X}, \bar{D})$. Before we can do so we need two more elementary lemmas.

Lemma 6.21: *Let $\mathcal{G}$ be a profinite group, $\mathcal{H}_1$ and $\mathcal{H}_2$ two closed subgroups and M a discrete $\mathcal{G}$-module. Suppose that $\mathcal{H}_1$ acts trivially on M. If the restriction of a class $\phi \in H^1(\mathcal{G}, M)$ to both $H^1(\mathcal{H}_1, M)$ as well as to $H^1(\mathcal{H}_2, M)$ is zero, its restriction to $H^1(\mathcal{H}_1\mathcal{H}_2, M)$ is zero. Here, $\mathcal{H}_1\mathcal{H}_2$ denotes the closed subgroup of $\mathcal{G}$ which is topologically generated by $\mathcal{H}_1$ and $\mathcal{H}_2$.*

Proof: Without loss of generality we may assume that $\mathcal{G} = \mathcal{H}_1\mathcal{H}_2$. Let x be a 1-cocycle representing ϕ. By assumption

$$\begin{aligned} x(h_1) &= 0 \qquad && \forall h_1 \in \mathcal{H}_1 \qquad \text{and} \\ \exists m \in M \text{ such that } x(h_2) &= h_2 m - m && \forall h_2 \in \mathcal{H}_2. \end{aligned}$$

Since $\mathcal{H}_1$ acts trivially on M, we can thus write

$$x(g) = gm - m$$

for all $g \in \mathcal{H}_1 \cup \mathcal{H}_2$. Because a cocycle is determined by its values on a generating subset of $\mathcal{G}$, the above equation holds for all $g \in \mathcal{G}$. □

Lemma 6.22: *The natural homomorphism*

$$H^1(\bar{X}_{\bar{b}}, \mu_l) \to H^1((\bar{X}_0)_{\bar{b}}, \mu_l)$$

is an isomorphism.

Proof: The stated homomorphism decomposes as follows:

$$H^1(\bar{X}_{\bar{b}}, \mu_l) \to H^1((\bar{X} \times_B B_0)_{\bar{b}}, \mu_l) \to H^1((\bar{X}_0)_{\bar{b}}, \mu_l).$$

The left hand homomorphism is an isomorphism as $\bar{X}_{\bar{b}} = (\bar{X} \times_B B_0)_{\bar{b}}$. Since the singularities of $\bar{X} \times_B B_0$ are rational by Proposition 4.19, the right hand homomorphism is also an isomorphism. □

We are interested in the following part of the long exact cohomology sequence associated to the short exact sequence (6.1):

$$\ldots \to H^0(B, \mathcal{F}) \to H^1(B, R^1\bar{\pi}_*\mu_l) \to H^1(B, \iota_* R^1\bar{\pi}_{\eta *}\mu_l) \to \ldots. \tag{6.3}$$

By Corollary 6.15 the limit over $\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}$ of $H^0(B', \mathcal{F}')$ vanishes. In order to show that the cohomology group $H^1(B, R^1\bar{\pi}_*\mu_l)$ vanishes in the limit it thus suffices to prove:

Lemma 6.23: *Assume that the following conditions are satisfied:*

(i) The scheme W is regular.

(ii) The action of $\mathcal{G}_\eta$ on the l-division points of the Jacobian of $X_{\bar{\eta}}$ factors through an l-primary quotient.

(iii) Every irreducible component of W has nontrivial intersection with D_v.

(iv) The pair $(\bar{X}, \bar{D})$ has enough tame coverings.

(v) Let $T \subset B$ be a finite (possibly empty) set of closed points. Denote by $\tilde{B}^l$ the universal l-covering of B and by $\tilde{T}^l$ the preimage of T in $\tilde{B}^l$. Then

$$\pi_1(\tilde{B}^l - \tilde{T}^l)(l) \cong \mathop{\ast}_{\bar{s} \in \tilde{T}^l} I_{\bar{s}}(l).$$

Then

$$H^1(B, \iota_* R^1\bar{\pi}_{\eta *}\mu_l)$$

vanishes in the limit over $\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}$.

Proof: The five term exact sequence of the Leray spectral sequence associated to $\iota : \eta \to B$ reads

$$0 \to H^1(B, \iota_* R^1\bar{\pi}_{\eta *}\mu_l) \to H^1(\eta, R^1\bar{\pi}_{\eta *}\mu_l) \to H^0(B, R^1\iota_*(R^1\bar{\pi}_{\eta *}\mu_l)) \to \ldots.$$

We have

$$H^1(\eta, R^1\bar{\pi}_{\eta *}\mu_l) = H^1(\mathcal{G}_\eta, H^1(\bar{X}_{\bar{\eta}}, \mu_l)),$$

and $H^0(B, R^1\iota_*(R^1\bar{\pi}_{\eta *}\mu_l))$ is the group of global sections of the sheaf associated to the presheaf

$$(B' \to B) \mapsto \prod_{\eta' \to B} H^1(\eta', R^1\bar{\pi}_{\eta *}\mu_l) = \prod_{\eta' \to B} H^1(\mathcal{G}_{\eta'}, H^1(\bar{X}_{\bar{\eta}}, \mu_l)),$$

where the product runs over all generic points η' of B' and $\mathcal{G}_{\eta'}$ denotes the absolute Galois group of η'. We view these groups as subgroups of the absolute Galois group G_η of the generic point η of B. A class $\varphi \in H^1(\mathcal{G}_\eta, H^1(\bar{X}_{\bar{\eta}}, \mu_l))$ maps to zero in $H^0(B, R^1\iota_*(R^1\bar{\pi}_{\eta*}\mu_l))$ if and only if there is an étale cover $B' \to B$ such that the restriction of φ to $H^1(\mathcal{G}_{\eta'}, H^1(\bar{X}_{\bar{\eta}}, \mu_l))$ vanishes for all generic points η' of B'. This is the case if and only if the restriction of φ to the inertia group of each geometric point of B vanishes.

Let $\eta_1|\eta$ be the minimal extension of η such that $H^1(\bar{X}_{\bar{\eta}}, \mu_l)$ is a trivial $\mathcal{G}_{\eta_1}$-module. It is finite Galois and by assumption it is an l-extension. The cohomology group $H^1(\mathcal{G}_{\eta_1}, H^1(\bar{X}_{\bar{\eta}}, \mu_l))$ parameterizes finite l-extensions of η_1 and vanishes automatically in the limit over all such extensions. We conclude that there is a finite l-extension $\eta''|\eta$ such that

$$H^1(\mathcal{G}_\eta, H^1(\bar{X}_{\bar{\eta}}, \mu_l)) \to H^1(\mathcal{G}_{\eta''}, H^1(\bar{X}_{\bar{\eta}}, \mu_l))$$

is the zero map.

Denote by $T \subseteq B$ the finite set of closed points p such that $\bar{X}_b$ is singular. Then $\bar{X}_\eta$ has good reduction at all closed points in $B - T$. Hence, by smooth base change (see [SGA4.5], Exp. V, Théorème 3.1), the action of $\mathcal{G}_\eta$ on $H^1(X_{\bar{\eta}}, \Lambda)$ factors through $\pi_1(B - T, \bar{\eta})$. In other words, the extension $\eta_1|\eta$ is unramified in $B - T$. Let φ be an element of the kernel of

$$H^1(\mathcal{G}_\eta, H^1(\bar{X}_{\bar{\eta}}, \mu_l)) \to H^0(B, R^1\iota_*(R^1\bar{\pi}_{\eta*}\mu_l)).$$

In particular, φ is in the kernel of

$$H^1(\mathcal{G}_\eta, H^1(\bar{X}_{\bar{\eta}}, \mu_l)) \to H^0(B - T, R^1\iota_*(R^1\bar{\pi}_{\eta*}\mu_l)).$$

By the above description there is an étale cover $B' \to B - T$ such that the restriction of φ to $H^1(\mathcal{G}_{\eta'}, H^1(\bar{X}_{\bar{\eta}}, \mu_l))$ vanishes for all generic points η' of B'. Without loss of generality we may assume that $\eta_1|\eta$ is a subextension of each $\eta'|\eta$. Then $\mathcal{G}_{\eta'}$ acts trivially on $H^1(\bar{X}_{\bar{\eta}}, \mu_l)$ and by repeated use of Lemma 6.21 we find a common subextension $\eta_2|\eta$ of the extensions $\eta'|\eta$ and of $\eta''|\eta$ such that the restriction of φ to $\mathcal{G}_{\eta_2}$ is trivial. The extension $\eta_2|\eta$ must necessarily be an l-extension which is unramified in $B - T$. We conclude that φ can be lifted to an element $\tilde{\varphi}$ of

$$H^1((\pi_1(\tilde{B}^l - \tilde{T}^l))(l), H^1(\bar{X}_{\bar{\eta}}, \mu_l)),$$

where $\tilde{B}^l$ denotes the universal l-covering of B and $\tilde{T}^l$ the preimage of T in $\tilde{T}^l$. In order to prove that φ vanishes in the limit over all desingularized l-coverings of $(\bar{X}, \bar{D})$ it suffices to show that $\tilde{\varphi}$ vanishes in this limit. By assumption we have

$$\pi_1(\tilde{B}^l - \tilde{T}^l)(l) \cong \mathop{\ast}_{s \in \tilde{T}^l} I_s(l).$$

The first cohomology group of such a free pro-l-product is described by the following exact sequence (see [NSW], Theorem 4.3.14):

$$\begin{aligned} 0 \to H^1(\bar{X}_{\bar{\eta}}, \mu_l)^{\pi_1(\tilde{B}^l - \tilde{T}^l)(l)} \to H^1(\bar{X}_{\bar{\eta}}, \mu_l) \to \bigoplus_{s \in \tilde{T}^l} H^1(\bar{X}_{\bar{\eta}}, \mu_l)/H^1(\bar{X}_{\bar{\eta}}, \mu_l)^{I_s(l)} \to \\ \to H^1(\pi_1(\tilde{B}^l - \tilde{T}^l)(l), H^1(\bar{X}_{\bar{\eta}}, \mu_l)) \to \bigoplus_{s \in \tilde{T}^l} H^1(I_s(l), H^1(\bar{X}_{\bar{\eta}}, \mu_l)) \to 0. \end{aligned}$$

Note that in the statement of Theorem 4.3.14 of [NSW] there occurs a modified direct sum denoted $\oplus'$ instead of the usual direct sum $\oplus$. However, as explained in the proof of the cited Theorem 4.3.14, the modified direct sum

$$\bigoplus_{s \in \tilde{T}^l}{}' H^1(I_s(l), H^1(\bar{X}_{\bar{\eta}}, \mu_l))$$

coincides with the usual direct sum as the module $H^1(\bar{X}_{\bar{\eta}}, \mu_l)$ is finite. The same reasoning applies for $\bigoplus_{s\in\tilde{T}^l} H^1(\bar{X}_{\bar{\eta}}, \mu_l)/H^1(\bar{X}_{\bar{\eta}}, \mu_l)^{I_s(l)}$.

