

THE TAME SITE OF A SCHEME

KATHARINA HÜBNER AND ALEXANDER SCHMIDT

CONTENTS

1. Introduction	1
2. Definition of the tame site	3
3. Topogical invariance and excision	6
4. Continuity	7
5. The tame fundamental group	9
6. Joins of hensel rings	11
7. Comparison with Čech cohomology	13
8. Comparison with étale cohomology	16
9. Finiteness in dimension 1	17
10. Review of Huber pairs	19
11. Joins of henselian Huber pairs	25
12. Riemann-Zariski morphisms	30
13. Čech cohomology of discretely ringed adic spaces	36
14. Comparison between algebraic and adic tame cohomology	38
15. Purity and homotopy invariance	41
16. Connection to Suslin homology	42
References	45

1. INTRODUCTION

The étale fundamental group of a scheme plays a similar role in algebraic geometry as the topological fundamental group in algebraic topology. For a scheme X of characteristic $p > 0$ however, the p -part of $\pi_1^{\text{ét}}(X)$ is not well-behaved, e.g., it is not (\mathbb{A}^1 -)homotopy invariant. Therefore the *tame* fundamental group has been studied in positive and mixed characteristic (Grothendieck-Murre [GM71], Kerz-Schmidt [KS10]). Unfortunately, lacking an associated tame cohomology theory, sometimes ad hoc arguments have to be used in applications. It would be helpful to have a tame Grothendieck topology whose associated fundamental group is the tame fundamental group. Such a topology would provide a tame cohomology theory and, in addition, higher tame homotopy groups. The latter seem to be even more desirable because the higher étale homotopy groups of affine varieties vanish in positive characteristic (Achinger [Ach17]). In this paper we give a definition of a tame site that hopefully will prove useful as a tool to separate tame from wild phenomena in scheme theory.

Let X be a scheme over a fixed base scheme S . The tame site $(X/S)_t$ which will be defined below has the same underlying category as the small étale site $X_{\text{ét}}$ but less coverings. Every Nisnevich covering is tame, so we have natural morphisms of sites

$$X_{\text{ét}} \xrightarrow{\alpha} (X/S)_t \xrightarrow{\beta} X_{\text{Nis}}.$$

Tameness of a covering should be thought of as “at most tamely ramified along the boundary of compactifications over S ”. We will use the associated valuation space as a convenient technical tool to make this precise.

A technical advantage of the tame site sitting between the étale and the Nisnevich site is that motivic techniques like sheaves with transfers à la Voevodsky and the \mathbb{A}^1 -homotopy category of Morel-Voevodsky work essentially without change also for the tame site. A technical disadvantage is that the topology is not local enough in the sense that we cannot sufficiently separate the valuations that occur in the definition of tameness. We will resolve this problem by comparing the tame cohomology of an S -scheme X with the tame cohomology of the adic space $Spa(X, S)$ as defined in Hübner [Hüb18].

Our wish list of properties one would expect of an “ideal” tame site $(X/S)_t$ contains the following:

- (1) (Topological invariance) If $X' \rightarrow X$ is a universal homeomorphism of S -subschemes, then the sites $(X/S)_t$ and $(X'/S)_t$ are isomorphic.
- (2) (Comparison with étale cohomology for invertible coefficients) Let $F \in Sh_{\text{ét}}(X)$ be an abelian sheaf with $mF = 0$ for some m which is invertible on S . Then the natural morphism $\alpha : X_{\text{ét}} \rightarrow (X/S)_t$ induces isomorphisms

$$H_t^n(X/S, \alpha_* F) \cong H_{\text{ét}}^n(X, F) \quad \text{for all } n \geq 0.$$

- (3) (Comparison with étale cohomology for proper schemes) Let $F \in Sh_{\text{ét}}(X)$ be an abelian sheaf and assume that X/S is proper. Then

$$H_t^n(X/S, \alpha_* F) \cong H_{\text{ét}}^n(X, F) \quad \text{for all } n \geq 0.$$

- (4) (Homotopy invariance of cohomology): The projection $p : \mathbb{A}_X^1 \rightarrow X$ induces isomorphisms

$$H_t^n(X/S, F) \xrightarrow{\sim} H_t^n(\mathbb{A}_X^1/S, p^* F)$$

for all $n \geq 0$ and all torsion sheaves $F \in Sh_t(X/S)$.

- (5) (Excision) Let $\pi : X' \rightarrow X$ be an étale morphism and $Z \hookrightarrow X$, $Z' \hookrightarrow X'$ closed immersions with open complements $U = X \setminus Z$, $U' = X' \setminus Z'$. Assume that $\pi(U') \subset U$ and that π induces an isomorphism $Z'_{\text{red}} \xrightarrow{\sim} Z_{\text{red}}$. Then the induced homomorphisms between tame cohomology groups with supports

$$H_{t,Z}^n(X/S, F) \longrightarrow H_{t,Z'}^n(X'/S, \pi^* F)$$

are isomorphisms for every $F \in Sh_t(X/S)$ and all $n \geq 0$.

- (6) (Purity, locally constant coefficients) If X is regular and of finite type over S , S is pure of characteristic $p > 0$, $U \subset X$ is a dense open subscheme and F is a locally constant p -torsion sheaf, then

$$H_t^n(X/S, F) \cong H_t^n(U/S, F|_U).$$

for all $n \geq 0$.

Moreover, we expect tame versions of the proper and the smooth base change theorem and purity for deRham-Witt sheaves. In addition, we expect finiteness properties of tame cohomology groups and cohomological dimension in those cases where they hold for the étale cohomology (e.g., schemes of finite type over \mathbb{Z} or over separably closed fields).

In this paper we will prove properties (1), (2), (3) and (5) in general. Moreover, we prove (4) for locally constant coefficients and (6) in the pure characteristic case assuming resolution of singularities. If X is locally noetherian, we show that the (curve-)tame fundamental group as defined in [KS10] occurs as the natural fundamental group of the site $(X/S)_t$. For schemes of dimension less or equal to

one, we prove finiteness properties in Section 9. Purity for deRham-Witt sheaves on curves has been addressed in [Hüb20].

The essential point in proving (4) and (6) is the comparison theorem 14.8 between the tame cohomology of an S -scheme X and the cohomology of the associated adic space $\mathrm{Spa}(X, S)$. At the moment, we can prove this only in pure characteristic. The method is to compare the associated Čech cohomologies. To prove that the corresponding sheaf and the Čech cohomologies coincide, we adapt M. Artin's method of joins of hensel rings [Art71] to the respective situations. In the adic case this is rather involved and requires a careful analysis of relative Riemann-Zariski spaces as introduced by M. Temkin [Tem11].

Finally, in Section 16 we give an application to Suslin homology which in fact was the initial motivation for our construction of the tame site. For a scheme X of finite type over an algebraically closed field k , we construct natural maps from tame to Suslin cohomology

$$\beta_n : H_i^n(X, \mathbb{Z}/m\mathbb{Z}) \longrightarrow H_S^n(X, \mathbb{Z}/m\mathbb{Z}), \quad n \geq 0, \quad m \geq 1.$$

We conjecture that all β_n are isomorphisms. This extends work of Suslin-Voevodsky [SV96] in the case $(m, \mathrm{char}(k)) = 1$ and of Geisser-Schmidt [GS16] for $n = 1$ and general m .

Acknowledgement: The authors want to thank M. Temkin for helpful discussions on Riemann-Zariski points. The construction of the map between tame and Suslin cohomology arose from discussions of the second author with T. Geisser while writing [GS16].

2. DEFINITION OF THE TAME SITE

By a valuation on a field we mean a non-archimedean valuation, not necessarily discrete or of finite rank. The trivial valuation is included. If v is a valuation, we denote by $\mathcal{O}_v, \mathfrak{m}_v$ and $k(v)$ the valuation ring, its maximal ideal and the residue field.

Definition 2.1. Let $f : X \rightarrow S$ be a scheme morphism. An S -valuation on X is a valuation v on the residue field $k(x)$ of some point $x \in X$ such that there exists a morphism $\varphi : \mathrm{Spec}(\mathcal{O}_v) \rightarrow S$ making the diagram

$$\begin{array}{ccc} \mathrm{Spec}(k(x)) & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \mathrm{Spec}(\mathcal{O}_v) & \xrightarrow{\varphi} & S \end{array}$$

commutative (if S is separated, φ is unique if it exists). The set of all S -valuations is denoted by $\mathrm{Val}_S X$. If $S = \mathrm{Spec} R$ is affine, we also write $\mathrm{Val}_R X$ for $\mathrm{Val}_S X$. We denote elements of $\mathrm{Val}_S X$ in the form (x, v) , $x \in X$, $v \in \mathrm{Val}_S(k(x))$.

Remarks 2.2. (1) Directly by definition, we have a disjoint union decomposition

$$\mathrm{Val}_S X = \coprod_{x \in X} \mathrm{Val}_S(k(x)).$$

- (2) If $S = \mathrm{Spec} k$ for a field k , then S -valuations are the valuations with trivial restriction to k .
- (3) If $S = \mathrm{Spec} \mathbb{Z}$, then every valuation is an S -valuation.

Given a commutative square of scheme morphisms

$$\begin{array}{ccc} X' & \xrightarrow{\varphi} & X \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{\psi} & S, \end{array}$$

there is an associated map

$$\mathrm{Val}_\psi \varphi : \mathrm{Val}_{S'} X' \longrightarrow \mathrm{Val}_S X, \quad (x', v') \longmapsto (x = \varphi(x'), v = v'|_{k(x)}).$$

Definition 2.3. The *tame site* $(X/S)_t$ consists of the following data:

The category $\mathrm{Cat}(X/S)_t$ is the category of étale morphisms $p : U \rightarrow X$.

A family $(U_i \rightarrow U)_{i \in I}$ of morphisms in $\mathrm{Cat}(X/S)_t$ is a covering if it is an étale covering and for every point $(u, v) \in \mathrm{Val}_S U$ there exists an index i and a point $(u_i, v_i) \in \mathrm{Val}_S U_i$ mapping to (u, v) such that v_i/v is (at most) tamely ramified in the sense of valuation theory.

We will use the notation $Sh_t^{\mathrm{Sets}}(X/S)$ resp. $Sh_t(X/S)$ for the category of sheaves of sets resp. abelian groups on $(X/S)_t$.

Remark 2.4. There are obvious morphisms of sites

$$X_{\mathrm{et}} \xrightarrow{\alpha} (X/S)_t \xrightarrow{\beta} X_{\mathrm{Nis}}.$$

In particular, every étale sheaf is a tame sheaf. We mention the following special cases:

For an abelian group A , the presheaf (constant Zariski-sheaf)

$$U \longmapsto \underline{A}(U) := \mathrm{Map}_{\mathrm{cts}}(U, A)$$

is a tame sheaf.

For a quasi-coherent sheaf Q on X_{Zar} , the presheaf $W(Q)$ given by

$$W(Q)(U \xrightarrow{h} X) := (h_{\mathrm{Zar}}^*(Q) \otimes_{h_{\mathrm{Zar}}^*(\mathcal{O}_X)} \mathcal{O}_U)(U)$$

is a sheaf on $(X/S)_t$.

Consider a commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

of scheme morphisms. Then the pullback $(U \rightarrow X) \mapsto (U' = U \times_X X' \rightarrow X')$ induces a morphism of sites $\varphi : (X'/S')_t \rightarrow (X/S)_t$.

Lemma 2.5. *The inverse image functors $\varphi_{\mathrm{Sets}}^* : Sh_t^{\mathrm{Sets}}(X/S) \rightarrow Sh_t^{\mathrm{Sets}}(X'/S')$ and $\varphi^* : Sh_t(X/S) \rightarrow Sh_t(X'/S')$ are exact. The direct image functor $\varphi_* : Sh_t(X'/S') \rightarrow Sh_t(X/S)$ sends injective abelian sheaves to injective abelian sheaves.*

Proof. The underlying categories of the tame sites are the same as for the small étale sites. Hence the first statement follows in the same way as for the étale topology from the fact that the index categories for the colimits which define the presheaf-pullback are filtered (cf. [Art62, II, Thm. 4.14]). The second assertion is a formal consequence of the adjunction $\varphi^* \dashv \varphi_*$. \square

Lemma 2.6. *Let $X \xrightarrow{f} S \xrightarrow{g} T$ be scheme morphisms. If g is proper, then the natural morphism of sites*

$$(X/S)_t \longrightarrow (X/T)_t$$

is an isomorphism.

Proof. If $g : S \rightarrow T$ is proper, then the valuative criterion for properness implies $\text{Val}_S U = \text{Val}_T U$ for every $p : U \rightarrow X$ étale. Whence the statement. \square

Definition 2.7. A *tame point* of $(X/S)_t$ is a pair (\bar{x}, \bar{v}) where $\bar{x} : \text{Spec } k(\bar{x}) \rightarrow X$ is a morphism from the spectrum of a field to X and \bar{v} is an S -valuation on $k(\bar{x})$ such that $k(\bar{x})$ does not admit nontrivial, finite, separable extensions for which \bar{v} is tamely ramified. In other words,

- (1) $k(\bar{x})$ is strictly henselian with respect to \bar{v} , and
- (2) the value group of \bar{v} is divisible by any integer prime to the residue characteristic.

If (1) holds and the residue field of \bar{v} has characteristic zero, then condition (2) means that $k(\bar{x})$ is separably closed. If the residue field of \bar{v} has characteristic $p > 0$, it means that the absolute Galois group of $k(\bar{x})$ is a pro- p -group.

If \bar{v} is the trivial valuation, we call the point (\bar{x}, \bar{v}) *trivial point*. Note that (regardless of the residue characteristic) the field $k(\bar{x})$ of a trivial point is separably closed.

A tame point (\bar{x}, \bar{v}) induces a morphism of sites

$$\bar{x} : (\text{Spec } k(\bar{x}) / \text{Spec } \mathcal{O}_{\bar{v}})_t \longrightarrow (X/S)_t.$$

By Lemma 2.5, the inverse image sheaf functor \bar{x}^* is exact. Since every tame covering of $\text{Spec}(k(\bar{x}))$ splits, the global sections functor on $(\text{Spec } k(\bar{x}) / \text{Spec } \mathcal{O}_{\bar{v}})_t$ is exact. Therefore the functor “stalk at \bar{x} ”

$$F \longmapsto F_{\bar{x}} := \Gamma(\text{Spec } k(\bar{x}), \bar{x}^* F),$$

is a topos-theoretical point of $(X/S)_t$. The trivial points correspond to the usual geometric points of the étale site (followed by $\alpha : X_{\text{ét}} \rightarrow (X/S)_t$).

Remark 2.8. The stalk of the structure sheaf at a point of the Nisnevich resp. étale site is the henselization resp. the strict henselization of the local ring of the underlying scheme theoretic point.

Let (\bar{x}, \bar{v}) be a tame point of X/S with underlying scheme theoretic point $x_0 \in X$. Let $k(\bar{x})_s/k(x_0)$ be the relative separable algebraic closure of $k(x_0)$ in $k(\bar{x})$ and choose a separable algebraic closure $k^s/k(\bar{x})_s$. Denote the strict henselization of \mathcal{O}_{X, x_0} with respect to $k^s/k(x_0)$ by $\mathcal{O}_{X, \bar{x}}^{sh}$. Then the stalk $\mathcal{O}_{X, (\bar{x}, \bar{v})}$ of the structure sheaf at (\bar{x}, \bar{v}) is the unique ind-étale subextension of

$$\mathcal{O}_{X, x_0}^h \hookrightarrow \mathcal{O}_{X, \bar{x}}^{sh}$$

which corresponds to the subextension $k(\bar{x})_s$ of $k^s/k(x_0)$.

Definition 2.9. We call the ring $\mathcal{O}_{X, (\bar{x}, \bar{v})}$ described in Remark 2.8 the *tame henselization* of X at the point (\bar{x}, \bar{v}) .

Lemma 2.10. *The site $(X/S)_t$ has enough points.*

Proof. We fix a separable closure $k(x)^s$ of $k(x)$ for every point $x \in X$. For each S -valuation v on $k(x)$ we choose a prolongation \bar{v} to $k(x)^s$ and consider the fixed field $K_v := (k(x)^s)^{R_{\bar{v}}}$ of its ramification group. Then the tame points $(\bar{x}_L, \bar{v}|_L)$ with

$$\bar{x}_L : \text{Spec } L \longrightarrow \text{Spec } k(x) \longrightarrow X,$$

where x runs through the points of X , v runs through the S -valuations of $k(x)$ and L runs through the finite subextensions of $k(x)^s/K_v$ establish a conservative set of points. Indeed, this follows immediately from the definition of coverings in $(X/S)_t$. \square

We learn that exactness of sequences of tame sheaves can be checked stalkwise at tame points. As in the étale case, we obtain

Lemma 2.11. *Let $\pi : X' \rightarrow X$ be a finite morphism between S -schemes. Then*

$$\pi_* : Sh_t(X'/S) \rightarrow Sh_t(X/S)$$

is exact. For $F' \in Sh_t(X'/S)$ we have $H_t^q(X/S, \pi_ F') \cong H_t^q(X'/S, F')$ for all $q \geq 0$.*

Proof. The proof is similar to that for the étale topology: The first statement follows by considering the stalks of $\pi_* F'$ and using the fact that a finite algebra over a local henselian ring is a finite product of local henselian rings. From this we obtain $R^q \pi_* F' = 0$ for $q > 0$ and the second statement follows from the Leray-Serre spectral sequence. \square

3. TOPOLOGICAL INVARIANCE AND EXCISION

We start by collecting properties of the tame site which follow from or in a similar way as the corresponding properties of the small étale site.