The group $I_s(l)$ is the maximal l-quotient of the inertia group of $\mathcal{G}_\eta$ at s. As we have seen above, the restriction of φ to the inertia group of $\mathcal{G}_\eta$ at s vanishes and hence, so does the restriction of $\tilde{\varphi}$ to $I_s(l)$. We conclude that $\tilde{\varphi}$ lifts to an element of $\bigoplus_{s\in\tilde{T}^l} H^1(\bar{X}_{\bar{\eta}}, \mu_l)/H^1(\bar{X}_{\bar{\eta}}, \mu_l)^{I_s(l)}$. Therefore, it suffices to prove that for a fixed geometric point s of B lying over T the quotient $H^1(\bar{X}_{\bar{\eta}}, \mu_l)/H^1(\bar{X}_{\bar{\eta}}, \mu_l)^{I_s(l)}$ vanishes in the limit over $\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}$.

Consider the sequence of stalks at s associated with the short exact sequence (6.1):

$$0 \to H^1(\bar{X}_s, \mu_l) \to H^1(\bar{X}_{\bar{\eta}}, \mu_l)^{I_s(l)} \to \mathcal{F}'_s \to 0.$$

We would like to have a similar sequence with $H^1(\bar{X}_{\bar{\eta}}, \mu_l)$ in the middle. Let $B_1 \to B$ be the normalization of B in η_1. Since we assumed W to be regular, the singularities of $\bar{X} \times_B B_1$ are all contained in $\bar{D} \times_B B_1$ (see Lemma 4.2). Let $(\bar{X}', \bar{D}') \to (\bar{X} \times_B B1, \bar{D} \times_B B_1)$ be a tidy desingularization. Then Lemma 6.12 also holds on $\bar{X}'$ and we obtain a short exact sequence

$$0 \to R^1\bar{\pi}'_*\mu_l \to \iota'_* R^1\bar{\pi}'_{\eta_1 *}\mu_l \to \mathcal{F}' \to 0.$$

Taking stalks at s and using functoriality yields a diagram of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(\bar{X}_s, \mu_l) & \longrightarrow & H^1(\bar{X}_{\bar{\eta}}, \mu_l)^{I_s(l)} & \longrightarrow & \mathcal{F}_s & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1(\bar{X}'_s, \mu_l) & \longrightarrow & H^1(\bar{X}_{\bar{\eta}}, \mu_l) & \longrightarrow & \mathcal{F}'_s & \longrightarrow & 0.
\end{array}$$

Note that by setting $B_0 = (B_1)^{sh}_s$ we are in the situation described after Lemma 6.15 and $\mathcal{F}'_s$ equals $\mathcal{F}_0(B_0)$. By Lemma 6.22 we have

$$H^1(\bar{X}_s, \mu_l) = H^1(\bar{X}'_s, \mu_l).$$

Lemma 6.13 allows us to assume that for every vertical component $\bar{Z}$ of $\bar{D}_{\bar{b}}$ we have $\bar{Z} \cdot W_{\bar{b}} \leqslant 1$. Then, by Lemma 6.20, $\mathcal{F}_s$ as well as $\mathcal{F}'_s$ vanish in the limit over $\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}$. Hence, for every element $\xi \in H^1(\bar{X}_{\bar{\eta}}, \mu_l)$ there is a desingularized l-covering $(\bar{X}'', \bar{D}'') \to (\bar{X}, \bar{D})$ such that the image of ξ in $H^1(\bar{X}''_{\bar{\eta}}, \mu_l)$ is contained in $H^1(\bar{X}''_{\bar{\eta}}, \mu_l)^{I''_s(l)}$. We conclude that the direct sum $\bigoplus_{s\in\tilde{T}^l} H^1(\bar{X}_{\bar{\eta}}, \mu_l)/H^1(\bar{X}_{\bar{\eta}}, \mu_l)^{I_s(l)}$ vanishes in the limit over $\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}$. Therefore, the cohomology class $\tilde{\varphi}$, which is an element of

$$ker(H^1(\pi_1(\tilde{B}^l - \tilde{T}^l)(l), H^1(\bar{X}_{\bar{\eta}}, \mu_l)) \to \bigoplus_{s\in\tilde{T}^l} H^1(I_s(l), H^1(\bar{X}_{\bar{\eta}}, \mu_l))),$$

vanishes in the limit and the same holds for the class φ in

$$H^1(B, \iota_* R^1\bar{\pi}_{\eta *}\mu_l) = ker(H^1(\eta, R^1\bar{\pi}_{\eta *}\mu_l) \to H^0(B, R^1\iota_*(R^1\bar{\pi}_{\eta *}\mu_l))).$$

This concludes the proof of the lemma. □

We can now collect the results of this section to prove

Proposition 6.24: *Assume that the following conditions are satisfied:*

(i) The class $\mathfrak{c}$ is the class of finite l-groups for a prime number l.

(ii) The scheme $\bar{X}$ as well as W is regular.

(iii) For every closed point b of B the prime l does not divide the greatest common divisor of the multiplicities of the irreducible components of $\bar{X}_b$.

(iv) The action of $\mathcal{G}_\eta$ on the l-division points of the Jacobian of $X_{\bar{\eta}}$ factors through an l-primary quotient.

(v) Every irreducible component of W has nontrivial intersection with D_v.

(vi) The pair $(\bar{X}, \bar{D})$ has enough tame coverings.

(vii) Let $T \subset B$ be a finite (possibly empty) set of closed points. Denote by $\tilde{B}^l$ the universal l-covering of B and by $\tilde{T}^l$ the preimage of T in $\tilde{B}^l$. Then

$$\pi_1(\tilde{B}^l - \tilde{T}^l)(l) \cong \mathop{\ast}_{\bar{s} \in \tilde{T}^l} I_{\bar{s}}(l).$$

Then

$$H^1(B, R^1\pi_*\Lambda)$$

vanishes in the limit over $\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}$.

Remember that hypotheses (i) and (iii) and the assertion that $\bar{X}$ is regular are required to hold throughout this section. This is the reason why they do not appear in the statement of Lemma 6.23. However, we included these conditions in Proposition 6.24 for later reference.

Proof: We choose an isomorphism $\Lambda \cong \mu_l$ and show the proposition with Λ replaced with μ_l. This allows a more natural interpretation of the cohomology groups involved. At first we show that we may replace X with $\bar{X}$ and π with $\bar{\pi}$. Consider the excision sequence

$$\ldots \to R^1\bar{\pi}_*\mu_l \to R^1\pi_*\mu_l \to R^2_{\bar{D}_h}\bar{\pi}_*\mu_l \to \ldots.$$

The sheaf $R^2_{\bar{D}_h}\bar{\pi}_*\mu_l$ is the sheaf associated to the presheaf

$$(B' \to B) \mapsto H^2_{\bar{D}_h \times_B B'}(\bar{X} \times_B B', \mu_l).$$

Let $(\bar{X}', \bar{D}') \to (\bar{X}, \bar{D})$ be a desingularized l-covering such that the ramification index of each irreducible component of $\bar{D}'_h$ is divisible by l. Such a covering exists as $(\bar{X}, \bar{D})$ has enough tame coverings. By Proposition 5.17

$$H^2_{\bar{D}_h \cap \bar{\pi}^{-1}(U)}(\bar{X} \times_B B', \mu_l) \to H^2_{\bar{D}'_h \times_B B'}(\bar{X} \times_B B', \mu_l)$$

is the zero map. Since this holds for all étale $B' \to B$, also

$$R^2_{\bar{D}_h}\bar{\pi}_*\mu_l \to R^2_{\bar{D}'_h}\bar{\pi}'_*\mu_l$$

is the zero map and thus the image of $R^1\pi_*\mu_l$ in $R^1\pi'_*\mu_l$ can be lifted to $R^1\bar{\pi}'_*\mu_l$. Propositions 7.2 and 7.3 will show that our assumptions a stable under desingularized l-coverings. Hence, it suffices to prove that $H^1(B, R^1\bar{\pi}_*\mu_l)$ vanishes in the limit over all desingularized l-coverings.