Proposition 3.1 (Topological invariance). *Let $f : X_0 \rightarrow X$ be a universal homeomorphism of S -schemes. Then the tame sites $(X_0/S)_t$ and $(X/S)_t$ are isomorphic. In particular, there are isomorphisms*

$$H_t^n(X/S, F) \xrightarrow{\sim} H_t^n(X_0/S, f^* F)$$

for all $n \geq 0$ and $F \in Sh_t(X/S)$.

Proof. By the topological invariance of the small étale site [Gro67, Theorem 18.1.2], $Y \mapsto Y \times_X X_0$ defines an equivalence $f^{-1} : \text{Cat}(X/S)_t \xrightarrow{\sim} \text{Cat}(X_0/S)_t$ of the underlying categories. By [Sta19, TAG 04DF], f is integral. Hence, for any point $x_0 \in X_0$ with image $x \in X$, the induced field extension $k(x_0)/k(x)$ is algebraic and a universal homeomorphism, hence purely inseparable. This implies that every S -valuation on $k(x)$ has a unique extension to $k(x_0)$. By [Sch20, Lemma 1.1], for a valuation v_0 on $k(x_0)$ with restriction v to $k(x)$, the induced homomorphism on ramification groups is an isomorphism. Hence, f^{-1} respects tame coverings. The result follows. \square

Let $X \rightarrow S$ be an S -scheme, $i : Z \hookrightarrow X$ a closed immersion and $U = X \setminus Z$ the open complement. The right derivatives of the left exact functor “sections with support in Z ”

$$F \mapsto \ker(F(X) \rightarrow F(U))$$

are called the *tame cohomology groups with support in Z* . Notation: $H_{t,Z}^*(X/S, F)$.

Exactly in the same way as in the étale case ([Art62, III, (2.11)]), one shows the existence of a long exact sequence

$$\dots \rightarrow H_{t,Z}^p(X/S, F) \rightarrow H_t^p(X/S, F) \rightarrow H_t^p(U/S, F) \rightarrow H_{t,Z}^{p+1}(X/S, F) \rightarrow \dots$$

Proposition 3.2 (Excision). *Let $\pi : X' \rightarrow X$ be a morphism of S -schemes, $Z \hookrightarrow X$, $Z' \hookrightarrow X'$ closed immersions and $U = X \setminus Z$, $U' = X' \setminus Z'$ the open complements. Assume that*

- $\pi : X' \rightarrow X$ is étale,
- π induces an isomorphism $Z'_{\text{red}} \xrightarrow{\sim} Z_{\text{red}}$,
- $\pi(U') \subset U$.

Then the induced homomorphism

$$H_{t,Z}^p(X/S, F) \xrightarrow{\sim} H_{t,Z'}^p(X'/S, \pi^* F)$$

is an isomorphism for every sheaf $F \in Sh_t(X/S)$ and all $p \geq 0$.

Proof. The standard proof for the étale topology applies: Since π belongs to $\text{Cat}(X/S)_t$, π^* has the exact left adjoint “extension by zero”. Hence π^* sends injectives to injectives. Since π^* is exact, it suffices to deal with the case $p = 0$. Without changing the statement, we can replace all occurring schemes by their reductions (see Proposition 3.1). By assumption, $X' \sqcup U \rightarrow X$, is a tame covering. For $\alpha \in H_{t,Z}^0(X, F)$ mapping to zero in $H_{t,Z'}^0(X', \pi^*F)$ we therefore obtain $\alpha = 0$.

Now let $\alpha' \in H_{t,Z'}^0(X', \pi^*F)$ be given. We show that α' and $0 \in H_t^0(U, F)$ glue to an element in $H_{t,Z}^0(X, F)$. The only nontrivial compatibility on intersections is $p_1^*(\alpha') = p_2^*(\alpha')$ for $p_1, p_2 : X' \times_X X' \rightarrow X'$. This can be checked on stalks noting that $Z' \xrightarrow{\sim} Z$ implies that the two projections $Z' \times_Z Z' \rightarrow Z'$ are the same. \square

4. CONTINUITY

The aim of this section is to prove theorems 4.5 and 4.6 below, which deal with the cohomology of colimits and limits. We start with the following finiteness result:

Theorem 4.1. *Assume that S is quasi-compact and quasi-separated and let X be a quasi-compact S -scheme. Then every tame covering of X admits a finite subcovering.*

For the proof we need some preparations. The definition of the tame site suggests to tackle the problem by endowing $\text{Val}_S X$ with a suitable topology and then to apply a compactness argument. A convenient way to do this is to move the problem to the setting of adic spaces, where it was solved in [Hüb18].

Following M. Temkin [Tem11], we associate with a morphism of schemes $f : X \rightarrow S$ the set $\text{Spa}(X, S)$, whose elements are triples (x, v, ε) , where x is a point of X , v is a valuation of $k(x)$ and $\varepsilon : \text{Spec } \mathcal{O}_v \rightarrow S$ is a morphism compatible with $\text{Spec } k(x) \rightarrow S$. If S is separated, then ε is uniquely determined (if it exists) by (x, v) . In other words, we have a natural forgetful map

$$\text{Spa}(X, S) \longrightarrow \text{Val}_S X,$$

which is bijective if S is separated. Here, $\text{Val}_S X$ is the earlier defined space of pairs (x, v) with $x \in X$ and v a valuation of $k(x)$ with center in S (but we do not specify a homomorphism $\text{Spec } \mathcal{O}_v \rightarrow S$).

Remark 4.2. Note that the space $\text{Val}_X(S)$ defined in [Tem11] is not the same as our $\text{Val}_S X$. Temkin’s $\text{Val}_X(S)$ denotes the relative Riemann Zariski space of X over S . It can be viewed as the subspace of all Riemann-Zariski points (see Definition 10.1 below) of $\text{Spa}(X, S)$.

If $X = \text{Spec } A$ and $S = \text{Spec } R$ are affine, then $\text{Spa}(X, S)$ coincides with the set of points of R. Huber’s affinoid adic space $\text{Spa}(A, A^+)$, where A^+ is the integral closure of R in A and A and A^+ are equipped with the discrete topology, cf. [Hub96]. By patching, we obtain on $\text{Spa}(X, S)$ the structure of a discretely ringed adic space. If

$$\begin{array}{ccc} X' & \xrightarrow{\varphi} & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

is a commutative diagram of scheme morphisms, then

$$\text{Spa}(\varphi) : \text{Spa}(X', S') \longrightarrow \text{Spa}(X, S)$$

is a morphism of adic spaces, in particular it is continuous. The topology of $\text{Spa}(X, S)$ is generated by the images of $\text{Spa}(X', S')$ in $\text{Spa}(X, S)$ coming from commutative diagrams as above with X' and S' affine, $X' \rightarrow X$ an open immersion and $S' \rightarrow S$ locally of finite type.

Recall that a topological space is called *spectral* if it is homeomorphic to $\text{Spec}(R)$ for some commutative ring R .

Lemma 4.3. *If S and X are quasi-compact and quasi-separated, then $\text{Spa}(X, S)$ is a spectral space. In particular, $\text{Spa}(X, S)$ is a quasi-compact and quasi-separated topological space.*

Proof. By [Hoc69, Corollary to Proposition 7], we have to show that $\text{Spa}(X, S)$ is locally spectral, quasi-compact and quasi-separated. Let

$$S = \bigcup_i S_i \quad \text{and} \quad X \times_S S_i = \bigcup_{j_i} X_{j_i}$$

be finite coverings by open, affine subschemes. Then

$$\text{Spa}(X, S) = \bigcup_{i, j_i} \text{Spa}(X_{j_i}, S_i).$$

The spaces $\text{Spa}(X_{j_i}, S_i)$ are spectral by [Hub93, Theorem 3.5]. Hence, $\text{Spa}(X, S)$ is locally spectral and quasi-compact. In order to show quasi-separatedness, fix i, i' and $j_i, j_{i'}$. The intersection

$$\text{Spa}(X_{j_i}, S_i) \cap \text{Spa}(X_{j_{i'}}, S_{i'}) = \text{Spa}((X_{j_i} \cap X_{j_{i'}}), S_i \cap S_{i'})$$

is quasi-compact by what we just saw and the quasi-separatedness assumptions on S and X . Therefore, $\text{Spa}(X, S)$ is quasi-separated. \square

If $\varphi : Y \rightarrow X$ is an étale morphism of S -schemes, then $\text{Spa}(\varphi) : \text{Spa}(Y, S) \rightarrow \text{Spa}(X, S)$ is an étale morphism of adic spaces in the sense of [Hub96, Definition 1.6.5]. By definition [Hüb18, Definition 3.1], $\text{Spa}(\varphi)$ is *tame* if for every point (y, v, ε) of $\text{Spa}(Y, S)$, the valuation v is (at most) tamely ramified in the finite separable field extension $k(y)/k(\varphi(y))$. In particular, tameness at a point of $\text{Spa}(Y, S)$ only depends on its image in $\text{Val}_S Y$. The following proposition is essential for us:

Proposition 4.4 (Openness of the tame locus, [Hüb18, Corollary 4.4]). *Let $\varphi : Y \rightarrow X$ be an étale morphism of S -schemes. Then the set of points where the morphism of adic spaces $\text{Spa}(\varphi) : \text{Spa}(Y, S) \rightarrow \text{Spa}(X, S)$ is tame is an open subset of $\text{Spa}(Y, S)$.*

Proof of Theorem 4.1. Since X admits a finite Zariski-covering by affine schemes, we may assume that X is affine. Let $(U_i \rightarrow X)$ be a tame covering. By Proposition 4.4, the tame locus of $\text{Spa}(U_i, S) \rightarrow \text{Spa}(X, S)$ is an open subspace \mathcal{U}_i of $\text{Spa}(U_i, S)$ for all i . By [Hub96, 1.10.12.i)], étale morphisms of adic spaces are open, hence the images $\mathcal{W}_i \subset \text{Spa}(X, S)$ of the \mathcal{U}_i are open. As $(U_i \rightarrow X)$ is a tame covering, the \mathcal{W}_i cover $\text{Spa}(X, S)$. Since X is affine, $\text{Spa}(X, S)$ is a spectral space by Lemma 4.3. In particular, $\text{Spa}(X, S)$ is quasi-compact.

We conclude that there is a finite subset $J \subset I$ with $\text{Spa}(X, S) = \bigcup_{j \in J} \mathcal{W}_j$. Therefore, $(U_j \rightarrow X)_{j \in J}$ is a finite subcovering of $(U_i \rightarrow X)_{i \in I}$. \square

As in [AGV72, VII, 3.2] for the étale site, we define the restricted tame site

$$(X/S)_t^{\text{res}}$$

as the restriction of $(X/S)_t$ to the subcategory of all $U \in \text{Cat}(X/S)_t$ such that $U \rightarrow X$ is of finite presentation.

Assume that X is quasi-compact and quasi-separated. Then the same is true for any such U and Theorem 4.1 shows that the restricted site is noetherian. Moreover, the categories of sheaves on (X/S) and $(X/S)_t^{\text{res}}$ are naturally equivalent. Hence the same argument as in the étale case [AGV72, VII, Proposition 3.3] shows

Theorem 4.5. *Assume that S and X are quasi-compact and quasi-separated and let (F_i) be a filtered direct system of abelian sheaves on $(X/S)_t$. Then*

$$\operatorname{colim}_i H_t^i(X/S, F_i) \cong H_t^i(X/S, \operatorname{colim}_i F_i).$$

Now we consider limits of schemes. The main result is:

Theorem 4.6. *Let $(f_i : X_i \rightarrow S_i)_{i \in I}$ be a filtered inverse system of scheme morphisms with affine transition morphisms and assume that all S_i and X_i are quasi-compact and quasi-separated. Denote by $X \rightarrow S$ its inverse limit. Then, for every étale surjection $U \rightarrow X$ of finite presentation which is a tame covering with respect to S , there exists an index $i \in I$ and an étale surjection $U_i \rightarrow X_i$ of finite presentation which is a tame covering with respect to S_i and whose base change to X is $U \rightarrow X$.*

The natural map from $(X/S)_t^{\text{res}}$ to the limit topology of the $(X_i/S_i)_t^{\text{res}}$ is an isomorphism.

Proof. By [AGV72, VI, 5.6], there is $i \in I$ such that $U \rightarrow X$ is the pullback of an étale surjection $U_i \rightarrow X_i$. For every $j \geq i$ denote by $U_j \rightarrow X_j$ its pullback to X_j . We have to show that there is $j \geq i$ such that $U_j \rightarrow X_j$ is tame over S_j . By Proposition 4.4, the set of points in $\operatorname{Spa}(U_j, S_j)$ where $\operatorname{Spa}(U_j, S_j) \rightarrow \operatorname{Spa}(X_j, S_j)$ is tame is open. Denote by $\mathcal{Z}_j \subset \operatorname{Spa}(X_j, S_j)$ the complement of the image of this set. Since étale morphisms of adic spaces are open, \mathcal{Z}_j is a closed subset. Since tameness is stable under base change, for $k \geq j \geq i$ the transition map $\operatorname{Spa}(X_k, S_k) \rightarrow \operatorname{Spa}(X_j, S_j)$ sends \mathcal{Z}_k to \mathcal{Z}_j . As $U \rightarrow X$ is a tame covering,

$$\lim_{j \geq i} \mathcal{Z}_j = \emptyset.$$

By Lemma 4.3, $\operatorname{Spa}(X_j, S_j)$ is spectral and hence by [Sta19, TAG 0901] compact in its constructible topology. In particular, the \mathcal{Z}_j are compact in the constructible topology. Since the inverse limit of nonempty compact spaces is nonempty, we conclude that $\mathcal{Z}_j = \emptyset$ for large enough j . In other words $U_j \rightarrow X_j$ is a tame covering. This shows the first statement.

For the second statement note that every tame covering in the limit topology of the $(X_i/S_i)_t$ defines a covering in $(X/S)_t$. To show the equivalence, it suffices to show that every morphisms

$$U \rightarrow V$$

of étale X -schemes of finite presentation which is a tame covering with respect to S comes by base change from a finite level. This follows from the first statement \square

As an immediate consequence we obtain:

Theorem 4.7. *In the situation of Theorem 4.6 assume that $i_0 \in I$ is a final object. Let F_0 be a sheaf of abelian groups on $(X_{i_0}/S_{i_0})_t$. For $i \in I$ denote by F_i its pullback to $(X_i/S_i)_t$ and by F its pullback to $(X/S)_t$. Then the natural map*

$$\operatorname{colim}_{i \in I} H_t^n(X_i/S_i, F_i) \longrightarrow H_t^n(X/S, F)$$

is an isomorphism for all $n \geq 0$.

5. THE TAME FUNDAMENTAL GROUP

In this section we assume that X is locally noetherian. Then the small étale site of X is locally connected [AM69, Prop. 9.5] and, having the same underlying category, the same is true for the site $(X/S)_t$. Therefore by [AM69, §9], after choosing a tame base point (\bar{x}, \bar{v}) , we have the tame homotopy (pro)groups

$$\pi_n^t(X/S, (\bar{x}, \bar{v})).$$

We call the first tame homotopy group $\pi_1^t(X/S, (\bar{x}, \bar{v}))$ the *tame fundamental group*. If X is connected, then by [AM69, Corollary 10.7], it classifies pointed torsors for the tame topology. Moreover, by [AM69, Corollary 10.8], there is a natural equivalence between the category of locally constant abelian tame sheaves and the category of abelian groups A with a homomorphism (of pro-groups) $\pi_1^t(X/S, (\bar{x}, \bar{v})) \rightarrow \text{Aut}(A)$.

For a group G , the set of isomorphism classes of tame G -torsors is a subset of the set of isomorphism classes of étale G -torsors. Hence, for a trivial point $\bar{x} = (\bar{x}, v_{\text{triv}})$, the morphism of pointed sites

$$(X, \bar{x})_{\text{et}} \longrightarrow (X/S, \bar{x})_t$$

induces a surjection $\pi_1^{\text{et}}(X, \bar{x}) \twoheadrightarrow \pi_1^t(X/S, \bar{x})$ (here π_1^{et} denotes the étale fundamental group of [AM69], which coincides with the *pro-groupe fondamentale élargi* of [DG70, X, §6]). We conclude that the finite quotients of $\pi_1^t(X/S, \bar{x})$ correspond to pointed, finite, connected étale Galois covers $f : (Y, \bar{y}) \rightarrow (X, \bar{x})$ that are coverings for the tame site. Since the Galois group acts transitively on the fibres, this means that for every $y \in Y$ and $w \in \text{Val}_S k(y)$ the extension $k(y)/k(f(y))$ is at most tamely ramified at w . We will call such covers *tame Galois covers*. If X is geometrically unibranch, then the group $\pi_1^{\text{et}}(X, \bar{x})$ is profinite by [AM69, Theorem 11.1], and hence the same holds for its factor group $\pi_1^t(X/S, \bar{x})$.

From now on we assume that S is integral, pure-dimensional (i.e., $\dim S = \dim \mathcal{O}_{S,s}$ for every closed point $s \in S$), separated and excellent. Furthermore, we assume that $X \rightarrow S$ is separated and of finite type. For integral X we put

$$\dim_S X := \deg.\text{tr.}(k(X)/k(T)) + \dim_{\text{Krull}} T,$$

where T is the closure of the image of X in S . If the image of X in S contains a closed point of T , then $\dim_S X = \dim_{\text{Krull}} X$ by [Gro67, 5.6.5]. We call X an *S-curve*, if $\dim_S X = 1$. For a regular connected S -curve C , let \bar{C} be the unique regular compactification of C over S . Then the finite tame Galois covers of C are exactly those étale Galois covers which are tamely ramified along $\bar{C} \setminus C$ in the classical sense (e.g. [GM71]). We recall the notion of curve-tameness from [KS10]:

Definition 5.1. Let $Y \rightarrow X$ be an étale cover of separated S -schemes of finite type. We say that $Y \rightarrow X$ is *curve-tame* if for any morphism $C \rightarrow X$ with C a regular S -curve, the base change $Y \times_X C \rightarrow C$ is tamely ramified along $\bar{C} \setminus C$.