Consider the cohomology sequence associated to the short exact sequence (6.1):

$$\ldots \to H^0(B, \mathcal{F}) \to H^1(B, R^1\bar{\pi}_*\mu_l) \to H^1(B, \iota_* R^1\bar{\pi}_{\eta*}\mu_l) \ldots.$$

By Lemma 6.23 the cohomology group $H^1(B, \iota_* R^1\bar{\pi}_{\eta*}\mu_l)$ vanishes in the limit. After replacing $(\bar{X}, \bar{D})$ with a desingularized l-covering, the elements of $H^1(B, R^1\bar{\pi}_*\mu_l)$ can thus be lifted to $H^0(B, \mathcal{F})$. Since by Propositions 7.2 and 7.3 from the next chapter our assumptions are stable under desingularized l-coverings, we are left with proving that $H^0(B, \mathcal{F})$ vanishes in the limit. This is precisely the assertion of Lemma 6.15. $\square$

6.5 The second higher direct image

At this point we have tackled all hypotheses of corollary 6.2. Under the conditions stated in the preceding sections the said corollary thus implies that

$$\varinjlim_{\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} H^n(X',\Lambda) = 0$$

for $n \geqslant 3$ and

$$\varinjlim_{\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} ker(H^2(X',\Lambda) \overset{edge}{\to} H^0(B', R^2\pi'_*\Lambda)) = 0.$$

The only thing left to prove is that in the limit over $\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}$ the elements of $H^2(U,\Lambda)$ cannot only be lifted to $H^2(X,\Lambda)$ but even to

$$ker(H^2(X,\Lambda) \overset{edge}{\to} H^0(B, R^2\pi_*\Lambda)).$$

Proposition 6.25: *Assume that $\bar{D}_h$ is nonempty and intersects all irreducible components of W. Let ϕ be in the image of*

$$H^2(X,\Lambda) \to H^2(U,\Lambda).$$

Assume further that $(\bar{X},\bar{D})$ has enough tame coverings. Then there is a desingularized $\mathfrak{c}$-covering $(\bar{X}',\bar{D}') \to (\bar{X},\bar{D})$ such that the image of ϕ in $H^2(U',\Lambda)$ can be lifted to an element ψ' of $H^2(X',\Lambda)$, which lies in the kernel of the edge morphism

$$H^2(X',\Lambda) \to H^0(B', R^2\pi_*\Lambda).$$

Proof: Consider the diagram

$$\begin{tikzcd}
H^2_D(X,\Lambda) \arrow[r] \arrow[d, "\sim"', "\text{edge}"] & H^2(X,\Lambda) \arrow[r] \arrow[d, "\text{edge}"] & H^2(U,\Lambda) \\
H^0(B, R^2_D\pi_*\Lambda) \arrow[r] & H^0(B, R^2\pi_*\Lambda).
\end{tikzcd}$$

The left vertical arrow is an isomorphism because due to purity $R^j_D\pi_*\Lambda = 0$ for $j \leqslant 1$. We conclude that it suffices to show that the cokernel of

$$H^0(B, R^2_D\pi_*\Lambda) \to H^0(B, R^2\pi_*\Lambda)$$

vanishes in the limit over $\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}$. We have a direct sum decomposition indexed by the irreducible components D_i of D

$$R^2_D\pi_*\Lambda = \bigoplus_i R^2_{D_i}\pi_*\Lambda.$$

It is sufficient to prove that the cokernel of the vertical part vanishes after a desingularized $\mathfrak{c}$-covering: We want to show that there is a desingularized $\mathfrak{c}$-covering $(\bar{X}',\bar{D}') \to (\bar{X},\bar{D})$ as above such that

$$coker(H^0(B, R^2_{D_v}\pi_*\Lambda) \to H^0(B, R^2\pi_*\Lambda)) \to coker(H^0(B', R^2_{D'_v}\pi'_*\Lambda) \to H^0(B', R^2\pi'_*\Lambda))$$

is the zero map. Both $R^2_{D_v}\pi_*\Lambda$ and $R^2\pi_*\Lambda$ are skyscraper sheaves with support in the singular locus of $X \to B$. We can treat each singular fibre separately and thus assume that B is a henselian discrete valuation ring. We only have to make sure that the constructed desingularized $\mathfrak{c}$-covering extends to a desingularized $\mathfrak{c}$-covering above the initial base scheme. We have the following diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & H^0(B, R^2_{D_v}\pi_*\mathbb{Z}_\mathfrak{c}) \arrow[r, "\cdot m"] \arrow[d, "\rho"] & H^0(B, R^2_{D_v}\pi_*\mathbb{Z}_\mathfrak{c}) \arrow[r] \arrow[d, "\rho"] & H^0(B, R^2_{D_v}\pi_*\Lambda) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0(B, R^2\pi_*\mathbb{Z}_\mathfrak{c}) \arrow[r, "\cdot m"] & H^0(B, R^2\pi_*\mathbb{Z}_\mathfrak{c}) \arrow[r] & H^0(B, R^2\pi_*\Lambda) \arrow[r] & 0.
\end{tikzcd} \tag{6.4}$$

The exactness of the above sequences can be checked by using the explicit description of the cohomology groups involved: Let $\bar{b}$ be a geometric point above the closed point b of B. Denote by B^{sh} the strict henselization of B at $\bar{b}$ and by X^{sh}, $\bar{X}^{sh}$ and D^{sh} the base change of X, $\bar{X}$, and D to B^{sh}. Let k be the residue field of B and $\bar{k}$ the residue field of B^{sh} and denote by $\mathcal{G}$ the Galois group of $\bar{k}|k$. Then

$$H^0(B, R^2_{D_v}\pi_*\mathbb{Z}_{\mathfrak{c}}) \cong H^2_{D_v^{sh}}(B^{sh}, \mathbb{Z}_{\mathfrak{c}})^{\mathcal{G}} \cong (\bigoplus_{C\subseteq D_{\bar{b}}} \mathbb{Z}_{\mathfrak{c}} \cdot C)^{\mathcal{G}},$$

where $\mathcal{G}$ acts on the direct sum by permuting the components C of D^{sh}. This already shows the injectivity of multiplication by m. Similarly,

$$H^0(B, R^2_{D_v}\pi_*\Lambda) \cong (\bigoplus_{C\subseteq D_v^{sh}} \Lambda \cdot C)^{\mathcal{G}}.$$

and $H^0(B, R^2_{D_v}\pi_*\mathbb{Z}_{\mathfrak{c}}) \to H^0(B, R^2_{D_v}\pi_*\Lambda)$ corresponds to the canonical projection

$$(\bigoplus_{C\subseteq D_{\bar{b}}} \mathbb{Z}_{\mathfrak{c}} \cdot C)^{\mathcal{G}} \to (\bigoplus_{C\subseteq D_{\bar{b}}} \Lambda \cdot C)^{\mathcal{G}},$$

which is surjective. In the lower row we have

$$H^0(B, R^2\pi_*\mathbb{Z}_{\mathfrak{c}}) \cong (\bigoplus_{C\subseteq X_{\bar{b}}} H^2(C, \mathbb{Z}_{\mathfrak{c}}))^{\mathcal{G}} \cong (\bigoplus_{C\subseteq X_{\bar{b}}, C/B^{sh} \text{ proper}} \mathbb{Z}_{\mathfrak{c}} \cdot C)^{\mathcal{G}}.$$

The irreducible components of $X_{\bar{b}}$ which are proper over B^{sh} correspond to the irreducible components of $\bar{X}_{\bar{b}}$ not intersecting $\bar{D}_h$. By assumption these irreducible components coincide with the irreducible components of $D_{\bar{b}}$. The rest of the calculation for the lower sequence is analogous.

In order to show that the cokernel of the right hand side vertical map in the diagram (6.4) vanishes after a desingularized $\mathfrak{c}$-covering it suffices to show that the cokernel of the middle vertical map does so. The stalk of the morphism $R^2_{D_v}\pi_*\hat{\mathbb{Z}}(\mathfrak{c}) \to R^2\pi_*\hat{\mathbb{Z}}(\mathfrak{c})$ at $\bar{b}$ is

$$H^2_{D_{\bar{b}}}(X^{sh}, \hat{\mathbb{Z}}(\mathfrak{c})) \to H^2(X^{sh}, \hat{\mathbb{Z}}(\mathfrak{c})).$$

By Lemma 6.7 it is given by the intersection matrix ρ of the components of $D_{\bar{b}}$. Since $D_{\bar{b}}$ does not contain all components of the geometric special fibre, ρ is injective. Denote by $\mathcal{F}$ the cokernel. By Corollary 6.10 there is a desingularized $\mathfrak{c}$-covering $(\bar{X}', \bar{D}') \to (\bar{X}, \bar{D})$ such that $\mathcal{F} \to \mathcal{F}'$ is the zero map (where $\mathcal{F}'$ is the respective cokernel defined on X'). Note that this covering extends to the initial base scheme by Remark 6.11. We have an exact sequence

$$0 \to H^0(B, R^2_{D_v}\pi_*\hat{\mathbb{Z}}(\mathfrak{c})) \to H^0(B, R^2\pi_*\hat{\mathbb{Z}}(\mathfrak{c})) \to H^0(\mathcal{G}, \mathcal{F}).$$

So the cokernel of $H^0(B, R^2_{D_v}\pi_*\hat{\mathbb{Z}}(\mathfrak{c})) \to H^0(B, R^2\pi_*\hat{\mathbb{Z}}(\mathfrak{c}))$ is a subgroup of $\mathcal{F}$. This shows the result. $\square$

Chapter 7

Construction of $K(\pi, 1)$-neighborhoods

In the preceding chapters we proved the vanishing in the limit over all desingularized tame coverings of cohomology groups $H^i(X, \Lambda)$, where X is an arithmetic surface (and similarly for cohomology groups with support). The vanishing of these cohomology groups in the limit was subject to various conditions on X. In this chapter we construct étale neighborhoods of a given geometric point satisfying these conditions. Moreover, we show that these conditions are stable under desingularized $\mathfrak{c}$-coverings

7.1 Stability of certain properties under desingularized tame coverings

In order to apply the results of chapter 5 and of chapter 6 it is not enough that an arithmetic surface X itself has the above mentioned properties. Since we are aspiring to show the vanishing of a limit over all desingularized $\mathfrak{c}$-coverings (for a full class of finite groups) of X, every desingularized $\mathfrak{c}$-covering of X has to have these properties. Therefore, we have to show that the necessary properties are stable under desingularized $\mathfrak{c}$-coverings. We start with the base scheme.

Lemma 7.1: *Let B be a global Dedekind scheme and $\bar{b} \to B$ a geometric point over a closed point $b \in B$. Let l be a prime number which is invertible on B. The following properties are stable under restriction to Zariski neighborhoods of b and étale l-coverings of B*

(i) $\mu_l \cong \mathbb{Z}/l\mathbb{Z}$ *on B, i. e., the quotient field of B contains the l^{th} roots of unity.*

(ii) For any closed point $b_1 \in B$ and geometric point $\bar{b}_1$ over b_1 the natural morphism

$$\pi_1(b_1, \bar{b}_1)(l) \to \pi_1(B, \bar{b}_1)(l)$$

is injective.