After choosing a geometric point \bar{x} , we obtain the *curve-tame fundamental group* $\pi_1^{\text{ct}}(X, \bar{x})$, which is the profinite quotient of $\pi_1^{\text{et}}(X, \bar{x})$ which classifies finite curve-tame Galois covers. This curve-tame fundamental group was considered in [Sch07], [KS09], [GS16].

Proposition 5.2. *The curve-tame fundamental group is the profinite completion of the tame fundamental group. If X is geometrically unibranch, then both are isomorphic:*

$$\pi_1^t(X, \bar{x}) \cong \pi_1^{\text{ct}}(X, \bar{x}).$$

For the proof we recall the “Key Lemma”, Lemma 2.4 of [KS10]. The normalization of an integral scheme Z in its function field is denoted by \tilde{Z} .

Lemma 5.3. *Let A be a local, normal and excellent ring and let $X' \subset X = \text{Spec}(A)$ be a nonempty open subscheme. Let $Y' \rightarrow X'$ be an étale Galois cover of prime degree p . Assume that $X \setminus X'$ contains an irreducible component D of codimension one in X such that $Y' \rightarrow X'$ is ramified along the generic point of D . Then there exists an integral, closed subscheme C of X of Krull-dimension one with $C' := C \cap X' \neq \emptyset$ such that the base change $Y' \times_{X'} \tilde{C}' \rightarrow \tilde{C}'$ is ramified along a point of $\tilde{C} \setminus \tilde{C}'$.*

Proof of Proposition 5.2. We have to show that a finite étale Galois cover is tame if and only if it is curve-tame. Both notions coincide for regular S -curves and are stable under base change. Hence finite tame Galois covers are curve-tame and it remains to show the converse.

So let $f : Y \rightarrow X$ be a finite étale Galois cover, $y \in Y$ a point, $x = f(y)$ and $w \in \text{Val}_S k(y)$ such that w is wildly ramified in $k(y)/k(x)$. We have to find an S -curve C and a morphism $\varphi : C \rightarrow X$ such that $C \times_X Y \rightarrow C$ is wildly ramified at some point in $\bar{C} \setminus C$.

We first assume that $\text{Gal}(Y/X)$ is cyclic of order $p := \text{char } k(w)$. Then, since w is ramified, $k(y)/k(x)$ is also of degree p . Let X' be the closure of $\{x\}$ in X with reduced scheme structure and $Y' = X' \times_X Y$. In order to find $\varphi : C \rightarrow X$, we may replace X by X' and Y by the connected component of y in Y' , i.e., we may assume that x and y are the generic points of the integral schemes X and Y . Replacing X by its normalization \tilde{X} and Y by its base change along $\tilde{X} \rightarrow X$, we may assume that X and Y are normal.

By quasi-purity of the branch locus [Sch20], we find a geometric discrete rank-one S -valuation v on $k(y)$ with $\text{char } k(v) = p$ that is (wildly) ramified in $k(y)/k(x)$. Then we can find a normal compactification (over S) \bar{X} of X such that the center D of v on \bar{X} is of codimension one. Applying Lemma 5.3 to the local ring of some closed point of \bar{X} contained in D , we obtain a morphism $\varphi : C \rightarrow X$ with the required property.

It remains to reduce the general case to the cyclic-order- p -case. For this let

$$A \subset R_w(k(y)/k(x)) \subset \text{Gal}(k(y)/k(x))$$

be a subgroup of order p (here R_w denotes the ramification group) and let $X' = Y_A$. Then Y/X' is wildly ramified and we find a morphism $\varphi : C \rightarrow X'$ with the required property. Then the composite of φ with the projection $X' \rightarrow X$ yields what we need. \square

6. JOINS OF HENSEL RINGS

In this section we recall some results of M. Artin [Art71]. We need them in slightly higher generality, therefore we have to go into a little more detail.

Let A be a normal integral domain, and $\mathfrak{p}, \mathfrak{q}$ prime ideals in A . Suppose we are given embeddings of the henselizations $A_{\mathfrak{p}}^h, A_{\mathfrak{q}}^h$ into a separable closure \bar{K} of K . We call the ring $D = [A_{\mathfrak{p}}^h, A_{\mathfrak{q}}^h] \subset \bar{K}$ generated by $A_{\mathfrak{p}}^h$ and $A_{\mathfrak{q}}^h$ their *join* (with respect to the chosen embeddings to \bar{K}):

$$\begin{array}{ccc} & \bar{K} & \\ & \downarrow & \\ & D = [A_{\mathfrak{p}}^h, A_{\mathfrak{q}}^h] & \\ & \swarrow \quad \searrow & \\ A_{\mathfrak{p}}^h & & A_{\mathfrak{q}}^h \\ & \swarrow \quad \searrow & \\ & A & \end{array}$$

Artin's main technical result is the following.

Theorem 6.1 ([Art71, Theorems 2.2, 2.5]). *The join $D = [A_{\mathfrak{p}}^h, A_{\mathfrak{q}}^h]$ is a local henselian ring. If neither of the primes $\mathfrak{p}, \mathfrak{q}$ contains the other, then D is strictly*

henselian and the pullback of its maximal ideal to A is strictly contained in \mathfrak{p} and \mathfrak{q} . \square

The following lemma will prove useful.

Lemma 6.2. *Let $A \subset D$ be an extension of integral domains and assume that D contains $A_{\mathfrak{p}}^h$ and $A_{\mathfrak{q}}^h$ for primes $\mathfrak{p}, \mathfrak{q} \subset A$ such that none of $\mathfrak{p}, \mathfrak{q}$ contains the other. Then -1 is a sum of squares in D . In particular, no residue field of D is formally real.*

Proof. We follow the argument of [Art71, Proof of Theorem 2.5] with a small modification. Assume $\text{char } k(\mathfrak{p}) = p > 0$. Then $A_{\mathfrak{p}}^h$ either contains \mathbb{F}_p or the ring of algebraic p -adic integers. In both cases, -1 is a sum of squares in $A_{\mathfrak{p}}^h$. It therefore remains the case that the residue fields of both \mathfrak{p} and \mathfrak{q} have characteristic zero.

If $\mathfrak{p} + \mathfrak{q} = A$, then we can choose $a \in A$ with $a \equiv 1 \pmod{\mathfrak{p}}$ and $a \equiv -1 \pmod{\mathfrak{q}}$. Then a is a unit and a square in $A_{\mathfrak{p}}^h$ and $-a$ is a square in $A_{\mathfrak{q}}^h$. Hence $-1 = a^{-1}(-a)$ is a square in D .

Finally, assume that $\mathfrak{p} + \mathfrak{q} \neq A$. Choose $g \in \mathfrak{p} \setminus \mathfrak{q}$, $h \in \mathfrak{q} \setminus \mathfrak{p}$ and set $f = g + h$. Then $f \in \mathfrak{p} + \mathfrak{q}$ but $f \notin \mathfrak{p} \cup \mathfrak{q}$. Replacing A by $A[f^{-1}] \subset A_{\mathfrak{p}} \cap A_{\mathfrak{q}} \subset D$ does not change $A_{\mathfrak{p}}^h$ and $A_{\mathfrak{q}}^h$. Since \mathfrak{p} and \mathfrak{q} are coprime in $A[f^{-1}]$, the argument of the last paragraph applies to show that -1 is a square in D . \square

Let A be a ring. The connected components of $\text{Spec}(A)$ are of the form $\text{Spec}(\bar{A})$, where \bar{A} is a colimit of rings of the form A/eA with idempotent elements $e \in A$. We call the rings \bar{A} occurring in this way the *components* of A .

Following M. Artin [Art71], we call a ring A *quasi-acyclic* if every component of A is a local henselian ring. By [Art71, Proposition 3.3], filtered colimits of quasi-acyclic rings and finite algebras over quasi-acyclic rings are quasi-acyclic. In particular, an integral algebra over a quasi-acyclic ring is quasi-acyclic. The main result of [Art71] (though formulated as the slightly weaker statement Theorem 3.4 there) is the following:

Theorem 6.3 (Artin). *Let A be a ring, $\mathfrak{p}_1, \mathfrak{p}_2 \subset A$ prime ideals and, for $i = 1, 2$, $A_{\mathfrak{p}_i}^h$ the henselization of the local ring $A_{\mathfrak{p}_i}$. Let, for $i = 1, 2$, B_i be local, integral $A_{\mathfrak{p}_i}^h$ -algebras. Then*

$$B = B_1 \otimes_A B_2$$

is quasi-acyclic. Let D be a component of B . Then one of the following holds.

- (1) *If both ring homomorphism $B_i \rightarrow D$ are not local, then the residue field D/\mathfrak{m}_D of D is separably closed.*
- (2) *If, say, $B_1 \rightarrow D$ is local, it is integral. In particular, D/\mathfrak{m}_D is an algebraic field extension of B_1/\mathfrak{m}_{B_1} .*

Proof. Because B is integral over $A_{\mathfrak{p}_1}^h \otimes_A A_{\mathfrak{p}_2}^h$, we may assume that $B_i = A_{\mathfrak{p}_i}^h$.

We follow the arguments given in [Art71]: If A is a factor ring of a ring A' and the statement holds for A' , then it holds for A . Therefore we may assume that A is a normal domain. Let K be the quotient field of A , \bar{K}/K a separable closure and \bar{A} the integral closure of A in \bar{K} . Then every component D of B is the limit of étale A -algebras, hence a normal domain. Therefore, we can embed D into \bar{K} . The induced maps $A_{\mathfrak{p}_1}^h \rightarrow D$ and $A_{\mathfrak{p}_2}^h \rightarrow D$ generate D because D is a quotient of $B = A_{\mathfrak{p}_1}^h \otimes_A A_{\mathfrak{p}_2}^h$. Hence we are in the situation of Theorem 6.1 and this theorem implies that D is a local henselian ring. Now let $\bar{\mathfrak{p}}_1, \bar{\mathfrak{p}}_2 \subset \bar{A}$ be the unique extensions of \mathfrak{p}_1 and \mathfrak{p}_2 to \bar{A} , such that $\bar{\mathfrak{p}}_i \bar{A}_{\bar{\mathfrak{p}}_i} \cap A_{\mathfrak{p}_i}^h = \mathfrak{p}_i A_{\mathfrak{p}_i}^h$. For a subextension K'/K of \bar{K}/K we let A' be the normalization of A in \bar{K} and $\mathfrak{p}'_i = \bar{\mathfrak{p}}_i \cap A'$.

If, say, $\bar{\mathfrak{p}}_1 \subset \bar{\mathfrak{p}}_2$, then $\mathfrak{p}_1 \subset \mathfrak{p}_2$ and we are in case (2). Otherwise there is a finite field extension K'/K such that \mathfrak{p}'_1 and \mathfrak{p}'_2 are not contained in one another.

Then $D' = [A'_{\mathfrak{p}_1}, A'_{\mathfrak{p}_2}]$ is the integral closure of D in the quotient field of D' . By Theorem 6.1, D' is henselian with separably closed residue field $D'/\mathfrak{m}_{D'}$. Since the extensions of the residue fields is finite, we conclude that D/\mathfrak{m}_D is separably closed, as well. Here, we have to exclude the possibility that D/\mathfrak{m}_D is real closed but not separably closed. This cannot happen by Lemma 6.2. \square

We call a ring homomorphism $f : A \rightarrow B$ *ind-étale* if it is a filtered colimit of étale ring homomorphisms $f_i : A \rightarrow B_i$. If $A \rightarrow B$ is ind-étale with B local, henselian, then B is an integral $A_{\mathfrak{p}}^h$ -algebra for some prime ideal $\mathfrak{p} \subset A$. Hence the following Theorem 6.4 is an immediate consequence of Theorem 6.3.

Theorem 6.4 (Artin). *Let A be a ring and let B_1, \dots, B_n be ind-étale quasi-acyclic A -algebras. Then their tensor product*

$$B = B_1 \otimes_A \dots \otimes_A B_n$$

is quasi-acyclic. For a maximal ideal $\mathfrak{m} \subset B$ one of the following holds

- (1) B/\mathfrak{m} is separably closed, or
- (2) B/\mathfrak{m} is a separable, algebraic extension of B_i/\mathfrak{m}_i , where $\mathfrak{m}_i \subset B_i$ is a maximal ideal for some i , $1 \leq i \leq n$.

7. COMPARISON WITH ČECH COHOMOLOGY

In this section we assume that all rings and schemes lie over a fixed quasi-compact and quasi-separated base scheme S .

Definition 7.1. A ring homomorphism $f : A \rightarrow B$ is called *tame* if it is étale and for every prime ideal $\mathfrak{p} \subset A$ and every S -valuation v of $k(\mathfrak{p})$ there exists a prime ideal $\mathfrak{q} \subset B$ over \mathfrak{p} and an S -valuation w of $k(\mathfrak{q})$ over v such that w/v is tamely ramified (in particular, f is faithfully flat).

Remark 7.2. A ring homomorphism $A \rightarrow B$ is tame if and only if the associated morphism $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is a covering in $(\text{Spec}(A)/S)_t$. The composite of tame morphisms is tame. Tame ring homomorphisms are stable under base change. The tensor product of tame A -algebras is a tame A -algebra.

Definition 7.3. A ring A is *tamely henselian* if $\text{Spec}(A)$ is connected and every tame ring homomorphism $A \rightarrow B$ splits.

Example 7.4. The tame henselizations of Definition 2.9 are tamely henselian rings.

Remark 7.5. A is tamely henselian if and only if $\text{Spec}(A)$ is connected and every tame covering of $\text{Spec}(A)$ splits. This follows since every tame covering of $\text{Spec}(A)$ has a finite subcovering by Theorem 4.1.

Lemma 7.6. *The following are equivalent.*

- (i) A is tamely henselian
- (ii) A is local, henselian and there exists an S -valuation v of the residue field $k = A/\mathfrak{m}_A$ such that k does not admit any nontrivial, finite separable field extension k'/k which is tamely ramified at some extension w of v to k' .

Proof. (ii) \Rightarrow (i): Let $A \rightarrow B$ be a tame homomorphism. Since A is local henselian, we have a decomposition

$$B = B_0 \times B_1 \times \dots \times B_n,$$

such that the maximal ideal \mathfrak{m} of A is not in the image of $\text{Spec } B_0 \rightarrow \text{Spec } A$, and $A \rightarrow B_i$ is finite étale with B_i local henselian for $i = 1, \dots, n$. Since $A \rightarrow B$ is tame, we can find an index i such that there is a valuation w of B_i/\mathfrak{m}_i lying over v

such that w/v is tamely ramified. By assumption $A/\mathfrak{m} \rightarrow B_i/\mathfrak{m}_i$ is an isomorphism, hence $A \rightarrow B_i$ splits.

(i) \Rightarrow (ii): Let A be tamely henselian. Assume that A is not local henselian and let $\mathfrak{m} \subset A$ be a maximal ideal. Then (by the construction of the henselization) there exists an étale A -Algebra B such that there is exactly one maximal ideal $\mathfrak{n} \subset B$ over \mathfrak{m} , $A/\mathfrak{m} \rightarrow B/\mathfrak{n}$ is a field isomorphism and $A \rightarrow B$ does not split. Then $(\text{Spec } A \setminus \{\mathfrak{m}\}) \cup \text{Spec } B \rightarrow \text{Spec } A$ is a tame covering of $\text{Spec}(A)$ which does not split. Hence A is not tamely henselian by Remark 7.5. We conclude that A is local henselian.

Assume now that for every S -valuation v of $k = A/\mathfrak{m}$ there is a finite, separable field extension k^v/k which is tamely ramified at some extension w of v to k^v . Then

$$\coprod_{v \in \text{Val}_S(k)} \text{Spec } k^v \longrightarrow \text{Spec } k$$

is a tame covering. By Theorem 4.1, there exists a finite subcovering. Hence we find a finite tame k -algebra $k \rightarrow R$ which does not split. Let $A \rightarrow B$ be the finite étale morphism lifting $k \rightarrow R$. Then $(\text{Spec } A \setminus \{\mathfrak{m}\}) \cup \text{Spec } B \rightarrow \text{Spec } A$ is a tame covering of $\text{Spec } A$ which does not split. Contradiction. \square

Corollary 7.7. *A finite algebra over a tamely henselian ring is the finite product of tamely henselian rings.*

Proof. A finite algebra over a local henselian ring is the finite product of local henselian rings. Moreover, the property of the residue field that it is closed under extensions which are tamely ramified with respect to a fixed valuation extends to finite field extensions. \square

Definition 7.8. A scheme X is *étale* (resp. *tamely*) *acyclic* if every étale (resp. tame) covering $X' \rightarrow X$ splits. A ring A is *étale* (resp. *tamely*) *acyclic* if the scheme $\text{Spec } A$ is *étale* (resp. *tamely*) *acyclic*.

Proposition 7.9. *A ring A is étale (resp. tamely) acyclic if and only if every component of A is a strictly (resp. tamely) henselian ring.*

Proof. The assertion in the étale case is given in [Art71, Proposition 3.2]. The proof in the tame case is word by word the same. \square

From Theorem 6.4, we obtain

Theorem 7.10. *Let A be a ring and let B_1, \dots, B_n be ind-étale A -algebras which are étale (resp. tamely) acyclic. Then their tensor product*

$$B = B_1 \otimes_A \dots \otimes_A B_n$$

is étale (resp. tamely) acyclic.