(iii) Let $T \subset B$ be a finite (possibly empty) set of closed points. Denote by $\tilde{B}^l$ the universal l-covering of B and by $\tilde{T}^l$ the preimage of T in $\tilde{B}^l$. Then $B - T$ is $K(\pi, 1)$ with respect to l and

$$\pi_1(\tilde{B}^l - \tilde{T}^l)(l) \cong \mathop{\ast}_{s \in \tilde{T}^l} I_s(l).$$

where I_s is the inertia group at s.

Proof: Property (i) is clearly stable under any étale morphism $B' \to B$. The same holds for property (ii): Let $B' \to B$ be étale and b_1' a closed point of B'. Let $\bar{b}_1 \to B'$ be a lift of $\bar{b}_1 \to B'$

with image b_1'. Consider the commutative diagram

$$\begin{tikzcd}
\pi_1(b_1',\bar{b}_1)(l) \arrow[r] \arrow[d, hook] & \pi_1(B',\bar{b}_1)(l) \arrow[d] \\
\pi_1(b_1,\bar{b}_1)(l) \arrow[r] & \pi_1(B,\bar{b}_1)(l)
\end{tikzcd}$$

If

$$\pi_1(b_1,\bar{b}_1)(l) \to \pi_1(B,\bar{b}_1)(l)$$

is injective, the same holds for $\pi_1(b_1',\bar{b}_1)(l) \to \pi_1(B',\bar{b}_1)(l)$.

Assume that property (iii) holds on B and let $B' \to B$ be an étale l-covering. Since by assumption B is $K(\pi,1)$ with respect to l (set $T = \varnothing$ in property (iii)), the same holds for B'. Also if $B' \to B$ is an open immersion, B' is $K(\pi,1)$ with respect to l by applying property (iii) with $T = B - B'$. Hence, in both cases property (iii) holds by [Sch II], Satz 8.1 and Bemerkung 8.2. □

Lemma 7.2: *Let $\bar{\pi} : \bar{X} \to B$ be a proper arithmetic surface and $\bar{D} \subset \bar{X}$ a tidy divisor. Set $X = \bar{X} - \bar{D}_h$ and $U = \bar{X} - \bar{D}$. Let $\mathfrak{c}$ be a full class of finite groups such that all integers in $\mathbb{N}(\mathfrak{c})$ are invertible on $\bar{X}$ and $\mu_l \cong \mathbb{Z}/l\mathbb{Z}$ on B for every prime number in $\mathbb{N}(\mathfrak{c})$. Then the following conditions are stable under desingularized $\mathfrak{c}$-coverings.*

(i) Every connected component of D has at least one horizontal component.

(ii) $\bar{D}_h$ is nonempty and intersects all irreducible components of W.

(iii) Every irreducible component of W has nontrivial intersection with D_v.

(iv) For every prime number $l \in \mathbb{N}(\mathfrak{c})$ and every geometric point $\bar{b}$ of B there is an irreducible component C of $W_{\bar{b}}$ such that l does not divide the multiplicity of C in $\bar{X}_{\bar{b}}$

(v) $(\bar{X},\bar{D})$ has enough tame coverings.

(vi) W is regular.

Proof: Let $(\bar{X}_1,\bar{D}_1) \to (\bar{X},\bar{D})$ be a $\mathfrak{c}$-covering and $(\bar{X}',\bar{D}') \to (\bar{X}_1,\bar{D}_1)$ a tidy desingularization. By Corollary 4.7 a connected component of $\bar{D}_1$ is mapped surjectively onto a connected component of $\bar{D}$. Furthermore, connected components of $\bar{D}'$ are mapped surjectively onto connected components of $\bar{D}_1$. Therefore, if $(\bar{X},\bar{D})$ satisfies condition (i), so does $(\bar{X}',\bar{D}')$.

By Lemma 4.2 and Corollary 4.7, the preimage of $\bar{D}_h$ in $\bar{X}'$ is $\bar{D}_h'$. If $\bar{D}_h$ is nonempty, the same holds for $\bar{D}_h'$. Furthermore, suppose C' is an irreducible component of W'. It maps surjectively onto an irreducible component C of W. If $x \in C$ is contained in $\bar{D}_h$, the (nonempty) preimage of x in C' is contained in $\bar{D}_h'$. This proves that condition (ii) is stable under desingularized $\mathfrak{c}$-coverings.

Let C' be an irreducible component of W'. It maps to an irreducible component C of W. Let x be a closed point of $\bar{X}$ in the intersection of C with D_v. Then the preimage of x in C' is contained in the intersection of C' with D_v'. This treats condition (iii).

Let $l \in \mathbb{N}(\mathfrak{c})$ be a prime number and $\bar{b}$ a geometric point of B. Assume that there is an irreducible component C of $W_{\bar{b}}$ such that l does not divide its multiplicity n_C. Let $(\bar{X}',\bar{D}') \to (\bar{X},\bar{D})$ be a desingularized $\mathfrak{c}$-covering and $\bar{b}'$ a geometric point of B' above $\bar{b}$. Then there is an irreducible component C' of $W_{\bar{b}'}'$ above C. The desingularized $\mathfrak{c}$-covering $(\bar{X}',\bar{D}') \to (\bar{X},\bar{D})$ induces a generically étale $\mathfrak{c}$-covering $C' \to C$. Therefore, C' has the same multiplicity n_C over $\bar{b}$ as C. The multiplicity of C' over $\bar{b}'$ divides n_C and is thus not divisible by l.

In order to deal with condition (v) assume that $(\bar{X},\bar{D})$ has enough tame coverings. Let $(\bar{X}',\bar{D}') \to (\bar{X},\bar{D})$ be a desingularized $\mathfrak{c}$-covering, $p' \in \bar{D}'$ a closed point and Z' an irreducible component of $\bar{D}'$ passing through p'. We have to find $f' \in K(\bar{X}')^\times$ with support in $\bar{D}'$ such that $deg_{Z'}(f') > 0$ and $deg_{C'}(f') = 0$ for all other irreducible components C' of $\bar{D}'$ passing through p'. Let $Z_1, \ldots, Z_n$ (for $n = 1$ or $n = 2$) denote the irreducible components of $\bar{D}$ passing through the image point $p \in \bar{D}$ of p'. Since $(\bar{X}, \bar{D})$ has enough tame coverings, for $i = 1, \ldots, n$ there is $f_i \in K(\bar{X})^\times$ with support in $\bar{D}$ such that $deg_{Z_i}(f_i) > 0$ and $deg_{Z_j}(f_i) = 0$ for $i \neq j$. The projections of $div\ f_i$ to

$$\mathbb{Q} \cdot Z_1 \oplus \ldots \oplus \mathbb{Q} \cdot Z_n$$

constitute a basis of this vector space. Let $Z' = Z'_1, \ldots, Z'_m$ denote the irreducible components of $\bar{D}'$ passing through p'. In section 4.5 we investigated the multiplicity homomorphism

$$\phi_{p'} : \mathbb{Q} \cdot Z_1 \oplus \ldots \oplus \mathbb{Q} \cdot Z_n \to \mathbb{Q} \cdot Z'_1 \oplus \ldots \oplus \mathbb{Q} \cdot Z'_m$$

at p' induced by pullback. By Lemma 4.24 all multiplicity homomorphisms of a desingularized $\mathfrak{c}$-covering are surjective. Therefore, there is a linear combination of $div\ f_1, \ldots, div\ f_n$ with coefficients in $\mathbb{Q}$ mapping to Z'_1 under the multiplicity homomorphism. Clearing denominators we obtain

$$d \cdot Z'_1 = \phi_{p'}(k_1 div\ f_1 + \ldots k_n div\ f_n)$$

with integers $d, k_1, \ldots, k_n$ such that $d > 0$. In other words, setting $f = f_1^{k_1} \cdot \ldots f_n^{k_n}$ we have in a neighborhood of p'

$$div\ f = d \cdot Z',$$

what we wanted to prove.

Let us treat the last property. The statement that W is regular is equivalent to $(U_b)_{red}$ being regular for every closed point b of B such that $\bar{X}_b$ is singular. Indeed, since W is the Zariski closure of the union of these $(U_b)_{red}$, we only have to show that W is always regular at the finitely many closed points in the complement of $(U_b)_{red}$. But these points are intersection points of W with $\bar{D}$ and $\bar{D}$ is tidy. In particular, $\bar{D}$ has normal crossings with W and thus W is regular at the intersection points. A desingularized $\mathfrak{c}$-covering $(\bar{X}', \bar{D}') \to (\bar{X}, \bar{D})$ induces an étale covering $(U'_{b'})_{red} \to (U_b)_{red}$ for every closed point b' of B' with image b in B. Therefore, if $(U_b)_{red}$ is regular, so is $(U'_{b'})_{red}$. □

In the global case we have to treat one more property concerning the l-division points of the Jacobian of the generic fibre of an arithmetic surface.

Proposition 7.3: *Let $\bar{\pi} : \bar{X} \to B$ be a proper arithmetic surface and $\bar{D} \subset \bar{X}$ a tidy divisor. Set $X = \bar{X} - \bar{D}_h$ and $U = \bar{X} - \bar{D}$. Let l be a prime number which is invertible on X. Let η denote the generic point of B. Let $\bar{\eta}$ be a geometric point above η and denote by $\mathcal{G}_\eta$ the Galois group of $\bar{\eta}|\eta$. The following property is stable under desingularized l-coverings: The action of $\mathcal{G}_\eta$ on the l-division points of the Jacobian of $U_{\bar{\eta}}$ factors through an l-primary quotient.*

For the proof of Proposition 7.3 we will need the following proposition which is Proposition 4 in [Fri I] in the situation where the pro-group $\mathcal{G}$ is profinite.

Proposition 7.4: *Let*

$$1 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 1$$

be a short exact sequence of profinite groups such that $H^1(\mathcal{F}, \mathbb{Z}/l\mathbb{Z})$ is finite and $H^2(\mathcal{F}, \mathbb{Z}/l\mathbb{Z}) = 0$. Then $\mathcal{F}(l)$ is free pro-l.

Assume in addition that $K(\mathcal{H}, 1)(l) \xrightarrow{\#} K(\mathcal{H}(l), 1)$ is a $\sharp$-isomorphism. Then the action of $\mathcal{H}$ on $H^1(\mathcal{F}, \mathbb{Z}/l\mathbb{Z})$ factors through $\mathcal{H}(l)$ if and only if

$$1 \to \mathcal{F}(l) \to \mathcal{G}(l) \to \mathcal{H}(l) \to 1$$

is exact.

Corollary 7.5: *Let*

$$1 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 1$$

be a short exact sequence of profinite groups such that $H^1(\mathcal{F},\mathbb{Z}/l\mathbb{Z})$ is finite and $\mathcal{F}$ has cohomological l-dimension 1. Let $\mathcal{G}' \subseteq \mathcal{G}$ be an open subgroup of l-primary index. Denote by $\mathcal{F}'$ the intersection of $\mathcal{F}$ with $\mathcal{G}'$ and by $\mathcal{H}'$ the image of $\mathcal{G}'$ in $\mathcal{H}$. If the action of $\mathcal{H}$ on $H^1(\mathcal{F},\mathbb{Z}/l\mathbb{Z})$ factors through $\mathcal{H}(l)$, the action of $\mathcal{H}'$ on $H^1(\mathcal{F}',\mathbb{Z}/l\mathbb{Z})$ factors through $(\mathcal{H}')(l)$.

Proof: It suffices to prove that for each homomorphism $h' : \mathbb{Z} \to \mathcal{H}'$ the induced action of $\mathbb{Z}$ on $H^1(\mathcal{F},\mathbb{Z}/l\mathbb{Z})$ factors through $\mathbb{Z}_l$. For a homomorphism $h' : \mathbb{Z} \to \mathcal{H}'$ we denote by h the composite homomorphism $\mathbb{Z} \to \mathcal{H}' \to \mathcal{H}$. Setting $\mathcal{G}_h := \mathcal{G} \times_h \mathbb{Z}$ and $\mathcal{G}'_{h'} := \mathcal{G}' \times_{h'} \mathbb{Z}$ the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{H} & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{G}' & \longrightarrow & \mathcal{H}' & \longrightarrow & 1
\end{array}$$

induces the following diagram of exact sequences

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G}_h & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \| & & \\
1 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{G}'_{h'} & \longrightarrow & \mathbb{Z} & \longrightarrow & 1.
\end{array}$$

Noting that $K(\mathbb{Z},1)(l) \xrightarrow{\#} K(\mathbb{Z}_l,1)$ is a $\sharp$-isomorphism we check that the upper exact sequence meets the conditions for Proposition 7.4. Hence, l-completion yields

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{F}(l) & \longrightarrow & \mathcal{G}_h(l) & \longrightarrow & \mathbb{Z}_l & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \| & & \\
 & & (\mathcal{F}')(l) & \longrightarrow & (\mathcal{G}'_{h'})(l) & \longrightarrow & \mathbb{Z}_l & \longrightarrow & 1.
\end{array}$$

The vertical maps are injective as $\mathcal{F}'$ and $\mathcal{G}'_{h'}$ have l-primary index in $\mathcal{F}$ and $\mathcal{G}_h$, respectively. It follows that $(\mathcal{F}')(l) \to \mathcal{G}_h(l)$ is injective and thus so is $(\mathcal{F}')(l) \to (\mathcal{G}'_{h'})(l)$. Let us check that the assumptions of Proposition 7.4 continue to hold for the lower exact sequence in the diagram implying that the action of $\mathbb{Z}_l$ on $H^1(\mathcal{F}',\mathbb{Z}/l\mathbb{Z})$ factors through $\mathbb{Z}_l$. By [NSW] Proposition 3.3.5 we have $cd_l\mathcal{F}' = cd_l\mathcal{F} = 1$ and thus in particular $H^2(\mathcal{F}',\mathbb{Z}/l\mathbb{Z}) = 0$. We see that $H^1(\mathcal{F}',\mathbb{Z}/l\mathbb{Z})$ is finite using the Hochschild-Serre spectral sequence for $\mathcal{F}' \subseteq \mathcal{F}$ and the finiteness of $H^1(\mathcal{F},\mathbb{Z}/l\mathbb{Z})$. □

Proof of Proposition 7.3: Let $\bar{x}$ be a geometric point of $U_{\bar{\eta}}$. Let $\eta_1|\eta$ be a finite Galois l-subextension of $\bar{\eta}|\eta$ such that $\mathcal{G}_{\eta_1}$ acts trivially on $H^1(U_{\bar{\eta}},\mathbb{Z}/l\mathbb{Z})$. We have the following exact sequence of pro-finite groups.

$$1 \to \pi_1(U_{\bar{\eta}},\bar{x}) \to \pi_1(U_{\eta_1},\bar{x}) \to \mathcal{G}_{\eta_1} \to 1.$$

A desingularized l-covering $(\bar{X}',\bar{D}') \to (\bar{X},\bar{D})$ induces by base change to U_{η_1} a finite étale l-covering $U'_{\eta_1} \to U_{\eta_1}$ and we obtain the following diagram of short exact sequences

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(U_{\bar{\eta}},\bar{x}) & \longrightarrow & \pi_1(U_{\eta_1},\bar{x}) & \longrightarrow & \mathcal{G}_{\eta_1} & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \pi_1(U'_{\bar{\eta}},\bar{x}) & \longrightarrow & \pi_1(U'_{\eta_1},\bar{x}) & \longrightarrow & \mathcal{G}_{\eta'_1} & \longrightarrow & 1.
\end{array}$$

Let us verify that Corollary 7.5 applies. Since $U_{\bar{\eta}}$ is an affine curve over an algebraically closed field, it is $K(\pi,1)$ with respect to l and has cohomological l-dimension less or equal to 1. We conclude that $\pi_1(U_{\bar{\eta}},\bar{x})$ has cohomological l-dimension less or equal to 1 and $H^1(\pi_1(U_{\bar{\eta}},\bar{x}),\mathbb{Z}/l\mathbb{Z}) =$

$H^1(U_{\bar\eta}, \mathbb{Z}/l\mathbb{Z})$ is finite. Hence, by Corollary 7.5 the action of $\mathcal{G}_{\eta'_1}$ on $H^1(U'_{\bar\eta}, \mathbb{Z}/l\mathbb{Z})$ factors through an l-primary quotient $\mathcal{G}_{\eta'_1}/\mathcal{G}_{\eta'_2}$. We obtain the following diagram of extensions of η:

$$\begin{tikzcd}[column sep=small]
 & \eta'_2 \arrow[d, no head] & \\
 & \eta'_1 = \eta'\eta_1 \arrow[dl, no head] \arrow[dr, no head] & \\
\eta' \arrow[dr, no head] & & \eta_1 \arrow[dl, no head] \\
 & \eta. &
\end{tikzcd}$$

Since $\eta_1|\eta$ is an l-extension, the same holds for $\eta'_1|\eta'$. Furthermore, $\eta'_2|\eta'_1$ is an l-extension and thus also $\eta'_2|\eta'$. □

7.2 Neighborhoods with enough tame coverings

For the construction of étale neighborhoods with enough tame coverings we need the following variant of prime evasion.

Lemma 7.6: *Let r and s be positive integers. Let A be a noetherian ring and $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ and $\mathfrak{q}_1, \dots, \mathfrak{q}_s$ prime ideals such that for $i \neq j$ $\mathfrak{q}_i$ is not contained in $\mathfrak{q}_j$. For $j \leqslant s$ define the integer m_j by*

$$\mathfrak{p}_1 \cdot \ldots \cdot \mathfrak{p}_r \subseteq \mathfrak{q}_j^{m_j} \backslash \mathfrak{q}_j^{m_j+1}.$$

Then there is $a \in A$ such that for $i \leqslant r$ $a \in \mathfrak{p}_i$ and for $j \leqslant s$ $a \notin \mathfrak{q}_j^{m_j+1}$.

Proof: For $i \neq j$ we can find $b_{ij} \in \mathfrak{q}_i \backslash \mathfrak{q}_j$. We define

$$b_j := \prod_{i \neq j} b_{ij}^{m_i+1}.$$

Then we have

$$b_j \notin \mathfrak{q}_j, \qquad b_j \in \mathfrak{q}_i^{m_i+1} \quad \text{for } i \neq j.$$

Furthermore, by assumption, we can choose $a_j \in \mathfrak{p}_1 \cdot \ldots \cdot \mathfrak{p}_r \backslash \mathfrak{q}_j^{m_j+1}$. Then $c_j := a_j b_j$ for $j \leqslant s$ satisfies

$$c_j \notin \mathfrak{q}_j^{m_j+1}, \qquad c_j \in \mathfrak{q}_i^{m_i+1} \quad \text{for } i \neq j, \qquad c_j \in \mathfrak{p}_i \quad \text{for } i \leqslant r.$$

Finally,

$$a := \sum_{j \leqslant r} c_j$$

has the required properties. □

For the rest of this section we use the following notation: For an integral closed subscheme Z of an affine scheme *Spec* A we denote by $\mathfrak{p}_Z$ the prime ideal of A corresponding to the generic point of Z. Moreover, we write $m_x(Z)$ for the multiplicity of a closed subscheme Z in a point x. It is defined as the maximal power of $\mathfrak{p}_x$ containing the ideal corresponding to Z.

Lemma 7.7: *Let X/B be an arithmetic surface such that B is local with generic point η isomorphic to the spectrum of a global field (i. e., B is the localization of a global Dedekind scheme at a closed point). Let $x_1, \dots, x_n$ be finitely many points of X. Then there are horizontal prime divisors $G_1, \dots, G_s, G_{s+1}, \dots, G_r$ such that $G_1, \dots, G_s$ and $G_{s+1}, \dots, G_r$ each generate the Weil divisor class group $CH^1(X)$ of X. Furthermore, the supports of G_i for $i = 1, \dots, r$ do not contain x_j for $j = 1, \dots, n$ and the supports of G_i and G_j for $i \leqslant s$ and $j > s$ are disjoint.*

Proof: The generic fibre X_η of $X \to B$ is a smooth curve over a global field. By the Mordell-Weil theorem (see [Wei]) its Weil divisor class group is finitely generated. Denote by $C_1,\dots,C_l$ the irreducible components of the special fibre. The Weil divisor class group of X is generated by the Weil divisor class group of X_η and by $C_1,\dots,C_1$. It is therefore also finitely generated, by prime divisors $D_1,\dots,D_m$, say.