Proof. By Theorem 6.4, B is quasi-acyclic. It remains to show that every component is strictly (resp. tamely) henselian. We already know that the components are local henselian. Hence it suffices to show that the residue fields are separably closed (resp. tamely closed with respect to some S -valuation). This is clear in case (1) of Theorem 6.4, where the residue field is separably closed and follows in case (2) from the assumption on the B_i . \square

Definition 7.11. We call a ring homomorphism $f : A \rightarrow B$ *ind-tame* if it is a filtered colimit of tame homomorphisms $f_i : A \rightarrow B_i$.

Since an inverse limit of finite nonempty sets is nonempty, f is ind-tame if and only if it is ind-étale and for every prime ideal $\mathfrak{p} \subset A$ and every S -valuation v of $k(\mathfrak{p})$ there exists a prime ideal $\mathfrak{q} \subset B$ over \mathfrak{p} and an S -valuation w of $k(\mathfrak{q})$ over v such that w/v is tamely ramified. In particular, f is faithfully flat.

Proposition 7.12. *For every ring A there exists a faithfully flat ind-étale (resp. ind-tame) A -algebra \tilde{A} which is étale (resp. tamely) acyclic.*

Proof. We restrict to the tame case and follow the method of Bhatt-Scholze, [BS15, Proof of Lemma 2.2.7]. Let I be the set of isomorphism classes of tame A -algebras. For each $i \in I$ pick a representative $A \rightarrow B_i$ and set A_1 to be their tensor product, i.e.,

$$A_1 := \operatorname{colim}_{J \subset I \text{ finite}} \bigotimes_{j \in J} B_j,$$

where the tensor product is taken over A and the (filtered) colimit is indexed by the poset of finite subsets of I . There is an obvious ind-tame map $A \rightarrow A_1$, and it is clear from the construction that any tame A -algebra B admits a map to A_1 , i.e., the map $A \rightarrow B$ splits after base change to A_1 . Iterating the construction with A_1 replacing A and proceeding inductively defines a tower $A \rightarrow A_1 \rightarrow A_2 \rightarrow \dots$ of A -algebras with ind-tame transition maps. Set $\tilde{A} = \operatorname{colim} A_n$. As tame morphisms of rings are finitely presented, one checks that any tame \tilde{A} -algebra has a section, so \tilde{A} is tamely acyclic. \square

We will make use of the following topological observation:

Proposition 7.13. *Let X be a spectral space such that every connected component of X contains a unique closed point. Then the composition*

$$X^c \hookrightarrow X \twoheadrightarrow \pi_0(X)$$

from the subspace of closed points of X to the quotient space of connected components is a homeomorphism of profinite sets.

Proof. By [Sta19, TAG 0906], $\pi_0(X)$ is profinite. By assumption, $X^c \rightarrow \pi_0(X)$ is bijective (and continuous). We conclude that X^c is Hausdorff. Moreover, being homeomorphic to $\operatorname{Spec}m(R)$ for some commutative ring R , X^c is quasi-compact. Therefore X^c is compact and the continuous bijection is a homeomorphism. \square

By an affine, pro-étale X -scheme, we mean an inverse limit of affine étale X -schemes. This is more restrictive than ‘pro-étale’ in the sense of [BS15].

Proposition 7.14. *Let X be a scheme having the property that every finite subset of X is contained in an affine open. Suppose we are given affine pro-étale X -schemes $U_i = \operatorname{Spec} A_i$, $i = 1, \dots, n$, and let*

$$U = U_1 \times_X \dots \times_X U_n$$

be their fibre product.

- (i) *If all A_i are étale acyclic, then U is affine, $U = \operatorname{Spec} A$ with A étale acyclic.*
- (ii) *Let X be an S -scheme. If all A_i are tamely acyclic, then U is affine, $U = \operatorname{Spec} A$ with A tamely acyclic.*

Remark 7.15. A scheme having the property that every finite subset is contained in an affine open is separated ([Sta19, TAG 01KP]).

Proof of Proposition 7.14. If there is an affine Zariski-open subscheme $V \subset X$ such that all U_i map to V , the result follows from Theorem 7.10.

In the general case, let $p_i \in U_i$, $i = 1, \dots, n$, be closed points. By assumption, there is an affine, Zariski-open subscheme $V = \operatorname{Spec} B \subset X$ containing the images of p_1, \dots, p_n . Since U_i is quasi-acyclic, every connected component of U_i has a unique closed point and it maps to V if and only if its closed point maps to V . By Proposition 7.13, $\varphi_i : \operatorname{Spec}m A_i \rightarrow \pi_0(U_i)$ is a homeomorphism of profinite spaces. Therefore, we find a closed and open subset $W_i \subset \operatorname{Spec}m A_i$ which contains p_i and maps to V . Then also the preimage $V_i \subset U_i$ of $\varphi_i(W_i) \subset \pi_0(U_i)$ under

$U_i \rightarrow \pi_0(U_i)$ maps to V . Hence V_i is a closed and open subset of U_i containing p_i and mapping to V . Being closed and open in U_i , V_i is the spectrum of an ind-étale, étale (resp. tamely) acyclic B -algebra. By the first part of the proof, $V(p_1, \dots, p_n) := V_1 \times_X \cdots \times_X V_n$ is a closed and open subscheme of U , affine, ind-étale and étale (resp. tamely) acyclic. Varying the points p_1, \dots, p_n , the $V(p_1, \dots, p_n)$ cover U . Since U is quasi-compact, we find a finite subcovering. Replacing the $V(p_1, \dots, p_n)$ by closed and open subschemes, which are ind-étale and étale (resp. tamely) acyclic, we may assume that the finite union is disjoint. Hence U is affine, ind-étale and étale (resp. tamely) acyclic. \square

Next we generalize Artin's theorem on the comparison between étale Čech and sheaf cohomology to the tame case. We also obtain a slightly sharper statement for the étale case by weakening the Noetherian assumption to quasi-compactness.

Theorem 7.16 (Comparison with Čech cohomology). *Let X be a quasi-compact scheme having the property that every finite subset of X is contained in an affine open, S a quasi-compact and quasi-separated scheme and $X \rightarrow S$ a morphism. Then, for every presheaf P of abelian groups on $X_{\text{ét}}$ with étale resp. tame sheafification $P^{\# \text{ét}}$ resp. $P^{\# t}$, the natural maps*

$$\check{H}_{\text{ét}}^n(X, P) \rightarrow H_{\text{ét}}^n(X, P^{\# \text{ét}}) \text{ and } \check{H}_t^n(X/S, P) \rightarrow H_t^n(X/S, P^{\# t})$$

are isomorphisms for all $n \geq 0$.

Proof. We give the argument in the tame case, the étale case is analogous. The statement of the theorem is a formal consequence of the assertion

$$(\dagger) \quad \check{H}_t^n(X/S, P) = 0 \text{ for all } n \geq 0 \text{ and every presheaf } P \text{ with } P^{\# t} = 0.$$

Indeed, if (\dagger) holds, then $\check{H}_t^\bullet(X/S, -)$ is a δ -functor on *sheaves*. Since injective sheaves are injective as presheaves, the functor is also erasing. This shows $\check{H}_t^n(X/S, F) \cong H_t^n(X/S, F)$ for every sheaf F . It remains to see that $\check{H}_t^n(X/S, P) \rightarrow \check{H}_t^n(X/S, P^{\# t})$ is an isomorphism for every presheaf P . Splitting the presheaf homomorphism $P \rightarrow P^{\# t}$ into

$$0 \rightarrow K \rightarrow P \rightarrow G \rightarrow 0, \quad 0 \rightarrow G \rightarrow P^{\# t} \rightarrow C \rightarrow 0,$$

we have $K^{\# t} = 0 = C^{\# t}$ and the statement follows from the long exact sequence for Čech cohomology of presheaves.

To prove (\dagger) , we cover X by finitely many affine Zariski-open schemes. By Proposition 7.12, and forming the disjoint union, we find an affine, tamely acyclic, pro-tame covering $U \rightarrow X$, i.e., $U = \lim U_i$ is a limit of affine, tame coverings $U_i \rightarrow X$. Since every tame covering of U splits, the system (U_i) is cofinal in the system of all tame coverings of X (partially ordered by refinement). For the Čech cohomology, we obtain

$$\check{H}_t^p(X/S, P) = \text{colim}_i \check{H}^p(U_i \rightarrow X, P) = H^p(\text{colim}_i \check{C}^\bullet(U_i \rightarrow X, P)).$$

For an X -scheme Y , we denote by Y^n the n -th self fibre product over X . Then, for $n \geq 1$, $U^n = \lim_i U_i^n$ is tamely acyclic by Proposition 7.14. Since P has trivial sheafification, we obtain $\text{colim}_i P(U_i^n) = 0$. Hence the colimit of the Čech complexes $\check{C}^\bullet(U_i \rightarrow X, P)$ vanishes. In particular, the colimit has trivial cohomology. \square

8. COMPARISON WITH ÉTALE COHOMOLOGY

In this section we compare tame and étale cohomology in two situations. First, when the coefficients are invertible on S and second, if $X \rightarrow S$ is proper.

Since the étale site is finer than the tame site and has the same underlying category, any étale sheaf can be considered as a tame sheaf. To put this more

formal, we denote the natural morphism of sites by $\alpha : X_{\text{et}} \rightarrow (X/S)_t$. Then α_*F is the same presheaf as F but considered as a tame sheaf.

Proposition 8.1 (Invertible coefficients). *Let $m \geq 1$ be an integer invertible on S and let F be an étale sheaf of $\mathbb{Z}/m\mathbb{Z}$ -modules on X . Then*

$$H_t^n(X/S, \alpha_*F) \cong H_{\text{et}}^n(X, F)$$

for all $n \geq 0$.

Proof. Using the Leray-Serre spectral sequence, it suffices to show that the higher direct image sheaves $R^q\alpha_*F$ vanish for $q \geq 1$. This can be checked on stalks at the conservative family of tame points of described in Lemma 2.10.

Let $x \in X$ be a (scheme-)point, $k(x)^s$ a separable closure of $k(x)$, v an S -valuation on $k(x)$, \bar{v} a prolongation to $k(x)^s$, $K_v := (k(x)^s)^{R_v}$ the fixed field of its ramification group and L a finite subextensions of $k(x)^s/K_v$. Let $\bar{x}_L : \text{Spec}(L) \rightarrow X$ be the induced morphism and $(\bar{x}_L, \bar{v}|_L)$ the associated tame point of X . Since, by [AGV72, VI, 5.8], étale cohomology commutes with inverse limits of quasi-compact and quasi-separated schemes with affine transition maps, we have an isomorphism

$$(R^q\alpha_*F)_{(\bar{x}_L, \bar{v}|_L)} \cong H_{\text{et}}^q(\text{Spec } \mathcal{O}_{(\bar{x}_L, \bar{v}|_L)}, F),$$

where $\mathcal{O}_{(\bar{x}_L, \bar{v}|_L)}$ is the stalk of the structure sheaf at $(\bar{x}_L, \bar{v}|_L)$ described in Remark 2.8. This ring is henselian with residue field L , hence

$$(R^q\alpha_*F)_{(\bar{x}_L, \bar{v}|_L)} \cong H_{\text{et}}^q(\text{Spec } L, (\bar{x}_L)^*F).$$

Let p be the residue characteristics of v . Then the absolute Galois group of L is a pro- p -group and since $(p, m) = 1$, $H_{\text{et}}^q(\text{Spec } L, (\bar{x}_L)^*F)$ vanishes for $q \geq 1$ by [NSW08, (1.6.2)]. \square

Proposition 8.2 (Proper morphisms). *Let X be a quasi-compact scheme having the property that every finite subset of X is contained in an affine open, S a quasi-compact and quasi-separated scheme and $X \rightarrow S$ a proper morphism. Then, for every sheaf F of abelian groups on $(X/S)_t$ with étale sheafification $F^{\# \text{et}}$ the natural map*

$$H_t^n(X/S, F) \rightarrow H_{\text{et}}^n(X, F^{\# \text{et}})$$

is an isomorphism for all $n \geq 0$.

Proof. By Theorem 7.16, the horizontal arrows in the commutative diagram

$$\begin{array}{ccc} \check{H}_t^n(X/S, F) & \longrightarrow & H_t^n(X/S, F) \\ \downarrow & & \downarrow \\ \check{H}_{\text{et}}^n(X, F) & \longrightarrow & H_{\text{et}}^n(X, F^{\# \text{et}}) \end{array}$$

are isomorphisms. Since $X \rightarrow S$ is proper, every étale covering of X is a tame covering. Hence the left vertical arrow is an isomorphism and therefore also the right one. \square

9. FINITENESS IN DIMENSION 1

Definition 9.1. Assume that X is a quasi-compact and quasi-separated scheme over S . Let $\text{Co}_t(X/S)$ be the smallest full abelian subcategory of $\text{Sh}_t(X/S)$ which contains all sheaves $j_*\mathbb{Z}/n\mathbb{Z}$, where $n \in \mathbb{N}$ and $j : U \rightarrow X$ is étale and of finite presentation. We call the objects of $\text{Co}_t(X/S)$ *constructible* tame sheaves.

The étale sheafification of a constructible tame sheaf is a constructible étale sheaf. If X is noetherian and of Krull dimension ≤ 1 , then $F \in Sh_t(X/S)$ is constructible if and only if it has finite stalks and there is an étale $U \rightarrow X$ with dense image such that $F|_U$ is constant.

We consider the arithmetic case.

Lemma 9.2. *Let K/\mathbb{Q} be a number field, $\mathfrak{p} \subset \mathcal{O}_K$ a prime ideal, $\mathcal{O}_{K,\mathfrak{p}}^h$ the henselization of $\mathcal{O}_{K,\mathfrak{p}}$ and $K_{\mathfrak{p}}^h$ its quotient field.*

Let $X = \text{Spec } \mathcal{O}_{K,\mathfrak{p}}^h$, $x \in X$ the closed point, $\eta \in X$ the generic point, and put $S = \text{Spec } \mathbb{Z}$. Then for every tame torsion sheaf F on X/S , we have $H_t^q(X/S, F) = 0$ for $q > 1$, $H_t^q(\eta/S, F) = 0$ for $q > 2$ and $H_{t,\{x\}}^q(X/S, F) = 0$ for $q > 3$. If F is constructible, these groups are finite for all q .

Proof. The only valuation on the finite field $k(x) = \mathcal{O}_K/\mathfrak{p}$ is the trivial one. Since $K_{\mathfrak{p}}^h$ is an algebraic extension of \mathbb{Q} , every valuation on $k(\eta) = K_{\mathfrak{p}}^h$ is trivial or of rank one. First, we have the valuation associated to the valuation ring $\mathcal{O}_{K,\mathfrak{p}}^h$. Let $K_{\mathfrak{p}}^t/K_{\mathfrak{p}}^h$ be the maximal tamely ramified extension (with respect to $\mathcal{O}_{K,\mathfrak{p}}^h$) of the henselian local field $K_{\mathfrak{p}}^h$ in an algebraic closure \bar{K} .

Then, by [NSW08, 7.2.5], $G = \text{Gal}(K_{\mathfrak{p}}^t/K_{\mathfrak{p}}^h)$ is the semidirect product of $\hat{\mathbb{Z}}$ with $\hat{\mathbb{Z}}^{(p')} = \prod_{\ell \neq p} \hat{\mathbb{Z}}_{\ell}$, where p is the prime number below \mathfrak{p} . In particular, G has cohomological dimension 2 and the cohomology of G with values in finite discrete modules is finite.

Let v be another valuation on $K_{\mathfrak{p}}^h$. If v is trivial, every extension of $K_{\mathfrak{p}}^h$ is tame with respect to v . If v is nontrivial, then, since different decomposition groups in the absolute Galois group of a number field have trivial intersection by [NSW08, 12.1.3], v is completely decomposed in $\bar{K}/K_{\mathfrak{p}}^h$. We conclude that $\text{Spec } K_{\mathfrak{p}}^t/\text{Spec } K_{\mathfrak{p}}^h$ is a profinite tame cover of $K_{\mathfrak{p}}^h$ with respect to $S = \text{Spec } \mathbb{Z}$. Moreover, every tame cover of $\text{Spec } K_{\mathfrak{p}}^t$ splits. Hence for every sheaf $F \in Sh_t(\text{Spec } K_{\mathfrak{p}}^t/\text{Spec } \mathbb{Z})$ the Hochschild-Serre spectral sequence induces isomorphisms

$$H_t^q(\text{Spec } K_{\mathfrak{p}}^h/S, F) \cong H^q(G, \Gamma(\text{Spec } K_{\mathfrak{p}}^t, F)).$$

This shows that these groups are trivial for $q > 2$ if F is a torsion sheaf and finite for all q if F is constructible.

A similar argument with $\text{Gal}(\mathcal{O}_{K,\mathfrak{p}}^h/\mathcal{O}_{K,\mathfrak{p}}) \cong \hat{\mathbb{Z}}$ shows that $H_t^q(\text{Spec } \mathcal{O}_{K,\mathfrak{p}}^h/S, F)$ is trivial for $q > 1$ if F is a torsion sheaf, and finite for all q if F is constructible. Finally, the statement on the cohomology groups with support follows from the result for the cohomology of $\text{Spec } \mathcal{O}_{K,\mathfrak{p}}^h$ and $\text{Spec } K_{\mathfrak{p}}^h$, and the long exact sequence connecting them. \square

Theorem 9.3. *Let $X \rightarrow S = \text{Spec } \mathbb{Z}$ be quasi-finite. Then the tame cohomology groups*

$$H_t^q(X/S, F)$$

are finite for every constructible sheaf $F \in Sh_t(X/S)$ and all q .

For an arbitrary torsion sheaf $F \in Sh_t(X/S)$ the following holds.

- (i) *If $\dim X = 0$, then $H_t^q(X/S, F) = 0$ for $q > 1$.*
- (ii) *If $\dim X = 1$ assume that $X(\mathbb{R}) = \emptyset$ or that F has trivial 2-torsion. Then $H_t^q(X/S, F) = 0$ for $q > 3$.*

Proof. Using Mayer-Vietoris and induction on the cardinality of a covering by separated open subschemes, we may assume that X is separated. By the topological invariance of tame cohomology (Proposition 3.1), we may assume that X is reduced. If $\dim X = 0$, X is finite over $\text{Spec } \mathbb{Z}$. If $\dim X = 1$, we can find a scheme \bar{X} which is finite over $\text{Spec } \mathbb{Z}$, contains X as a dense open subscheme and is regular at all

points $x \in \bar{X} \setminus X$. We denote the open immersion by $j : X \hookrightarrow \bar{X}$. Then excision for the sheaf $j_!F$ on \bar{X} yields the long exact sequence

$$\cdots \rightarrow \bigoplus_{x \in \bar{X} \setminus X} H_{t, \{x\}}^q(\bar{X}_x^h/S, j_!F) \rightarrow H_t^q(\bar{X}/S, j_!F) \rightarrow H_t^q(X/S, F) \rightarrow \cdots$$

In view of Lemma 9.2, we are reduced to showing the result for $X = \bar{X}$.

For finite X , the assertion of the theorem follows by Proposition 8.2 from the corresponding well-known result for étale cohomology [Maz73]. \square

The same arguments also show

Theorem 9.4. *Let k be a separably closed (resp. finite) field and $X \rightarrow S = \text{Spec } k$ a separated scheme of finite type of dimension ≤ 1 . Then for every torsion sheaf $F \in \text{Sh}_t(X/S)$ the tame cohomology groups*

$$H_t^q(X/S, F)$$

are zero for $q > 2$ (resp. $q > 3$). If F is constructible, these groups are finite for all q .

10. REVIEW OF HUBER PAIRS

We refer to [Hub96] for basic notions on adic spaces and to [Hüb18] for the definition of the tame site on such spaces.

In this paper, by a *Huber pair* we mean a pair of rings (A, A^+) with $A^+ \subset A$ an integrally closed subring. We equip A and A^+ with the discrete topology. We call an adic space \mathcal{X} *discretely ringed*, if it is locally of the form $\text{Spa}(A, A^+)$ with (A, A^+) a Huber pair.

Points of $\text{Spa}(A, A^+)$ are by definition (multiplicatively written) valuations $v : A \rightarrow \Gamma_v \cup \{0\}$ such that $v(a_+) \leq 1$ for all $a_+ \in A^+$. We will frequently consider v also as a valuation on the quotient field $Q(A/\text{supp } v)$, where $\text{supp } v = \{a \in A \mid v(a) = 0\}$ is the *support* of v . The topology on $\text{Spa}(A, A^+)$ is generated by the sets

$$\{v \in \text{Spa}(A, A^+) \mid v(a) \leq v(b) \neq 0\} \quad (a, b \in A).$$

Recall that a subgroup H of the ordered group Γ_v is *convex* if each γ with $1 \leq \gamma \leq \delta \in H$ already belongs to H . A convex subgroup $H \subset \Gamma_v$ *bounds* A if for all $a \in A$ there exists an $h \in H$ with $v(a) \leq h$. In this case, the assignment

$$v|_H : A \rightarrow H \cup \{0\}, \quad a \mapsto \begin{cases} v(a) & \text{if } v(a) \in H \\ 0 & \text{if } v(a) \notin H \end{cases}$$

defines a valuation in $\text{Spa}(A, A^+)$ which is a specialization of v . We have $\text{supp } v \subset \text{supp } v|_H$ and $v|_H$ is called a *horizontal* (or *primary*) *specialization* of v . The set of horizontal specializations of v is totally ordered and has a minimal element \bar{v} corresponding to the “characteristic subgroup” $c\Gamma_v$ of Γ_v , the convex subgroup generated by the subset $\{v(a) \mid a \in A, v(a) \geq 1\}$.

Definition 10.1. $v \in \text{Spa}(A, A^+)$ is a *Riemann-Zariski point* if it has no nontrivial horizontal specializations.

Remark 10.2. The terminology “Riemann-Zariski” is motivated by the fact that these points are in bijection to the points of the relative Riemann-Zariski space of $\text{Spec } A^+$ with respect to $\text{Spec } A$, cf. [Tem11, Corollary 3.4.7].

The minimal horizontal specialization \bar{v} of v is the unique Riemann-Zariski point among the horizontal specializations of v . Assume that $\bar{v} = \text{tr}_{\mathfrak{p}}$ is the trivial valuation on its support $\mathfrak{p} = \text{supp } \bar{v}$. Then for any $\mathfrak{q} \in \text{Spec}(A)$, $\mathfrak{p} \subset \mathfrak{q}$, the trivial valuation $\text{tr}_{\mathfrak{q}}$ with support \mathfrak{q} is a specialization of \bar{v} , hence of v . A *generalized*

horizontal specialization is a horizontal specialization or a specialization $v \rightsquigarrow \text{tr}_{\mathfrak{q}}$ as above.

For an arbitrary convex subgroup $H \subset \Gamma_v$ the assignment

$$v/H : A \rightarrow (\Gamma/H) \cup \{0\}, \quad a \mapsto \begin{cases} v(a) \bmod H & \text{if } v(a) \neq 0 \\ 0 & \text{if } v(a) = 0 \end{cases}$$

defines a valuation with $\text{supp}(v/H) = \text{supp } v$ and v/H is a generalization of v . A specialization of type $v/H \rightsquigarrow v$ is called a *vertical* (or secondary) *specialization*.

Lemma 10.3. *Let (A, A^+) be a Huber pair and let $v, w \in \text{Spa}(A, A^+)$.*

- (i) *w is a vertical specialization of v if and only if $\text{supp } v = \text{supp } w$ and $\mathcal{O}_w \subset \mathcal{O}_v \subset Q(A/\text{supp } v)$. In this case \mathcal{O}_v is the localization of \mathcal{O}_w at some prime ideal.*
- (ii) *w is a horizontal specialization of v if and only if there is a prime ideal $\mathfrak{p} \subset \mathcal{O}_v$ such that*
 - (a) *$A/\text{supp } v \subset (\mathcal{O}_v)_{\mathfrak{p}}$ (as subrings in $Q(A/\text{supp } v)$),*
 - (b) *$\text{supp } w$ is the preimage of \mathfrak{p} under $A \rightarrow A/\text{supp } v \hookrightarrow (\mathcal{O}_v)_{\mathfrak{p}}$, and*
 - (c) *$\mathcal{O}_w = (\mathcal{O}_v/\mathfrak{p}) \cap Q(A/\text{supp } w)$ (intersection in $Q(\mathcal{O}_v/\mathfrak{p})$).*

$$\begin{array}{ccccc} \mathcal{O}_v & \longrightarrow & \mathcal{O}_v/\mathfrak{p} & & \\ \downarrow & & \downarrow & \swarrow & \\ (\mathcal{O}_v)_{\mathfrak{p}} & \longrightarrow & Q(\mathcal{O}_v/\mathfrak{p}) & & \mathcal{O}_w \\ \downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\ Q(A/\text{supp } v) & \longleftarrow & A/\text{supp } v & \longrightarrow & Q(A/\text{supp } w). \end{array}$$

Proof. (i) is obvious by valuation theory.

(ii) Let w be a horizontal specialization of v and let $c\Gamma_v \subset H \subset \Gamma_v$ be a convex subgroup as in the definition. Let $\mathfrak{p} \subset Q(A/\text{supp } v)$ be the set of all x with $v(x) \notin H$. Since $H \supset c\Gamma_v$, we have $\mathfrak{p} \subset \mathcal{O}_v$.

- (1) $a \in \mathcal{O}_v, b \in \mathfrak{p} \Rightarrow ab \in \mathfrak{p}$: Otherwise, $v(ab) \in H$ and $v(ab) \leq v(b) \leq 1$ would imply $v(b) \in H$, a contradiction.
- (2) $a, b \in \mathfrak{p} \Rightarrow a + b \in \mathfrak{p}$: $v(a + b) \leq \max(v(a), v(b)) \leq 1$: By convexity, $v(a + b) \in H$ implies $(v(a) \in H) \vee (v(b) \in H)$.
- (3) $a, b \notin \mathfrak{p} \Rightarrow ab \notin \mathfrak{p}$: clear since $v(ab) = v(a)v(b)$ and H is a group.

We conclude that $\mathfrak{p} \subset \mathcal{O}_v$ is a prime ideal. For $a \in A/\text{supp } v$ with $v(a) \leq 1$ we have $a \in \mathcal{O}_v$ by definition. If $v(a) > 1$, then $v(a) \in H$, hence $v(a^{-1}) \in H$ and $v(a^{-1}) \leq 1$. We conclude $a^{-1} \in \mathcal{O}_v \setminus \mathfrak{p}$, hence $a \in (\mathcal{O}_v)_{\mathfrak{p}}$. This shows (a). Moreover, $\text{supp } w = \{a \in A \mid v(a) \notin H\}$ is the preimage of \mathfrak{p} under $A \rightarrow A/\text{supp } v \subset Q(A/\text{supp } v)$, showing (b). Finally, $\mathcal{O}_v/\mathfrak{p}$ is the valuation ring of the valuation v' on $Q(\mathcal{O}_v/\mathfrak{p})$ given on \mathcal{O}_v by $v'(a) = 0$ for $v(a) \notin H$ and $v(a)$ otherwise. We conclude that w is the restriction of v' to $Q(A/\text{supp } w)$, hence $\mathcal{O}_w = \mathcal{O}_v/\mathfrak{p} \cap Q(A/\text{supp } w)$.

Conversely, assume there exists a prime ideal $\mathfrak{p} \subset \mathcal{O}_v$ such that (a)–(c) hold. As is known from valuation theory, there is a convex subgroup $H \subset \Gamma_v$ such that $\mathfrak{p} = \{x \in \mathcal{O}_v \mid v(x) \notin H\}$. By (a), for all $x \in A$, there is a $b \in \mathcal{O}_v \setminus \mathfrak{p}$ with $xb \in \mathcal{O}_v$. We conclude that $v(x) \leq v(b^{-1}) \in H$, hence H bounds A . By (b) we see that $\text{supp } w = \text{supp } v|_H$, and then (c) implies $w = v|_H$. \square

Corollary 10.4. *Every vertical generalization of a Riemann-Zariski point is a Riemann-Zariski point.*

Proof. If v' is a vertical generalization of v , then by Lemma 10.3 (i), $\text{supp } v = \text{supp } v'$ and $\mathcal{O}_{v'} = (\mathcal{O}_v)_{\mathfrak{q}}$ for some prime ideal $\mathfrak{q} \subset \mathcal{O}_v$. Assume that v' has a

non-trivial horizontal specialization. Then there exist a prime $(0) \subsetneq \mathfrak{p} \subset \mathcal{O}'_v$ as in Lemma 10.3 (ii). But $(\mathcal{O}_{v'})_{\mathfrak{p}} = (\mathcal{O}_v)_{\mathfrak{P}}$ for some prime $\mathfrak{P} \subset \mathcal{O}_v$ with $(0) \subsetneq \mathfrak{P} \subset \mathfrak{q}$. The diagram in Lemma 10.3 (ii) shows that there is a non-trivial horizontal specialization of v , as well. \square

We will use the following fact below:

Proposition 10.5 ([HK94, Proposition 1.2.4]). *Any specialization in $\text{Spa}(A, A^+)$ is a vertical specialization of a generalized horizontal specialization.* \square

A homomorphism of Huber pairs $f : (A, A^+) \rightarrow (B, B^+)$ is a ring homomorphism $f : A \rightarrow B$ with $f(A^+) \subset B^+$. It is called *integral*, if $A \rightarrow B$ and $A^+ \rightarrow B^+$ are integral. It is called *finite*, if it is integral and $A \rightarrow B$ is finite. A homomorphism of Huber pairs is called *étale*, *tame* or *strongly étale* if the associated morphism of adic spaces $\text{Spa}(B, B^+) \rightarrow \text{Spa}(A, A^+)$ has this property. Explicitly, this means the following:

The homomorphism $f : (A, A^+) \rightarrow (B, B^+)$ is *étale* if $A \rightarrow B$ is an étale ring homomorphism and B^+ is the integral closure in B of a finitely generated A^+ -subalgebra.

The homomorphism f is *tame*, resp. *strongly étale* if it is étale and for every point w of $\text{Spa}(B, B^+)$ with image $v \in \text{Spa}(A, A^+)$ the extension of valuations w/v is tamely ramified resp. unramified in the separable algebraic field extension $k(\text{supp } w)/k(\text{supp } v)$.

Remark 10.6. In contrast to the scheme case, we do not assume that $\text{Spa}(B, B^+) \rightarrow \text{Spa}(A, A^+)$ is a tame covering.

Let A be a local ring and $A^+ \subset A$ the preimage of a valuation ring \mathcal{O} of the residue field k of A . Then A^+ is local and, since valuation rings are integrally closed, A^+ is integrally closed in A . Hence (A, A^+) is a Huber pair. We call such Huber pairs *local*. If (A, A^+) is a local Huber pair, then \mathfrak{m}_A is a prime ideal of A^+ and the natural map $A_{\mathfrak{m}_A}^+ = (A^+ \setminus \mathfrak{m}_A)^{-1}A^+ \rightarrow A$ is an isomorphism. A homomorphism $f : (A, A^+) \rightarrow (B, B^+)$ of local Huber pairs is *local* if $A^+ \rightarrow B^+$ and $A \rightarrow B$ are local ring homomorphisms.

For a Huber pair (A, A^+) and $v \in \text{Spa}(A, A^+)$ we denote by $(A, A^+)_v$ the *localization* of (A, A^+) at v . It is obtained as follows. Let $B := A_{\text{supp } v}$ be the localization of A at $\text{supp } v$. In its residue field k we have the valuation ring \mathcal{O} corresponding to v . Let B^+ be the pre-image of \mathcal{O} in B . Then $(A, A^+)_v = (B, B^+)$.

The adic space $\text{Spa}((A, A^+)_v)$ is the intersection of all open neighborhoods of v in $\text{Spa}(A, A^+)$. A point $w \in \text{Spa}(A, A^+)$ specializes to v if and only if $(A, A^+) \rightarrow (A, A^+)_w$ factors through $(A, A^+) \rightarrow (A, A^+)_v$.

We start the discussion of henselizations with the following observation.

Lemma 10.7. *Let A be a local ring with residue field k , $pr : A \rightarrow k$ the projection and $\mathcal{O} \subset k$ a local subring. Set*

$$A^+ = pr^{-1}(\mathcal{O}) \subset A.$$

Then A^+ is henselian if and only if A and \mathcal{O} are henselian.

Proof. First observe that \mathfrak{m}_A is a prime ideal of A^+ . If \mathcal{O} and A are henselian, then applying Hensel's lemma twice shows that A^+ is henselian: Let f be a monic polynomial over A^+ and $f = gh$ modulo \mathfrak{m}_{A^+} a decomposition with $g \bmod \mathfrak{m}_{A^+}$ and $h \bmod \mathfrak{m}_{A^+}$ monic and coprime. We have $A^+/\mathfrak{m}_{A^+} = \mathcal{O}/\mathfrak{m}_{\mathcal{O}}$ and we can lift the decomposition to \mathcal{O} . Since A is henselian with residue field $k \supset \mathcal{O}$, we can lift the decomposition to A and the coefficients lie modulo \mathfrak{m}_A in \mathcal{O} . Hence they are in A^+ .

Now assume that A^+ is henselian. Then, being a quotient ring of A^+ , also \mathcal{O} is henselian. Assume that $A \rightarrow A'$ is finite with A' connected. Then A' is semi-local and, in order to show that A is henselian, we have to show that A' is local. For this it suffices to show that $A'/\mathfrak{m}_A A'$ is local. Since this ring is a finite algebra over the field A/\mathfrak{m}_A , it suffices to show that $A'/\mathfrak{m}_A A'$ is connected. Let A'^+ be the integral closure of A^+ in A' . Since all idempotents of A' lie in A'^+ and A' is connected, A'^+ is connected, as well. Hence A'^+ is local henselian and the same holds for $A'^+/\mathfrak{m}_A A'^+$.

Claim: The image of $\phi : A'^+/\mathfrak{m}_A A'^+ \rightarrow A'/\mathfrak{m}_A A'$ is integrally closed in $A'/\mathfrak{m}_A A'$. Using the claim we see that $\text{im}(\phi)$ has the same connected components as $A'/\mathfrak{m}_A A'$. But $\text{im}(\phi)$ is (henselian) local, hence connected, showing that $A'/\mathfrak{m}_A A'$ is connected.

It remains to show the claim. First we observe that $\mathfrak{m}_A A' \subset A'^+$. Now assume that for $x \in A'$ there exists a monic polynomial $f \in A'^+[T]$ with $f(x) = a' \in \mathfrak{m}_A A' \subset A'^+$. Then x is a zero of $f - a' \in A'^+[T]$, hence $x \in A'^+$. \square

Definition 10.8. Let (A, A^+) be a local Huber pair, $k = A/\mathfrak{m}_A$ and \mathcal{O} the image of A^+ in k .