Since X is quasi-projective over an affine scheme, there is an affine open subscheme $Spec\ A \subseteq X$ containing $x_1,\dots,x_n$, as well as the generic points of $D_1,\dots,D_m$ and of $C_1,\dots,C_l$ (see [Liu], Proposition 3.3.36). By Lemma 7.6 we can choose $f_1,\dots,f_m \in A$ such that for $i = 1,\dots,m$

$$
\begin{aligned}
f_i &\in \mathfrak{p}_{D_i}\backslash\mathfrak{p}_{D_i}^2,\\
f_i &\notin \mathfrak{p}_{C_j}^{m_{C_j}(D_i)+1} \quad \text{for } j = 1,\dots,l,\\
f_i &\notin \mathfrak{p}_{x_j}^{m_{x_j}(D_i)+1} \quad \text{for } j = 1,\dots,n.
\end{aligned}
$$

Viewing f_i as elements of $K(X)^\times$ we obtain divisors $D_1 - div\ f_1,\dots,D_m - div\ f_m$ generating the Weil divisor class group. The supports of the divisors $D_i - div\ f_i$ do not contain $x_1,\dots,x_n$ and the coefficients of $C_1,\dots,C_l$ are zero, i.e., $D_i - div\ f_i$ are horizontal. Denote by $G_1,\dots,G_s$ the prime divisors in the support of $D_1 - div\ f_1,\dots,D_m - div\ f_m$. Then $G_1,\dots,G_s$ are horizontal prime divisors generating the Weil divisor class group whose supports do not contain $x_1,\dots,x_n$.

Denote by $z_1,\dots,z_t$ the intersection points of $G_1,\dots,G_s$ with the special fibre. Choose an affine open subscheme $Spec\ A' \subseteq X$ containing $x_1,\dots,x_n,z_1,\dots,z_t$ as well as the generic points of $D_1,\dots,D_m$ and of $C_1,\dots,C_1$. Using Lemma 7.6 again we find $g_1,\dots,g_m \in A'$ such that for $i = 1,\dots,m$

$$
\begin{aligned}
g_i &\in \mathfrak{p}_{D_i}\backslash\mathfrak{p}_{D_i}^2,\\
g_i &\notin \mathfrak{p}_{C_j}^{m_{C_j}(D_i)+1} \quad \text{for } j = 1,\dots,l,\\
g_i &\notin \mathfrak{p}_{x_j}^{m_{x_j}(D_i)+1} \quad \text{for } j = 1,\dots,n,\\
g_i &\notin \mathfrak{p}_{z_j}^{m_{z_j}(D_i)+1} \quad \text{for } j = 1,\dots,t.
\end{aligned}
$$

Denote by $G_{s+1},\dots,G_r$ the prime divisors in the support of $D_1 - div\ g_1,\dots,D_m - div\ g_m$. As above $G_{s+1},\dots,G_r$ are horizontal prime divisors generating the Weil divisor class group whose supports do not contain $x_1,\dots,x_n$. Moreover, $z_1,\dots,z_t$ are not contained in the support of $G_{s+1},\dots,G_r$. Hence, the support of G_i for $i \leqslant s$ is disjoint from the support of G_j for $j > s$ as $z_1,\dots,z_t$ are the only possible intersection points of G_i with another divisor. □

Lemma 7.8: *Let X/B be an arithmetic surface and let $G_1,\dots,G_s$ be horizontal prime divisors generating the Weil divisor class group $CH^1(X)$ of X. Let p be a closed point of X such that X is regular at p and p is not contained in any G_j for $j = 1,\dots,s$. Denote by $X' \to X$ the blowup of X in p. Let G be a horizontal prime divisor disjoint from $G_1,\dots,G_s$ with nontrivial intersection with the exceptional locus E. Then $G_1,\dots,G_s,G$ generate $CH^1(X')\otimes\mathbb{Q}$.*

Proof: The Weil divisor class group of X' is generated by $G_1,\dots,G_s$ and E. Let G_0 denote the image of G in X. Since $G_1,\dots,G_s$ generate the Weil divisor class group of X, there are $n_j \in \mathbb{Z}$ such that

$$G_0 = \sum_{j=1}^{s} n_j G_j$$

in $CH^1(X)$. By [Liu], Chapter 9, Proposition 2.23 the pullback of G_0 to X' is given by

$$G + m_p(G_0)\cdot E.$$

Since $p \in G_0$, the multiplicity $m_p(G_0)$ is positive. In $CH(X') \otimes \mathbb{Q}$ we thus have

$$E = \frac{1}{m_p(G_0)}(\sum_{j=1}^{s} n_j G_j - G).$$

□

Lemma 7.9: *Let Y/B be an arithmetic surface such that B is local with its generic point isomorphic to the spectrum of a global field and $x \in Y$ a closed point. Let $\bar{Y}$ be a compactification of Y over B. Then there is an open neighborhood $V \subseteq Y$ of x and a compactification $\bar{X}/B$ of V dominating $\bar{Y}$ such that $\bar{D} = \bar{X} - V$ is a tidy divisor and such that the following assertion holds: For every closed point $y \in \bar{X}$ and every prime divisor Z of $\bar{X}$ passing through y there is $f \in K(\bar{X})^\times$ with support in $Z \cup \bar{D}$ such that $deg_Z(f) > 0$ and $deg_W(f) = 0$ for all other prime divisors W passing through y.*

Proof: Take an open neighborhood V' of x such that the complement contains all singular points except x (if x is singular) and all vertical prime divisors not passing through x and set $D' = \bar{Y} - V'$. By [Lip] we can replace $(\bar{Y}, D')$ by a desingularization (in the strong sense) and thus assume that x is the only possible singular point of $\bar{Y}$ and D' is a Cartier divisor. Choose prime divisors $G_1, \ldots, G_r$ of $\bar{Y}$ not passing through x as in Lemma 7.7. Making V' smaller we may assume that $G_1, \ldots, G_r$ are contained in D'.

Let $(\bar{X}, D_0) \to (\bar{Y}, D')$ be a tidy desingularization, which exists by Proposition 2.15. Since $\bar{Y}$ is regular at every point in D', the morphism $\bar{X} \to \bar{Y}$ is a consecutive blowup in closed points over D'. Moreover, the exceptional fibre of each blowup in a closed point p is isomorphic to $\mathbb{P}^1_{k(p)}$ (see [Liu], Chapter 8, Theorem 1.19). Denote by $E_1, \ldots, E_n$ the irreducible components of the exceptional locus of $\bar{X} \to \bar{Y}$. For each $i = 1, \ldots, n$ choose two different closed points $p_i, q_i \in E_i$ in the regular locus of D_0 and (horizontal) prime divisors D_i and K_i intersecting E_i transversally at p_i and q_i, respectively. Since a horizontal prime divisor consists of only two points, namely the special and the generic point, D_i and K_i are regular and do not intersect D_0 in any other point. Denote by $\bar{D}$ the sum of D_0 and the prime divisors $D_1, \ldots, D_n, K_1, \ldots, K_n$ as above and set $V = \bar{X} - \bar{D}$.

We claim that $(\bar{X}, \bar{D})$ has the required properties. By the definition of a tidy desingularization, D_0 is a tidy divisor. The property of being tidy is invariant under adding horizontal prime divisors intersecting the special fibre transversally in a regular point of D_0. Therefore, also $\bar{D}$ is tidy. Let $y \in \bar{X}$ be a closed point and Z a prime divisor of $\bar{X}$ passing through y. Either $G_1, \ldots, G_s$ or $G_{s+1}, \ldots, G_r$ do not pass through the image point of y, say $G_1, \ldots, G_s$. Furthermore, either $D_1, \ldots, D_n$ or $K_1, \ldots, K_n$ do not pass through y, say $D_1, \ldots, D_n$. By Lemma 7.8 the prime divisors $G_1, \ldots, G_s, D_1, \ldots, D_n$ generate the first Chow group $CH^1(\bar{X}) \otimes \mathbb{Q}$. Hence, there are $m, m_1, \ldots, m_n, n_1, \ldots, n_s \in \mathbb{Z}$ with $m > 0$ and $f \in K(\bar{X})^\times$ such that

$$mZ = \sum_{j=1}^{n} m_j D_j + \sum_{j=1}^{s} n_j G_j + div\ f.$$

The prime divisors $D_1, \ldots, D_n$ and $G_1, \ldots, G_s$ do not pass through y. Therefore, $deg_W(f) = 0$ for all prime divisors W different from Z passing through y and $deg_Z(f) = m > 0$. Furthermore, $D_1, \ldots, D_n, G_1, \ldots, G_s$ are contained in $\bar{D}$ and thus f has support in $Z \cup \bar{D}$. □

As a direct consequence of Lemma 7.9 we obtain:

Corollary 7.10: *In the situation of Lemma 7.9 let $U \subseteq V$ be a neighborhood of x such that $D' = \bar{X} - U$ is a tidy divisor. Then $(\bar{X}, D')$ has enough tame coverings.*

7.3 Construction of suitable étale neighborhoods

Definition 7.11: *Let Y/B be an arithmetic surface and $\bar{x}$ a geometric point of Y inducing a geometric point $\bar{b}$ on B. An étale neighborhood of $\bar{x}$ in Y/B is an arithmetic surface U/B' fitting into a (not necessarily cartesian) commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & Y \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B, \end{array}$$

such that $U \to Y$ is an étale neighborhood of $\bar{x}$ and $B' \to B$ is an étale neighborhood of $\bar{b}$.

Given an arithmetic surface Y/B and a geometric point $\bar{x}$ of Y we want to construct an étale neighborhood U/B' of $\bar{x}$ in Y/B such that B' satisfies properties (i)-(iii) of Lemma 7.1 and U satisfies properties (i)-(vi) of Proposition 7.2.