- (1) (A, A^+) is called *henselian* if A^+ is a henselian ring. By Lemma 10.7, this is equivalent to A and \mathcal{O} being henselian.
- (2) (A, A^+) is called *strongly henselian* if A^+ is a strictly henselian ring. By Lemma 10.7, this is equivalent to A being henselian and \mathcal{O} being strictly henselian.
- (3) (A, A^+) is called *tamely henselian* if (A, A^+) is henselian and k does not admit any nontrivial, finite separable field extension which is tamely ramified with respect to the (henselian) valuation corresponding to \mathcal{O} . Equivalently: (A, A^+) is strongly henselian and k is separably closed if the residue field κ of \mathcal{O} has characteristic zero, resp. the absolute Galois group of k is a pro- p -group if κ has characteristic $p > 0$.
- (4) (A, A^+) is called *strictly henselian* if A is a strictly henselian ring. In this case \mathcal{O} , and, by Lemma 10.7, also A^+ is strictly henselian.

Lemma 10.9. Let (A, A^+) be a Huber pair such that $\mathcal{X} = \text{Spa}(A, A^+)$ is connected. Then

- (i) (A, A^+) is local if and only if every open covering of \mathcal{X} splits.
- (ii) (A, A^+) is strongly henselian if and only if every strongly étale covering of \mathcal{X} splits.
- (iii) (A, A^+) is tamely henselian if and only if every tame covering of \mathcal{X} splits.
- (iv) (A, A^+) is strictly henselian if and only if every étale covering of \mathcal{X} splits.

Proof. That the properties local, etc. imply the corresponding splitting property follows directly from the definitions.

Assume that every open covering of \mathcal{X} splits. Then \mathcal{X} has exactly one closed point, say v , to which all other points specialize. Every open neighborhood of v is equal to \mathcal{X} , hence $(A, A^+) = (A, A^+)_v$ is local. This shows (i) and that (A, A^+) is local in cases (ii)–(iv). We denote the image of A^+ in $k = A/\mathfrak{m}_A$ by \mathcal{O} (a valuation ring of k).

Now assume that every strongly étale covering of \mathcal{X} splits. We first show that A is henselian. Let $\text{Spec } B \rightarrow \text{Spec } A$ be a connected étale covering such that there is a point $y \in \text{Spec } B$ lying over $\text{supp } v$ with trivial residue field extension $k(y) = k(\text{supp } v)$. Let B^+ be the integral closure of A^+ in B . Then the natural projection $f : \text{Spa}(B, B^+) \rightarrow \mathcal{X}$ is étale and strongly étale at every point with support y . Let $\mathcal{Y} \subset \text{Spa}(B, B^+)$ be the strongly étale locus of f , which is open by

[Hüb18, Corollary 4.3]. Then $f(\mathcal{Y})$ is open in \mathcal{X} and contains v , hence $f(\mathcal{Y}) = \mathcal{X}$. We conclude that $f|_{\mathcal{Y}} : \mathcal{Y} \rightarrow \mathcal{X}$ is a strongly étale covering, hence admits a splitting. This induces a splitting of $\text{Spec } B \rightarrow \text{Spec } A$. We conclude that A is a local henselian ring. In order to show (ii), it remains to show that \mathcal{O} is strictly henselian.

If \mathcal{O} were not strictly henselian, there would exist a connected étale \mathcal{O} -algebra \mathcal{O}' which does not split. Being normal and connected, \mathcal{O}' is a domain. Let k' denote its quotient field. Then k'/k is a nontrivial finite separable extension (and \mathcal{O}' is a localization of the normalization of \mathcal{O} in k'). Let $A \rightarrow B$ be the finite étale extension of local henselian rings associated with k'/k . Let $\mathcal{O}' = \mathcal{O}[\bar{x}_1, \dots, \bar{x}_n]$, x_1, \dots, x_n preimages of the \bar{x}_i in B , and $B^+ \subset B$ the integral closure in B of $A[x_1, \dots, x_n]$. Then $\text{Spa}(B, B^+) \rightarrow \mathcal{X}$ is étale and strongly étale over the closed point v . Since the strongly étale locus is open, we obtain a strongly étale covering of \mathcal{X} which does not split. This shows (ii).

Finally, using (ii), the verification of (iii) and (iv) is easy and left to the reader. \square

For a Huber pair (A, A^+) and a point $x \in \text{Spa}(A, A^+)$ the *henselization* of (A, A^+) at x is obtained as follows: Let A^h be the henselization of A at $\text{supp } x$. In its residue field k we have the valuation ring \mathcal{O} corresponding to x . The henselization \mathcal{O}^h of \mathcal{O} is a valuation ring, its quotient field k^h is a separable algebraic field extension of k . We define C as the ind-(finite étale) A^h -algebra corresponding to k^h/k and C^+ as the preimage of \mathcal{O}^h in C .

Definition 10.10. We call $(A, A^+)_x^h := (C, C^+)$ the *henselization* of (A, A^+) at x .

The induced homomorphism $(A, A^+)_x \rightarrow (A, A^+)_x^h$ is universal for local homomorphisms of $(A, A^+)_x$ to henselian Huber pairs.

Now we fix a separable closure \bar{k}/k^h . The henselian valuation of k^h associated with \mathcal{O}^h has a unique extension to \bar{k} . Inside this we have the maximal tamely ramified and unramified extensions $k^t/k^{nr}/k^h$. The normalization of \mathcal{O}^h in k^{nr} is the strict henselization \mathcal{O}^{sh} of \mathcal{O}^h . We denote the normalization of \mathcal{O}^h in k^t by \mathcal{O}^t and call it the tame henselization. Denoting the normalization of \mathcal{O}^h in \bar{k} by $\bar{\mathcal{O}}$, we obtain local ring extensions $\mathcal{O}^h \hookrightarrow \mathcal{O}^{nr} \hookrightarrow \mathcal{O}^t \hookrightarrow \bar{\mathcal{O}}$.

Definition 10.11. The *strong henselization* $(B, B^+) = (A, A^+)_x^{sh}$ is given by setting B the unique ind-(finite étale) subextension of A^{sh}/A^h (strict henselization and henselization of A at $\text{supp } x$) associated to k^{nr}/k and B^+ the preimage of $\mathcal{O}^{sh} \subset k^{nr}$ in B .

The *tame henselization* $(B, B^+) = (A, A^+)_x^t$ is given by setting B the unique ind-(finite étale) subextension of A^{sh}/A^h associated to k^t/k and B^+ the preimage of $\mathcal{O}^t \subset k^t$ in B .

The *strict henselization* $(B, B^+) = (A, A^+)_x^{strict}$ is given by setting $B = A^{sh}$ (strict henselization of A at $\text{supp } x$) and B^+ the preimage of $\bar{\mathcal{O}} \subset \bar{k}$ in B .

The homomorphisms

$$(A, A^+)_x^h \rightarrow (A, A^+)_x^{sh} \rightarrow (A, A^+)_x^t \rightarrow (A, A^+)_x^{strict}$$

are integral by Lemma 10.13 (ii) below.

Lemma 10.12. *Let (A, A^+) be a henselian Huber pair and $(A, A^+) \rightarrow (B, B^+)$ a homomorphism of Huber pairs with $A \rightarrow B$ integral, ind-étale and local. Then $\mathfrak{m}_B \subset B^+$.*

Proof. Every element of $x \in \mathfrak{m}_B$ satisfies an equation of the form $x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0$ with $a_0, \dots, a_{n-1} \in \mathfrak{m}_A$ (the a_i are up to sign the elementary symmetric

functions in the conjugates of x in some Galois closure of B/A). Since $\mathfrak{m}_A \subset A^+$, x is integral over A^+ , hence $x \in B^+$. \square

Lemma 10.13. *Let $f : (A, A^+) \rightarrow (B, B^+)$ be a local homomorphism of local Huber pairs.*

- (i) *If (A, A^+) is henselian (resp. strongly, tamely or strictly henselian) and f is integral, then (B, B^+) is henselian (resp. strongly, tamely or strictly henselian).*
- (ii) *If f is ind-étale and (A, A^+) and (B, B^+) are henselian, then f is integral.*

Proof. (i) B^+ , being an integral, local algebra over the henselian ring A^+ , is henselian, as well. If A^+ is strictly henselian, then also B^+ is strictly henselian. Let k_A be the residue field of A and k_B that of B . Then k_B/k_A is an algebraic field extension. We obtain an injection $G_{k_B} \subset G_{k_A}$ of absolute Galois groups showing the statement also in the strict and tame case.

(ii) We may assume that f is étale. Since f is local and A is henselian, B is a local, finite étale A -algebra. Let A' be the integral closure of A^+ in B . We have to show that the inclusion $A' \rightarrow B^+$ is surjective. By Lemma 10.12, we have $\mathfrak{m}_B \subset A'$. It therefore suffices to show that $A'/\mathfrak{m}_B \rightarrow B^+/\mathfrak{m}_B$ is surjective. Both are subrings of the field $B/\mathfrak{m}_B = k_B$, which is a finite separable extension of $k_A = A/\mathfrak{m}_A$.

We claim that A'/\mathfrak{m}_B is the integral closure of $A^+/\mathfrak{m}_A \subset k_A$ in k_B . By definition of A' , every element of A'/\mathfrak{m}_B is integral over A^+/\mathfrak{m}_A . On the other hand, if for $y \in B$ the element $\bar{y} \in k_B$ is integral over A^+/\mathfrak{m}_A , then there is a congruence $y^n + a_{n-1} + \dots + a_0 \equiv 0 \pmod{\mathfrak{m}_B}$ with all $a_i \in A^+$. Since $\mathfrak{m}_B \subset A'$, y is integral over A' and hence $y \in A'$.

By assumption, A^+/\mathfrak{m}_A is a henselian valuation ring, hence so is A'/\mathfrak{m}_B . Therefore $A'/\mathfrak{m}_B \rightarrow B^+/\mathfrak{m}_B$ is a local homomorphism of valuation rings with the same quotient field, hence an isomorphism. This concludes the proof. \square

Now we consider Huber pairs (A, A^+) via their adic spectrum $\text{Spa}(A, A^+)$.

Definition 10.14. A Huber pair (A, A^+) is *étale (strongly étale, tamely) acyclic* if every étale (strongly étale, tame) covering of $\text{Spa}(A, A^+)$ splits.

Proposition 10.15. *A Huber pair (A, A^+) is étale (strongly étale, tamely) acyclic if and only if every connected component of $\text{Spa}(A, A^+)$ is the adic spectrum of a strictly (strongly, tamely) henselian Huber pair.*

Proof. We consider the canonical inclusion $\text{Spec } A \hookrightarrow \text{Spa}(A, A^+)$, $x \mapsto (x, \text{tr}_x)$. Then $\text{Spec } A$ carries the subspace topology and is dense in $\text{Spa}(A, A^+)$. Hence the connected components of $\text{Spa}(A, A^+)$ are in bijection with the connected components of $\text{Spec } A$.

Let $\mathfrak{p} \subset A$ be a prime ideal and let e run through all idempotents of A contained in \mathfrak{p} . Then the connected component of $\mathfrak{p} \in \text{Spec } A$ is equal to $\text{Spec } B$ with $B = \varinjlim_e A/e$. Note that B can also be obtained from A by inverting all elements $1 - e$. Since A^+ is integrally closed in A , all idempotents of A are in A^+ . Therefore the components of $\text{Spa}(A, A^+)$ are of the form $\text{Spa}(B, B^+)$, where $B = \varinjlim_e A/e$ is a component of A and $B^+ = \varinjlim_e A^+/e$.

Now the statement of the proposition reduces to Lemma 10.9 by exactly the same limit argument as in the proof of the analogous statement for schemes [Art71, Proposition 3.2]. \square

We call a Huber pair (A, A^+) *quasi-acyclic* if every connected component of $\text{Spa}(A, A^+)$ is the adic spectrum of a henselian Huber pair.

11. JOINS OF HENSELIAN HUBER PAIRS

Our next aim is to compare the tame cohomology of an S -scheme X with the tame cohomology of its associated adic space $\mathrm{Spa}(X, S)$. This is rather easy for Čech cohomology and we have proven in Section 7 that tame cohomology coincides with tame Čech cohomology for many schemes. Hence our next task is to prove a similar statement for discretely ringed adic spaces. This turns out to be more complicated as the analysis of joins is more involved.

Let (A, A^+) be a Huber pair such that A is a normal domain. Let $K = Q(A)$ and \bar{K}/K a separable closure. Let $v_1, v_2 \in \mathrm{Spa}(A, A^+)$ be points and $(B_i, B_i^+) = (A, A^+)_{v_i}^h$. We embed B_1 and B_2 into \bar{K} , hence also B_1^+ and B_2^+ are embedded into \bar{K} . Let $D = [B_1, B_2] \subset \bar{K}$ be the subring over A generated by B_1 and B_2 and let $D^+ \subset D$ be the integral closure of $[B_1^+, B_2^+]$ in D . We call the Huber pair (D, D^+) the *join* of the Huber pairs (B_1, B_1^+) and (B_2, B_2^+) (with respect to the chosen embeddings into \bar{K}):

$$\begin{array}{ccc}
 & \bar{K} & \\
 & \downarrow & \\
 & (D, D^+) = [(B_1, B_1^+), (B_2, B_2^+)] & \\
 (B_1, B_1^+) & \swarrow \quad \searrow & (B_2, B_2^+) \\
 & (A, A^+) &
 \end{array}$$

Now we can formulate the adic analogue of Artin's Theorem 6.1.

Theorem 11.1. *With the assumptions above, the join (D, D^+) is a henselian Huber pair. If v_1, v_2 are Riemann-Zariski points such that none of the v_i specializes to the other, then (D, D^+) is the strong henselization of (A, A^+) at some Riemann-Zariski point $w \in \mathrm{Spa}(A, A^+)$: $(D, D^+) = (A, A^+)_{w}^{sh}$.*

We start with some lemmas.

Lemma 11.2. *Let (A, A^+) be a Huber pair such that A is of finite type over A^+ . Then any finite subset M of $\mathrm{Spa}(A, A^+)$ has a common affinoid neighborhood $\mathrm{Spa}(B, B^+)$ such that $\mathrm{Spec} B^+ \rightarrow \mathrm{Spec} B$ is an open immersion. Moreover, any open affinoid contained in $\mathrm{Spa}(B, B^+)$ has this property.*

Proof. Let $\mathrm{Spec} A \xrightarrow{i} S \xrightarrow{p} \mathrm{Spec} A^+$ be a compactification with an open immersion i and p projective. Let $\{c_m\}_{m \in M}$ be the set of centers in S of the points in M . Then we can find a common open affine neighborhood $\mathrm{Spec} B'$ of the c_m 's. Setting $B = B' \otimes_{A^+} A$ and B^+ the normalization of B' in B , we obtain the required affinoid neighborhood $\mathrm{Spa}(B, B^+)$.

For an affinoid open $\mathrm{Spa}(C, C^+) \subset \mathrm{Spa}(B, B^+)$ consider the diagram

$$\begin{array}{ccc}
 \mathrm{Spec} C & \longrightarrow & \mathrm{Spec} B \\
 \downarrow & & \downarrow \\
 \mathrm{Spec} C^+ & \longrightarrow & \mathrm{Spec} B^+.
 \end{array}$$

The upper horizontal and the right vertical arrows are open immersions. Hence, so is the left vertical arrow. \square

Lemma 11.3. *Let M be a finite set of Riemann-Zariski points of an affinoid adic space $\text{Spa}(A, A^+)$. Assume that no element of M specializes to another. Then among all open affinoid neighborhoods $\text{Spa}(B, B^+)$ of M those are cofinal where the centers c_m of the elements $m \in M$ in $\text{Spec } B^+$ do not satisfy any specialization relation.*

Proof. Consider the map

$$c : \text{Spa}(A, A^+) \longrightarrow \lim S,$$

where S runs through all $\text{Spec } A$ -modifications of $\text{Spec } A^+$, i.e., factorizations

$$\text{Spec } A \xrightarrow{i} S \xrightarrow{p} \text{Spec } A^+$$

with i an open dense immersion and p proper, and the map c is given by sending a valuation to the system of its centers on the various S . Let $RZ(A, A^+)$ be the subspace of all Riemann-Zariski points in $\text{Spa}(A, A^+)$. By [Tem11, Corollary 3.4.7], the restriction of c to $RZ(A, A^+)$ is a homeomorphism

$$RZ(A, A^+) \xrightarrow{\sim} \lim S.$$

By Temkin's generalization of Chow's Lemma, [Tem11, Corollary 3.4.8], among the modifications, those with $p : S \rightarrow \text{Spec } A^+$ projective are cofinal. Since the $m \in M \subset RZ(A, A^+)$ do not specialize to each other, we find a projective $\text{Spec } A$ -modification $S \rightarrow \text{Spec } A^+$ such that the centers $(c_m)_{m \in M}$ on S do not specialize to each other. Now choose a common open affine neighborhood $\text{Spec } B' \subset S$ of the c_m . Set $B = B' \otimes_{A^+} A$ and B^+ the normalization of B' in B . This provides the required open affinoid neighborhood $\text{Spa}(B, B^+)$. It is clear that any smaller open affinoid neighborhood of M has the same property. \square

Lemma 11.4. *Let K be a field.*

- (i) *If $\mathcal{O} \subset K$ is a henselian valuation ring of K and $\mathcal{O}' \supset \mathcal{O}$ is an overring of \mathcal{O} in K , then also \mathcal{O}' is a henselian valuation ring.*
- (ii) *If $\mathcal{O}, \mathcal{O}' \subset K$ are henselian valuation rings of K such that none of $\mathcal{O}, \mathcal{O}'$ contains the other, then their join $\mathcal{O}\mathcal{O}'$ is strictly henselian.*

Proof. (i) Being an overring of \mathcal{O} , \mathcal{O}' is a valuation ring. Let us denote the associated valuations by v, v' . Then v' is a generalization of v and \mathcal{O}' is the localization of \mathcal{O} at some prime ideal \mathfrak{p} . In order to show that v' is henselian, we have to show that there is at most one extension of v' to L in any finite separable field extension L/K . Let A be the normalization of \mathcal{O} in L . Since \mathcal{O} is henselian, A is local, hence a valuation ring of L . The extensions of v' to L are in bijection to the prime ideals of A lying over $\mathfrak{p} \subset \mathcal{O}$. But there is only one since there are no containment relations between those primes and the set of primes of the valuation ring A is totally ordered with respect to inclusion.