Lemma 7.12: *Let B be a global Dedekind scheme and $\bar{b} \to B$ a geometric point over a closed point $b \in B$. Let l be a prime number different from the residue characteristic of b. Then there is an étale neighborhood $B' \to B$ of $\bar{b}$ satisfying the following conditions:*

(i) The prime l is invertible on B' and $\mu_l \cong \mathbb{Z}/l\mathbb{Z}$ on B', i. e., the quotient field of B' contains the l^{th} roots of unity.

(ii) For all closed points $b' \in B'$ the natural morphism

$$\pi_1(b', \bar{b})(l) \to \pi_1(B', \bar{b})(l)$$

is injective.

(iii) Let $T' \subset B'$ be a finite (possibly empty) set of closed points. Denote by $\tilde{B}'^l$ the universal l-covering of B' and by $\tilde{T}'^l$ the preimage of T' in $\tilde{B}'^l$. Then $B' - T'$ is $K(\pi, 1)$ with respect to l and

$$\pi_1(\tilde{B}'^l - \tilde{T}'^l)(l) \cong \mathop{\ast}_{s' \in \tilde{T}'^l} I_{s'}(l), \tag{7.1}$$

where $I_{s'}$ is the inertia group at s'.

Proof: Choose an open neighborhood $B_1 \subseteq B$ of b on which l is invertible. This is possible as l is prime to the residue characteristic of b. Etale locally on B_1 the l^{th} roots of unity are isomorphic to $\mathbb{Z}/l\mathbb{Z}$. We can thus find an étale neighborhood $B' \to B$ of $\bar{b}$ such that every étale neighborhood $B'' \to B'$ of $\bar{b}$ satisfies condition (i). We can further assume that the fraction field K' of B' is totally imaginary in case $l = 2$ and K' is a number field. We claim that any étale neighborhood $B'' \to B'$ of $\bar{b}$ also satisfies conditions (ii). For a closed point b'' of B'' the maximal l-extension of the finite field $k(b'')$ is globally realized by the cyclotomic l-extension of B'', which is unramified over B'' as l is invertible on B''. This shows that B'' satisfies condition (ii). After [Sch II], Theorem 1.1(iv) we can shrink B' further such that B' is $K(\pi, 1)$ with respect to l. Then, by [Sch II], Satz 8.1 and Bemerkung 8.2 $B' - T'$ is $K(\pi, 1)$ with respect to l for any finite set of closed points T' and

$$\pi_1(\tilde{B'}^l - \tilde{T'}^l)(l) \cong \mathop{\ast}_{s' \in \tilde{T'}^l} \pi_1(K^{sh}_{s'})(l). \tag{7.2}$$

□

Having constructed a base scheme B with the necessary properties, we now show the existence of étale neighborhoods on an arithmetic surface over B satisfying properties (i)-(v) of Proposition 7.2.

Proposition 7.13: *Let $\pi : Y \to B$ be an arithmetic surface of local or global type and $\bar{x}$ a geometric point above a closed point $x \in Y$. Let $\mathfrak{c}$ be a full class of finite groups such that the residue characteristic of x is not contained in $\mathbb{N}(\mathfrak{c})$ and in the global case assume $\mathfrak{c} = \mathfrak{c}(l)$ for a prime number l. Then there is an étale neighborhood U/B' of $\bar{x}$ and a compactification $U \subseteq \bar{X}$ of $U \to B'$ such that the complement $\bar{D}$ of U in $\bar{X}$ is a tidy divisor with the following properties.*

(i) Every connected component of D has at least one horizontal component.

(ii) $\bar{D}_h$ is nonempty and intersects all irreducible components of W.

(iii) Every irreducible component of W has nontrivial intersection with D_v.

(iv) For every prime number $l \in \mathbb{N}(\mathfrak{c})$ and every geometric point $\bar{b}$ of B there is an irreducible component C of $W_{\bar{b}}$ such that l does not divide the multiplicity of C in $\bar{X}_{\bar{b}}$.

(v) $(\bar{X}, \bar{D})$ has enough tame coverings.

Proof: Denote by b the image of x in B. It suffices to prove the proposition in case B is local with generic point the spectrum of a global field. Indeed, if Y/B is of global type, denote by Y_0/B_0 the base change of Y/B to the localization of B at b. Suppose there is an étale neighborhood $B_0' \to B_0$ of b, an étale neighborhood $U_0 \to Y_{B_0'}$ of $\bar{x}$ and a compactification $U_0 \subseteq \bar{X}_0$ of $U_0 \to B_0'$ such that the complement $\bar{D}_0$ of U_0 in $\bar{X}_0$ is a tidy divisor with properties (i)-(v). We may assume without loss of generality that B_0' is local with closed point b'. Then $B_0' \to B_0$ is the localization of an étale morphism $B' \to B$ at a closed point b' above b and U_0 and $\bar{X}_0$ are the base changes of arithmetic surfaces U and $\bar{X}$ over B'. Making B' smaller we may assume that l is invertible on B', $\bar{X} \to B'$ is proper, U is an open subscheme of $\bar{X}$ and $\bar{X}_{b'}$ is the only possible singular fibre. Similarly, we may assume that $\bar{D} = \bar{X} - U$ is a tidy divisor whose vertical components are all contained in $\bar{X}_{b'}$. The irreducible components of $\bar{D}$ are then in one to one correspondence with the irreducible components of $\bar{D}_0$. Hence, conditions (i)-(iii) are satisfied. Moreover, the geometric fibres at $\bar{b}'$ of $\bar{X} \to B'$ and $\bar{X}_0 \to B_0'$ are identical, whence property (iv).

In order to achieve that $(\bar{X}, \bar{D})$ has enough tame coverings, we choose finitely many closed points $p_1, \dots, p_n$ in $\bar{D}_0$ including all special points of $\bar{D}_0$ such that there is at least one point p_i in each irreducible component of $\bar{D}_0$. Furthermore, denote by $D_1, \dots, D_m$ the irreducible components of $\bar{D}_0$. For every irreducible component D_j of $\bar{D}_0$ passing through one of the points p_i there is $f_{ij} \in K(\bar{X}_0)^\times$ with support in $\bar{D}_0$ such that $deg_{D_j}(f_{ij}) > 0$ and $deg_Z(f_{ij}) = 0$ for all other prime divisors passing through p_i. Viewing f_{ij} as an element of $K(\bar{X})^\times$ it has support in the union of $\bar{D}$ with finitely many vertical prime divisors mapping to closed points different from b' in B'. Removing these points from B results in f_{ij} having support in $\bar{D}$. Now $(\bar{X}, \bar{D})$ has enough tame coverings: Let p be a closed point of $\bar{D}$ and $K \subseteq \bar{D}$ a prime divisor passing through p, which corresponds to an irreducible component D_j of $\bar{D}_0$. If p is a special point of $\bar{D}$, it equals p_i for some i and f_{ij} serves our purposes. If p is a regular point of $\bar{D}$, we can take f_{ij} for any of the closed points p_i lying on D_j.

If Y/B is of local type, it is the base change to B of an arithmetic scheme Y_0/B_0 such that B_0 is local with generic point the spectrum of a global field and B is the completion of B_0 at its closed point. Taking completion does not affect the tidiness of a divisor, nor does it disturb properties (i)-(v). Therefore, also in this case it suffices to prove the proposition for B local with generic point the spectrum of a global field.

For the rest of the proof we assume that B is the localization of a global Dedekind scheme at a closed point b. Choose a geometric point $\bar{b}$ above b compatible with $\bar{x}$. Replacing B by an étale neighborhood of $\bar{b}$ we may assume that all irreducible components of the special fibre are geometrically irreducible. The irreducible components of the special fibre are then in one to one correspondence with the irreducible components of the geometric special fibre $Y_{\bar{b}}$. Replacing Y by an open subscheme we may further assume that all irreducible components of the special fibre Y_b contain x.

Let m be the greatest integer in $\mathbb{N}(\mathfrak{c})$ dividing all multiplicities of components of the closed fibre Y_b. We may assume that $\mu_m \cong \mathbb{Z}/m\mathbb{Z}$ on B. Let t be a uniformizer of B and denote by Y' the normalization of Y in $K(Y)[\sqrt[m]{t}]$. Let

$$Y' \to B' \to B$$

be the Stein factorization of $Y' \to B$. The morphism $Y' \to Y$ can ramify only in irreducible components of Y_b. Let C be a component of Y_b and y a closed point of C in the smooth locus of C. Locally at y we can write

$$t = \alpha u^{mn}$$

with a positive integer n, α a unit at x, and *div* $u = C$. Locally at x the covering $Y' \to Y$ is thus given by adjoining an m^{th}-root of the unit α, which is étale. Moreover, the multiplicity of a vertical prime divisor of Y' above C is n. We conclude that $Y' \to Y$ is an étale $\mathfrak{c}$-covering and the greatest integer in $\mathbb{N}(\mathfrak{c})$ dividing all multiplicities over B' of the vertical prime divisors in Y'_b is 1. Let x' be the image of $\bar{x}$ in Y', i.e., x' is a preimage of x. For every vertical prime divisor C of Y there is a vertical prime divisor C' of Y' over C passing through x' (Remember that all irreducible components of the special fibre Y_b contain x). Thus, the greatest integer in $\mathbb{N}(\mathfrak{c})$ dividing all multiplicities over B' of the vertical prime divisors in Y'_b passing through x' is still 1. After changing notation, we may assume that this is true already on Y. We conclude that for any (Zariski) open neighborhood U of x and any compactification $\bar{X}$ of U condition (iv) is satisfied.

Possibly replacing Y by an open neighborhood of x, we may assume that x is the only singular point of Y. We use Theorem 2.8 in order to construct a compactification $\bar{Y}$ of Y. After a tidy desingularization of $(\bar{Y}, \bar{Y} - Y)$ we may assume that the complement of Y is a tidy divisor. Property (iii) can be achieved by removing from Y one closed point (different from x) on each irreducible component of W and blowing up these points. Then condition (iii) continues to hold for every Zariski neighborhood of x contained in Y.

Choose an open neighborhood V of x and a compactification $\bar{X}/B$ as in Lemma 7.9. By Corollary 7.10, for any open neighborhood U of x which is contained in V and such that $\bar{D} = \bar{X} - U$ is a tidy divisor the pair $(\bar{X}, \bar{D})$ has enough tame coverings, i.e., condition (v) is satisfied.

On every irreducible component C of $\bar{X}_b$ choose a closed point $p_C \neq x$ in the smooth locus of C and not contained in any other irreducible component of $\bar{X}_b$. For each irreducible component of $\bar{X}_b$ remove from V a (horizontal) prime divisor intersecting C transversally at p_C. The complement $\bar{D}$ in $\bar{X}$ of the resulting open neighborhood U of x is tidy by construction and thus $(\bar{X}, \bar{D})$ has enough tame coverings. Moreover, $(\bar{X}, \bar{D})$ has properties (i) and (ii). $\square$

Chapter 8

The main results

In this chapter we collect the work of Chapters 4, 5, and 6 in order to prove the main theorems of this thesis. The first theorem treats the local case and the second one the global case.

Theorem 8.1: *Let Y/B be an arithmetic surface of local type and $\bar{y} \to Y$ a geometric point. Let $\mathfrak{c}$ be a full class of finite groups such that the residue characteristic of B is not contained in $\mathbb{N}(\mathfrak{c})$ and for all but finitely many primes $l \in \mathbb{N}(\mathfrak{c})$ the extension $B[\mu_l] \to B$ is a $\mathfrak{c}$-extension. Then Y has a basis of étale neighborhoods at $\bar{y}$ which are $K(\pi, 1)$ with respect to $\mathfrak{c}$.*

Proof: For every étale neighborhood $V \to Y$ of $\bar{y}$ we have to construct an étale neighborhood $U \to V$ of $\bar{y}$ which is $K(\pi, 1)$ with respect to $\mathfrak{c}$. Since U is again an arithmetic surface of local type over some B' over B, we may replace Y by U and B by B'. It thus suffices to show the existence of an étale neighborhood $U \to Y$ of $\bar{y}$ which is $K(\pi, 1)$ with respect to $\mathfrak{c}$.

Let $l_1, \ldots, l_n$ be the finitely many primes in $\mathbb{N}(\mathfrak{c})$ such that $B[\mu_{l_i}] \to B$ is not a $\mathfrak{c}$-extension. Consider the finite étale morphism

$$B' := B[\mu_{l_1}, \ldots, \mu_{l_n}] \to B.$$

Then

$$Y \times_B B' \xrightarrow{\pi'} B' \to B$$

constitutes the Stein factorization of $Y \times_B B' \to B$. Replacing $\pi : Y \to B$ with $\pi' : Y' \to B'$ we may assume $\mu_l \cong \mathbb{Z}/l\mathbb{Z}$ on B for all prime numbers $l \in \mathbb{N}(\mathfrak{c})$.

By Proposition 7.13 there is an étale neighborhood $U \to Y$ of $\bar{y}$ and a compactification $U \subseteq \bar{X}$ of $U \to B$ such that the complement of U in $\bar{X}$ is a tidy divisor satisfying properties (i)-(v) in the statement of Proposition 7.13.

By Corollary 2.5 and Lemma 2.22 we have to show

$$\varinjlim_{(\bar{X}', \bar{D}') \in \mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}} H^n(U', \Lambda) = 0 \tag{8.1}$$

for $\Lambda = \mathbb{Z}/l\mathbb{Z}$ with $l \in \mathbb{N}(\mathfrak{c})$ prime. For $n = 1$ the cohomology group $H^n(U, \Lambda)$ parameterizes finite étale l-coverings and thus

$$\varinjlim_{(\bar{X}', \bar{D}') \in \mathfrak{I}_{\bar{X}, \bar{D}, \bar{x}}} H^1(U', \Lambda) = 0$$

is automatically satisfied.

Let us show equality (8.1) for $n \geqslant 2$. We have chosen $\bar{X}$ and $\bar{D}$ such that the conditions (i)-(v) of Proposition 7.13 are satisfied. By Proposition 7.2, for every desingularized $\mathfrak{c}$-covering $(\bar{X}', \bar{D}') \to (\bar{X}, \bar{D})$ these conditions continue to hold on $\bar{X}'$. By propositions 5.17, 5.19, 5.20 for $n = 2$, $n = 3$, and $n \geqslant 4$, respectively, the restriction

$$\varinjlim_{(\bar{X}',\bar{D}')\in\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} H^n(X',\Lambda) \to \varinjlim_{(\bar{X}',\bar{D}')\in\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} H^n(U',\Lambda)$$

is surjective. Note that the assumptions made in these three propositions are among the conditions (i)-(v) of Proposition 7.13. It is thus legitimate to apply them in our situation.

We are left with treating the cohomology groups $H^n(X',\Lambda)$. Let us first examine the case $n \geqslant 3$. By Corollary 6.2 it suffices to prove that

$$\varinjlim_{(\bar{X}',\bar{D}')\in\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} H^i(B', R^j\pi'_*\Lambda) = 0$$

for $i+j=n$. This is true by Lemma 6.4.

Suppose now that $n=2$. By Lemma 6.4

$$\varinjlim_{(\bar{X}',\bar{D}')\in\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} H^i(B', R^j\pi'_*\Lambda) = 0$$

for $(i,j)=(1,1)$ and $(i,j)=(2,0)$. By Proposition 6.25 the composition

$$\varinjlim_{\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} ker(H^2(X',\Lambda) \to H^0(B', R^2\pi_*\Lambda)) \hookrightarrow \varinjlim_{\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} H^2(X',\Lambda) \to \varinjlim_{\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} H^2(U',\Lambda)$$

remains surjective. Again, the hypotheses of Proposition 6.25 are satisfied because they are part of conditions (i)-(v) of Proposition 7.13. We can now apply the part of Corollary 6.2 concerning $n=2$, which says that

$$\varinjlim_{(\bar{X}',\bar{D}')\in\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} ker(H^2(X',\Lambda) \to H^0(B', R^2\pi_*\Lambda)) = 0,$$

and thus also

$$\varinjlim_{(\bar{X}',\bar{D}')\in\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} H^2(U',\Lambda)$$

vanishes. This concludes the proof. □

Theorem 8.2: *Let Y/B be a regular arithmetic surface of global type and $\bar{x} \to Y$ a geometric point lying over a closed point $x \in Y$ mapping to $b \in B$. We assume that x is contained in the regular locus of $(Y_b)_{red}$. Let l be a prime number different from the residue characteristic of x. Let X_0 denote the completion of the generic fibre Y_η of $Y \to B$. Suppose that the action of the inertia group at $\bar{b}$ on the l-division points of the Jacobian of X_0 factors through an l-primary quotient. Then Y has a basis of étale neighborhoods at $\bar{x}$ which are $K(\pi,1)$ with respect to l.*

Proof: As in the local case we have to show the existence of an étale neighborhood $U \to Y$ of $\bar{y}$ which is $K(\pi,1)$ with respect to $\mathfrak{c}$. We may assume that B satisfies properties (i)-(iii) of Lemma 7.12. By Proposition 7.13 there is an étale neighborhood $U \to Y$ of $\bar{x}$ and a compactification $U \subseteq \bar{X}$ of $U \to B$ such that the complement of U in $\bar{X}$ is a tidy divisor with properties (i)-(v) of Proposition 7.2. Since by hypothesis x is not a special point, we may assume that all special points of the fibre $\bar{X}_b$ are contained in $\bar{D}$, i.e., W is regular. By shrinking B we may assume that $\bar{X}_b$ is the only possibly singular fibre of $\bar{X} \to B$. Let $\bar{b}$ be the geometric point of B induced by $\bar{x}$ and denote by η the generic point of B. Denote by $\eta_1|\eta$ the minimal extension of η such that $\mathcal{G}_{\eta_1}$ acts trivially on the l-division points of the Jacobian of $U_{\bar{\eta}}$. After replacing B with an étale neighborhood of $\bar{b}$ we may assume that $\eta_1|\eta$ is purely ramified at $\bar{b}$. By assumption, the extension $\eta_1|\eta$ is thus an l-extension. By Proposition 7.3 this property continues to hold on every desingularized l-covering of $(\bar{X},\bar{D})$. Furthermore, by Proposition 7.2 the above mentioned properties (i)-(v) are also stable under desingularized l-coverings.

Set $\Lambda = \mathbb{Z}/l\mathbb{Z}$. As in the local case we have to show that

$$\varinjlim_{(\bar{X}',\bar{D}')\in\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} H^n(U',\Lambda) = 0$$

for $n \geqslant 2$.

By the same reason as in the local case

$$\varinjlim_{(\bar{X}',\bar{D}')\in\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} H^n(\bar{X}',\Lambda) \to \varinjlim_{(\bar{X}',\bar{D}')\in\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} H^n(U',\Lambda)$$

is surjective (use Propositions 5.17, 5.19, and 5.20 as in the local case).

Again, we have to examine the cohomology groups $H^n(\bar{X}',\Lambda)$. Let us first treat the case $n \geqslant 3$. By Corollary 6.2 it suffices to prove that

$$\varinjlim_{(\bar{X}',\bar{D}')\in\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} H^i(B',R^j\pi'_*\Lambda) = 0$$

for $i+j=n$. This is true by Lemma 6.3 and Corollary 6.6. Note that for Corollary 6.6 we need the assumption on the Jacobian of the generic fibre.

Suppose now that $n=2$. By Lemma 6.3 and Proposition 6.24

$$\varinjlim_{(\bar{X}',\bar{D}')\in\mathfrak{I}_{\bar{X},\bar{D},\bar{x}}} H^i(B',R^j\pi'_*\Lambda) = 0$$

for $(i,j)=(2,0)$ and $(i,j)=(1,1)$, respectively. For Proposition 6.24 we need the assumption on the Jacobian again and moreover the hypothesis that W is regular. As in the local case the theorem now follows by Proposition 6.25 and Corollary 6.2. $\square$

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