(ii) Let $\mathfrak{m} \subset \mathcal{O}$ and $\mathfrak{m}' \subset \mathcal{O}'$ be the maximal ideals. Put $A = \mathcal{O} \cap \mathcal{O}'$ and $\mathfrak{p} = \mathfrak{m} \cap A$, $\mathfrak{p}' = \mathfrak{m}' \cap A$. By [EP05, Lemma 3.2.6], we have $\mathcal{O} = A_{\mathfrak{p}}$, $\mathcal{O}' = A_{\mathfrak{p}'}$. Since none of $\mathcal{O}, \mathcal{O}'$ contains the other, none of $\mathfrak{p}, \mathfrak{p}'$ contains the other. Hence the statement follows from Theorem 6.1. \square

Proof of Theorem 11.1. By a straightforward limit argument, we may assume that A is finitely generated over \mathbb{Z} .

Let \bar{A} and \bar{A}^+ be the integral closures of A and A^+ in \bar{K} . Let v'_i be the unique closed point of (B_i, B_i^+) . Then v'_i maps to v_i in $\text{Spa}(A, A^+)$. On the other hand, since (B_i, B_i^+) is henselian, there is a unique point $\bar{v}_i \in \text{Spa}(\bar{A}, \bar{A}^+)$ lying over v'_i .

By assumption, none of \bar{v}_1, \bar{v}_2 specializes to each other. The ring D is a component of the tensor product $B_1 \otimes_A B_2$ (and every component of the tensor product

arises in this way). By Theorem 6.3, D is local, henselian. It remains to show that $D^+ \subset D$ is the pre-image of a strictly henselian valuation ring $\mathcal{O} \subset k = D/\mathfrak{m}_D$.

Assume we are in case (1) of Theorem 6.3. Then both ring homomorphisms $B_1 \rightarrow D$, $B_2 \rightarrow D$ are not local and k is separably closed by loc. cit. Let \mathfrak{m}_1 , \mathfrak{m}_2 be the maximal ideals of B_1 and B_2 . Since the homomorphisms are not local, the ideal $\mathfrak{m}_1\mathfrak{m}_2 \subset D$ is not contained in \mathfrak{m}_D . Hence it is the whole ring D . Since B_1^+ and B_2^+ contain \mathfrak{m}_1 and \mathfrak{m}_2 , respectively, we conclude that $D^+ = D$. Hence (D, D^+) is strongly henselian (it is the strong henselization of (A, A^+) at some trivial valuation).

Assume we are in case (2) of Theorem 6.3 and, say, $B_1 \rightarrow D$ is a local, integral and ind-étale ring homomorphism. By Lemma 10.12, we have $\mathfrak{m}_D \subset D^+$. Let $\mathcal{O} := D^+/\mathfrak{m}_D \subset k = D/\mathfrak{m}_D$ and $\mathcal{O}_1 = B_1^+/\mathfrak{m}_1 \subset k_1 = B_1/\mathfrak{m}_1$. Then \mathcal{O} contains the integral closure \mathcal{O}'_1 of \mathcal{O}_1 in the field extension k/k_1 . Since \mathcal{O}_1 is a henselian valuation ring, the same is true for \mathcal{O}'_1 . Hence \mathcal{O} is an overring of a henselian valuation ring of k , hence itself a henselian valuation ring by Lemma 11.4(i). For $x \in D$ with $x \bmod \mathfrak{m}_B \in \mathcal{O}$, we find $y \in D^+$ with $x - y \in \mathfrak{m}_D \subset D^+$. This shows that D^+ is the full pre-image of \mathcal{O} in D . Hence (D, D^+) is local.

We conclude that (D, D^+) is a henselian Huber pair. For the rest of the proof we assume that v_1, v_2 are Riemann-Zariski and that we are in case (2) of Theorem 6.3. Let w' be the unique closed point of $\text{Spa}(D, D^+)$ and let $w \in \text{Spa}(A, A^+)$ be its image. Since $\text{Spa}(D, D^+)$ is henselian, there is a unique point $\bar{w} \in \text{Spa}(\bar{A}, \bar{A}^+)$ over w' . We learn:

1. Since $(A, A^+) \rightarrow (D, D^+)$ is ind-strongly étale and (D, D^+) is henselian, (D, D^+) lies somewhere between $(A, A^+)_w^h$ and $(A, A^+)_w^{sh}$. (We want to show that $(D, D^+) = (A, A^+)_w^{sh}$.)
2. \bar{w} is a vertical generalization of \bar{v}_1 and a generalization of \bar{v}_2 .
3. We are in the case that $B_1 \rightarrow D$ is local, hence (cf. the proof of Theorem 6.3) in the case $\text{supp } \bar{v}_1 \rightsquigarrow \text{supp } \bar{v}_2$ in $\text{Spec}(\bar{A})$.

If $\text{supp } \bar{v}_1 = \text{supp } \bar{v}_2$, then the statement reduces to the case of valuation rings, where it follows from Lemma 11.4. Hence we may assume that $\text{supp } \bar{v}_1 \subsetneq \text{supp } \bar{v}_2$. Since v_1 is a Riemann-Zariski point, the same is true for \bar{v}_1 and hence \bar{w} is a Riemann-Zariski point by Corollary 10.4(ii). But \bar{v}_2 is a specialization of \bar{w} . By Proposition 10.5, this is only possible if $\bar{w} = \text{tr}_{\text{supp } \bar{w}}$ and the specialization $\bar{w} \rightsquigarrow \bar{v}_2$ is of the form $\text{tr}_{\text{supp } \bar{w}} \rightsquigarrow \text{tr}_{\text{supp } \bar{v}_2} \rightsquigarrow \bar{v}_2$, where the second specialization is vertical.

We learn that $D = D^+$ and have to show that this henselian ring is strictly henselian. For this we may replace A by the normalization of $A/\text{supp } v_1$, i.e., we may assume that $\text{supp } v_1$ is the generic point of $\text{Spec } A$ and $D = D^+$ is a field. Since v_1 does not specialize to v_2 , v_1 is not the trivial valuation (moreover, in this case the assertion would be true by trivial reasons).

Let \bar{v}_3 be the trivial valuation with support $\text{supp } \bar{v}_2$, v_3 the restriction of \bar{v}_3 to A and $(B_3, B_3^+) = (A, A^+)_{v_3}^h$. Then B_2 is an integral étale extension of $B_3 = B_3^+$. In order to show the assertion, we therefore may replace v_2 by the trivial valuation v_3 .

After all these reductions, we arrive at the following situation:

- A is a normal domain of finite type over \mathbb{Z} with quotient field K
- $\text{supp } v_1$ is the generic point of $\text{Spec } A$, $(B_1, B_1^+) = (A, A^+)_{v_1}^h$
- B_1 is a field (intermediate extension of \bar{K}/K) and $B_1^+ \subsetneq B_1$ a valuation ring
- v_2 is a trivial valuation, $(B_2, B_2^+) = (A, A^+)_{v_2}^h$, i.e., $B_2 = B_2^+ = A_{\text{supp } v_2}^h$
- $D = D^+$ is a field (intermediate extension of \bar{K}/B_1).

Furthermore, by Lemma 11.2 we may assume that $\text{Spec } A \rightarrow \text{Spec } A^+$ is an open immersion and by Lemma 11.3 we may assume that the centers $c_1, c_2 \in \text{Spec } A^+$ of v_1, v_2 do not specialize to each other.

Moreover, by Lemma 6.2, we are still permitted to replace (A, A^+) by its normalization in a finite field extension L/K inside \bar{K} (and replace v_1, v_2 by points over them).

The local homomorphisms $A_{c_1}^+ \rightarrow \mathcal{O}_{v_1}, A_{c_2}^+ \xrightarrow{\sim} A_{\text{supp } v_2}$ induce maps $A_{c_1}^{+h} \rightarrow B_1^+, A_{c_2}^{+h} \xrightarrow{\sim} B_2^+$, hence a homomorphism

$$C := [A_{c_1}^{+h}, A_{c_2}^{+h}] \hookrightarrow D = [B_1, B_2].$$

By Theorem 6.1, C is strictly henselian. Since C is the inductive limit of étale A^+ -algebras, it is the strict henselization of A^+ at some prime \mathfrak{p}^+ . As $A \subset A_{\text{supp } v_2} = A_{c_2}^+ \subset C$, the morphism $\text{Spec } C \rightarrow \text{Spec } A^+$ factors through the open subset $\text{Spec } A$ of $\text{Spec } A^+$. Hence C can also be obtained as the strict henselization of A at $\mathfrak{p} = \mathfrak{p}^+A$. In particular, C is noetherian and has finite Krull-dimension. If C were a (separably closed) field, the inclusions $A \subset C \subset D \subset \bar{K}$ would prove that D is separably closed, hence our assertion. We will achieve this by decreasing induction on $\dim C$. Assume $\dim C > 0$. Then \mathfrak{p} is not the zero ideal, i.e., we find $0 \neq x \in \mathfrak{p} \subset A$.

The support of v_1 is the generic point of $\text{Spec } A$, hence $\mathcal{O}_{v_1} \subset K$. By valuation theory, the join $[\mathcal{O}_{v_1}, A] \subset K$ is the localization of \mathcal{O}_{v_1} at some prime ideal. If $[\mathcal{O}_{v_1}, A]$ were a proper subring of K , Lemma 10.3(ii) would imply the existence of a nontrivial horizontal specialization of v_1 . Since v_1 is Riemann-Zariski, we obtain

$$K = [\mathcal{O}_{v_1}, A].$$

Hence we can write

$$x^{-1} = \sum_{\text{finite}} y_i z_i, \quad y_i \in \mathcal{O}_{v_1}, \quad z_i \in A.$$

Now the homeomorphism $RZ(A, A^+) \cong \lim S$ above respects stalks. This means for a Riemann-Zariski point $v \in \text{Spa}(A, A^+)$ with $(A, A^+)_v = (E, E^+)$, the ring E^+ is the colimit of the (scheme-) stalks of the models at the respective centers, i.e., $E^+ = \text{colim } \mathcal{O}_{S, c(v)}$. Applying this to v_1 , we can (in the manner as we did above), modify (A, A^+) to (A', A'^+) to achieve that all y_i lie in A'_{c_1} . Defining C' similar to C but for (A', A'^+) , we obtain the commutative diagram

$$\begin{array}{ccc} & D = D^+ & \\ & \swarrow & \nwarrow \\ C & \xrightarrow{\quad} & C' \\ \uparrow & & \uparrow \\ A & \xrightarrow{\quad} & A', \end{array}$$

in which the lower horizontal map induces an open immersion on Spec . Since all y_i, z_i are in C' , x is invertible in C' , hence the image of the closed point of $\text{Spec } C'$ in $\text{Spec } A$ is a nontrivial generalization of \mathfrak{p} . This process stops since A has finite Krull-dimension. \square

Recall that the tensor product of Huber pairs

$$(B, B^+) = (B_1, B_1^+) \otimes_{(A, A^+)} \cdots \otimes_{(A, A^+)} (B_n, B_n^+)$$

is defined as follows: B is the (usual) tensor product $B_1 \otimes_A \cdots \otimes_A B_n$ and B^+ is the integral closure of $B_1^+ \otimes_{A^+} \cdots \otimes_{A^+} B_n^+$ in B .

Theorem 11.5. *Let (A, A^+) be a Huber pair, $v_1, v_2 \in \text{Spa}(A, A^+)$ and, for $i = 1, 2$, (B_i, B_i^+) a local, integral $(A, A^+)_{v_i}^h$ -algebra. Then*

$$(B, B^+) = (B_1, B_1^+) \otimes_{(A, A^+)} (B_2, B_2^+)$$

is quasi-acyclic. Assume that $v_1, v_2 \in \text{Spa}(A, A^+)$ are Riemann-Zariski points and let (D, D^+) be a component of (B, B^+) . Then the closed point of (D, D^+) maps to a Riemann-Zariski point of $\text{Spa}(A, A^+)$ and one of the following holds.

- (1) If both homomorphisms $(B_i, B_i^+) \rightarrow (D, D^+)$ are not local, then the residue field D^+/\mathfrak{m}_{D^+} of D^+ is separably closed.
- (2) If, say, $(B_1, B_1^+) \rightarrow (D, D^+)$ is local, it is integral. In particular, D^+/\mathfrak{m}_{D^+} is an algebraic field extension of $B_1^+/\mathfrak{m}_{B_1^+}$.

Proof. This follows in exactly the same way from Theorem 11.1 as Theorem 6.3 follows from Theorem 6.1. \square

We call a homomorphism of Huber pairs $f : (A, A^+) \rightarrow (B, B^+)$ *ind-étale* if it is a filtered colimit of étale homomorphisms $f_i : (A, A^+) \rightarrow (B_i, B_i^+)$.

Theorem 11.6. *Let (A, A^+) be a Huber pair and let $(B_1, B_1^+), \dots, (B_n, B_n^+)$ be ind-étale (A, A^+) -algebras which are quasi-acyclic. Then the tensor product*

$$(B, B^+) = (B_1, B_1^+) \otimes_{(A, A^+)} \dots \otimes_{(A, A^+)} (B_n, B_n^+)$$

is an ind-étale quasi-acyclic (A, A^+) -algebra. Suppose that all closed points of the $\text{Spa}(B_i, B_i^+)$ map to Riemann-Zariski points in $\text{Spa}(A, A^+)$. Then the closed points of $\text{Spa}(B, B^+)$ map to Riemann-Zariski points in $\text{Spa}(A, A^+)$. For a component (D, D^+) of (B, B^+) one of the following holds

- (1) D^+/\mathfrak{m}_{D^+} is separably closed, or
- (2) D^+/\mathfrak{m}_{D^+} is an algebraic extension of $D_i^+/\mathfrak{m}_{D_i^+}$, where (D_i, D_i^+) is a component of (B_i, B_i^+) for some i , $1 \leq i \leq n$.

Proof. Regarding Lemma 10.13(ii), this follows in exactly the same way from Theorem 11.5 as Theorem 6.4 follows from Theorem 6.3. \square

Corollary 11.7. *Let (A, A^+) be a Huber pair and let $(B_1, B_1^+), \dots, (B_n, B_n^+)$ be ind-étale quasi-acyclic (A, A^+) -algebras. Then the tensor product*

$$(B, B^+) = (B_1, B_1^+) \otimes_{(A, A^+)} \dots \otimes_{(A, A^+)} (B_n, B_n^+)$$

is quasi-acyclic.

- (i) If all (B_i, B_i^+) are étale acyclic, then also (B, B^+) is étale acyclic.
- (ii) If all $\text{Spa}(B_i, B_i^+)$ are strongly étale acyclic and all morphisms $\text{Spa}(B_i, B_i^+) \rightarrow (A, A^+)$ map closed points to Riemann-Zariski points, then (B, B^+) is strongly étale acyclic.
- (iii) If A is an \mathbb{F}_p -algebra for some prime number p , all (B_i, B_i^+) are tamely acyclic and all morphisms $\text{Spa}(B_i, B_i^+) \rightarrow (A, A^+)$ map closed points to Riemann-Zariski points, then (B, B^+) is tamely acyclic.

Proof. The first assertion follows immediately from Theorem 11.6. Assertion (i) follows from Theorem 6.4 and (ii) follows again from Theorem 11.6. It remains to show (iii). Assume that A is an \mathbb{F}_p -algebra and let (D, D^+) be a component of (B, B^+) . Since we already know that (D, D^+) is strongly henselian, it suffices to show that the absolute Galois group G_k of $k = D/\mathfrak{m}_D$ is a pro- p -group. Note that D is a component of B . Hence, by Theorem 6.4, either k is separably closed (in which case we are done) or there is a component D_i of B_i for some i such that $D_i \rightarrow D$ is local and k is an algebraic extension of the residue field D_i/\mathfrak{m}_{D_i} . The absolute Galois group of D_i/\mathfrak{m}_{D_i} is a pro- p -group by assumption. Therefore its subgroup G_k is also a pro- p -group. \square

12. RIEMANN-ZARISKI MORPHISMS

The main technical problem in proving a comparison theorem between the sheaf cohomology and Čech cohomology of discretely ringed adic spaces along the lines of the proof for schemes as given in Section 7 is the assumption on Riemann-Zariski points in Theorem 11.6 and Corollary 11.7. Therefore we have to carry out a more thorough analysis of Riemann-Zariski points first.

Let \mathcal{X} be a discretely ringed adic space. We say that $P \in \mathcal{X}$ specializes to $Q \in \mathcal{X}$ (notation: $P \rightsquigarrow Q$) if $Q \in \overline{\{P\}}$. Let \mathcal{X}_P be the local adic space of \mathcal{X} in P , i.e., $\mathcal{X}_P = Spa(A, A^+)_P$ for an open affinoid neighbourhood $Spa(A, A^+)$ of P in \mathcal{X} . Then $P \rightsquigarrow Q$ if and only if the morphism $\mathcal{X}_P \rightarrow \mathcal{X}$ factors through $\mathcal{X}_Q \rightarrow \mathcal{X}$.

Definition 12.1. Let $P \rightsquigarrow Q$. We say Q is a vertical resp. horizontal specialization of P if this is the case in one/every affinoid open of \mathcal{X} containing P and Q . A point $x \in \mathcal{X}$ is called a *Riemann-Zariski point* if it has no nontrivial horizontal specializations.

We call a morphism $f : \mathcal{X}' \rightarrow \mathcal{X}$ *Riemann-Zariski* if for every Riemann-Zariski point $x' \in \mathcal{X}'$, $f(x')$ is a Riemann-Zariski point in \mathcal{X} .

In Section 4 we recalled Temkin's construction of the discretely ringed adic space $Spa(X, S)$ associated to a scheme morphism $X \rightarrow S$. Points of $Spa(X, S)$ are triples (x, v, ε) , where x is a point of X , v is a valuation of $k(x)$ and $\varepsilon : Spec \mathcal{O}_v \rightarrow S$ is a morphism compatible with $Spec k(x) \rightarrow S$.

Lemma 12.2. Let $p : X \rightarrow S$ be a scheme morphism, $\mathcal{X} = Spa(X, S)$ and $(x, v, \varepsilon), (y, w, \mu) \in \mathcal{X}$.

- (i) (y, w, μ) is a vertical specialization of (x, v, ε) if and only if $x = y$, $\mathcal{O}_w \subset \mathcal{O}_v$ ($\subset k(x)$), and the diagram

$$\begin{array}{ccc} Spec \mathcal{O}_v & & \\ \downarrow & \searrow \varepsilon & \\ Spec \mathcal{O}_w & \xrightarrow{\mu} & S \end{array}$$

commutes (automatic if S is separated).

- (ii) (y, w, μ) is a horizontal specialization of (x, v, ε) if and only if there exists a point $P \in Spec \mathcal{O}_v$ and an x -morphism $f : (Spec \mathcal{O}_v)_P \rightarrow X$ such that
- (a) $f(P) = y$.
 - (b) The valuation w is the restriction to $k(y)$ of the valuation on $k(P)$ given by the valuation ring \mathcal{O}_v/P .
 - (c) The diagram (which exists by (b) since \mathcal{O}_v/P is the ring of global sections of the affine scheme $\overline{\{P\}}$.)

$$\begin{array}{ccc} \overline{\{P\}} & \longrightarrow & Spec \mathcal{O}_w \\ \downarrow & & \downarrow \mu \\ Spec \mathcal{O}_v & \xrightarrow{\varepsilon} & S \end{array}$$

commutes. (Note that (c) is automatic if S is separated.)

Here is a diagram elaborating the conditions:

$$\begin{array}{ccccc}
 \text{Spec } \mathcal{O}_v & \longleftarrow & \overline{\{P\}} & & \\
 \uparrow & & \uparrow & \searrow & \\
 (\text{Spec } \mathcal{O}_v)_P & \longleftarrow & P & & \text{Spec } \mathcal{O}_w \\
 \uparrow & \searrow & \downarrow & \searrow & \uparrow \\
 x & \xleftarrow{\varepsilon} & X & \xleftarrow{\mu} & y \\
 & & \downarrow p & & \\
 & & S & &
 \end{array}$$

Proof. (i) is obvious.

(ii) If (y, w, μ) is a horizontal specialization of (x, v, ε) , then both points lie in a common open affinoid and the conditions hold by Lemma 10.3. The same argument shows that the given conditions imply that (y, w, μ) is a horizontal specialization of (x, v, ε) if we find a common open affinoid neighborhood of the two points. Since $f(P) = y$ by (a), we have $x \rightsquigarrow y$ in X and $p(x) \rightsquigarrow p(y)$ in S . Denoting the closed point of $\text{Spec } \mathcal{O}_w$ by c_w , we moreover have $p(y) \rightsquigarrow \mu(c_w)$. Let S' be an open affine neighborhood of $\mu(c_w)$ in S and X' an open affine neighborhood of y in $p^{-1}(S')$. Then $\text{Spa}(X', S')$ is an open affinoid neighborhood of (y, w, μ) in $\text{Spa}(X, S)$. We are done if we can show that (x, v, ε) lies in $\text{Spa}(X', S')$. First, $x \rightsquigarrow y$ implies $x \in X'$. Hence it suffices to show that $\varepsilon : \text{Spec } \mathcal{O}_v \rightarrow S$ factors through S' . For this it suffices to show $\varepsilon(c_v) \in S'$, where c_v is the closed point of $\text{Spec } \mathcal{O}_v$. This follows since the closed point of $\overline{\{P\}}$ maps to c_v and c_w respectively and condition (c) implies $\varepsilon(c_v) = \mu(c_w)$. \square

Corollary 12.3. *If X and S are separated, then the set of horizontal specializations of a point in $\text{Spa}(X, S)$ is totally ordered.*

Proof. Let $(x, v, \varepsilon) \rightsquigarrow (y, w, \mu)$ be a horizontal specialization in $\text{Spa}(X, S)$. We show that, with the notation of part (ii) of Lemma 12.2, the triple (y, w, μ) is uniquely given by $P \in \text{Spec } \mathcal{O}_v$.

First of all, since X is separated, the morphism $f : (\text{Spec } \mathcal{O}_v)_P \rightarrow X$, and hence $y = f(P)$ is unique. Further, $\mathcal{O}_w = (\mathcal{O}_v/P) \cap k(y)$ (intersection in $k(P)$), hence w is given by P . Finally, since S is separated, $\mu : \text{Spec } \mathcal{O}_w \rightarrow S$ is uniquely given by y , hence by P . The result follows since the set of prime ideals in a valuation ring is totally ordered. \square

Corollary 12.4. *A point (x, v, ε) in $\text{Spa}(X, S)$ is Riemann-Zariski if and only if x is a closed point in $X \times_S \text{Spec } \mathcal{O}_v$.*

Proof. Assume that x is not closed in $X \times_S \text{Spec } \mathcal{O}_v$. Then there exists a nontrivial specialization \tilde{y} of x in $X \times_S \text{Spec } \mathcal{O}_v$. Hence we can find a valuation ring $\mathcal{O} \subsetneq k(x)$ such that $x \rightarrow X \times_S \text{Spec } \mathcal{O}_v$ factors as $x \rightarrow \text{Spec } \mathcal{O} \rightarrow X \times_S \text{Spec } \mathcal{O}_v$ and \tilde{y} is the image of the closed point in $\text{Spec } \mathcal{O}$. Projecting to the second factor shows $\mathcal{O}_v \subset \mathcal{O}$, hence $\mathcal{O} = (\mathcal{O}_v)_P$ for a point $P \in \text{Spec } \mathcal{O}_v$. Projecting to the first factor, we find an x -morphism $f : \text{Spec } \mathcal{O} \rightarrow X$ and hence by Lemma 12.2 a horizontal specialization of (x, v, ε) . Since $\mathcal{O} \subsetneq k(x)$, the image of the closed point of $\text{Spec } \mathcal{O}$ in X is not x , hence the horizontal specialization of (x, v, ε) is nontrivial. We conclude that (x, v, ε) is not a Riemann-Zariski point.

If (x, v, ε) is not a Riemann-Zariski point, it has a nontrivial horizontal specialization. Then the x -morphism $f : (\text{Spec } \mathcal{O}_v)_P \rightarrow X$ of Lemma 12.2 and the canonical morphism $(\text{Spec } \mathcal{O}_v)_P \rightarrow \text{Spec } \mathcal{O}_v$ give a morphism $(\text{Spec } \mathcal{O}_v)_P \rightarrow X \times_S \text{Spec } \mathcal{O}_v$. It maps P to a proper specialization of x . \square

Lemma 12.5. *If X is quasi-compact, then every point in $\text{Spa}(X, S)$ has a horizontal specialization which is a Riemann-Zariski point. Moreover, if X and S are separated, this Riemann-Zariski point is unique.*

Proof. Let (x, v, ϕ) be a point in $\text{Spa}(X, S)$. The set of horizontal specializations is partially ordered. As X is quasi-compact, it is covered by finitely many affinoids $\text{Spa}(A_i, A_i^+)$ and thus any chain of horizontal specializations of (x, v, ϕ) is contained entirely in one of the $\text{Spa}(A_i, A_i^+)$. We can then view v as a valuation of A_i and all specializations of the chain specialize to $v|_{c\Gamma_v}$. Hence, by Zorn's lemma, there is a maximal horizontal specialization (y, w, ψ) of (x, v, ϕ) . In particular, (y, w, ψ) is a Riemann-Zariski point. If X and S are separated, (y, w, ψ) is unique as the horizontal specializations are totally ordered by Corollary 12.3. \square

Definition 12.6. A commutative square of morphisms of schemes

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

is said to have universally closed diagonal if the induced morphism $X' \rightarrow X \times_S S'$ is universally closed.

Lemma 12.7. *Let*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

be a square with universally closed diagonal. Then the induced morphism

$$\text{Spa}(X', S') \rightarrow \text{Spa}(X, S)$$

is Riemann-Zariski. The converse holds if X' is quasi-compact and all residue field extensions of $X' \rightarrow X$ are algebraic (e.g., if $X' \rightarrow X$ is étale).

Proof. Suppose the square has universally closed diagonal. Let (x', v', π') be a Riemann-Zariski point of $\text{Spa}(X', S')$ and assume that its image (x, v, ϕ) in $\text{Spa}(X, S)$ is not a Riemann-Zariski point. This means there exists a localization $\mathcal{O} \subsetneq k(x)$ of the valuation ring \mathcal{O}_v such that the dotted arrow exists in the following diagram:

$$\begin{array}{ccc} \text{Spec } k(x) & \longrightarrow & X \\ \downarrow & \searrow & \downarrow \\ \text{Spec } \mathcal{O} & \longrightarrow & \text{Spec } \mathcal{O}_v \xrightarrow{\phi} S. \end{array}$$

There is a localization $\mathcal{O}' \subset k(x')$ of $\mathcal{O}_{v'}$ with $\mathcal{O}' \cap k(x) = \mathcal{O}$. This yields a diagram

$$\begin{array}{ccccc} \text{Spec } \mathcal{O}' & & & & \\ & \searrow & & & \\ & & X \times_S S' & \longrightarrow & X \\ & \searrow & \downarrow & & \downarrow \\ & & S' & \longrightarrow & S. \end{array}$$

The valuative criterion for universal closedness for the morphism $X' \rightarrow X \times_S S'$ provides a lift:

$$\begin{array}{ccccc}
 \text{Spec } \mathcal{O}' & & & & \\
 \swarrow & \dashrightarrow & & \searrow & \\
 & X' & \longrightarrow & X & \\
 \searrow \phi' & \downarrow & & \downarrow & \\
 & S' & \longrightarrow & S &
 \end{array}$$

But this means that (x', v', ϕ') is not a Riemann-Zariski point, a contradiction.

Assume now that X' is quasi-compact and all residue field extensions of $X' \rightarrow X$ are algebraic, and that $\text{Spa}(X', S') \rightarrow \text{Spa}(X, S)$ maps Riemann-Zariski points to Riemann-Zariski points. We have to show that for any diagram

$$\begin{array}{ccc}
 \text{Spec } k & \longrightarrow & X' \\
 \downarrow & \dashrightarrow & \downarrow \\
 \text{Spec } \mathcal{O} & \longrightarrow & X \times_S S',
 \end{array}$$

where \mathcal{O} is a valuation ring of the field k , the dotted arrow exists.

Without loss of generality we may assume that k equals the residue field of the image of $\text{Spec } k \rightarrow X'$. The solid diagram then defines a point (x', v', ϕ') in $\text{Spa}(X', S')$ whose image (x, v, ϕ) in $\text{Spa}(X, S)$ has center in X . Applying Lemma 12.5 we obtain a horizontal specialization (y', w', ψ') of (x', v', ϕ') which is a Riemann-Zariski point.

By assumption it maps to a Riemann-Zariski point (y, w, ψ) in $\text{Spa}(X, S)$. We know that (y, w, ψ) is a horizontal specialization of (x, v, ϕ) and that (x, v, ϕ) has center in X . Therefore, also (y, w, ψ) has center in X . Being a Riemann-Zariski point with center in X , (y, w, ψ) has to correspond to a trivial valuation. By our hypothesis on residue field extensions this implies that (y', w', ψ') corresponds to a trivial valuation, as well. In particular, (x', v', ϕ') has center in X' which is moreover compatible with the given center of (x, v, ϕ) in X . In other words, the dotted arrow exists. \square

Lemma 12.8. (i) (Composition) If the squares

$$\begin{array}{ccc}
 X'' & \longrightarrow & X' \\
 \downarrow & & \downarrow \\
 S'' & \longrightarrow & S'
 \end{array}
 \qquad
 \begin{array}{ccc}
 X' & \longrightarrow & X \\
 \downarrow & & \downarrow \\
 S' & \longrightarrow & S
 \end{array}$$

have universally closed diagonal, then the same is true for the square

$$\begin{array}{ccc}
 X'' & \longrightarrow & X \\
 \downarrow & & \downarrow \\
 S'' & \longrightarrow & S.
 \end{array}$$

(ii) (Base change) Consider two squares

$$\begin{array}{ccc}
 Y & \longrightarrow & X \\
 \downarrow & & \downarrow \\
 T & \longrightarrow & S
 \end{array}
 \qquad
 \begin{array}{ccc}
 X & \longleftarrow & X' \\
 \downarrow & & \downarrow \\
 S & \longleftarrow & S'
 \end{array}$$

and assume that the right hand one has universally closed diagonal. Then the same is true for the square

$$\begin{array}{ccc} Y & \longleftarrow & Y' = Y \times_X X' \\ \downarrow & & \downarrow \\ T & \longleftarrow & T' = T \times_S S' \end{array}$$

(iii) (Limits) Let $X \rightarrow S$ be a scheme morphism and let $(X_i \rightarrow S_i)_{i \in I}$ be a family of morphisms over $X \rightarrow S$ such that all X_i are quasi-compact and all squares

$$\begin{array}{ccc} X_i & \longrightarrow & X \\ \downarrow & & \downarrow \\ S_i & \longrightarrow & S \end{array}$$

have universally closed diagonal. Then the square

$$\begin{array}{ccc} \lim X_i & \longrightarrow & X \\ \downarrow & & \downarrow \\ \lim S_i & \longrightarrow & S \end{array}$$

has universally closed diagonal.

Proof. We leave the standard verifications of (i) and (ii) to the reader. For (iii) note that by the valuative criterion for universal closedness [Sta19, TAG 01KF], it suffices to show that in every solid diagram

$$\begin{array}{ccc} \text{Spec } K & \longrightarrow & \lim_{i \in I} X_i \\ \downarrow & \dashrightarrow & \downarrow \\ \text{Spec } \mathcal{O} & \longrightarrow & X \times_S \lim_{i \in I} S_i, \end{array} \quad (*)$$

where \mathcal{O} is a valuation ring with quotient field K , the dotted arrow exists. For every $i \in I$ denote by M_i the set of all possible dotted arrows in

$$\begin{array}{ccc} \text{Spec } K & \longrightarrow & X_i \\ \downarrow & \dashrightarrow & \downarrow \\ \text{Spec } \mathcal{O} & \longrightarrow & X \times_S S_i. \end{array}$$

By assumption M_i is non-empty. Since X_i is quasi-compact, it is furthermore finite. The reason is that given a finite affine cover $X_i = \bigcup_k U_k$ every dotted arrow factors through some affine open U_k and for every U_k there is at most one dotted arrow. The M_i thus form an inverse system of finite sets. Hence, their limit $M = \lim_i M_i$ is non-empty and every element of M provides a dotted arrow for the diagram (*). \square

Lemma 12.9. *Let S be quasi-compact and quasi-separated and $X \rightarrow S$ a separated morphism of finite type. Then there is an open covering of $\text{Spa}(X, S)$ by finitely many affinoids $\text{Spa}(A_i, A_i^+)$ coming from squares*

$$\begin{array}{ccc} \text{Spec } A_i & \longrightarrow & X \\ \downarrow & & \downarrow \\ \text{Spec } A_i^+ & \longrightarrow & S \end{array}$$

with universally closed diagonal such that $\text{Spec } A_i \rightarrow X$ is a quasi-compact open immersion and $\text{Spec } A_i^+ \rightarrow S$ is of finite type. In particular, $\text{Spa}(A_i, A_i^+) \rightarrow \text{Spa}(X, S)$ is a quasi-compact open immersion which is a Riemann-Zariski morphism.

Proof. By Deligne’s generalization of Nagata’s compactification theorem [Con07, Theorem 4.1], we can find a compactification \bar{X} of X over S . Blowing up the complement of X in \bar{X} if necessary, we may assume that $\bar{X} \setminus X$ is the support of a Cartier divisor. Hence, we can find a covering of \bar{X} by finitely many affine open subschemes \bar{U}_i such that $U_i = \bar{U}_i \cap X$ is affine as well. Then the affinoids $\text{Spa}(U_i, \bar{U}_i)$ form an open covering of $\text{Spa}(X, S)$. Moreover, since $\bar{X} \rightarrow S$ is separated, $X \hookrightarrow \bar{X} \times_S X$ is a closed immersion and applying $\bar{U}_i \times_{\bar{X}} -$, we see that also $U_i \rightarrow \bar{U}_i \times_S X$ is a closed immersion for all i . Hence the squares

$$\begin{array}{ccc} U_i & \longrightarrow & X \\ \downarrow & & \downarrow \\ \bar{U}_i & \longrightarrow & S \end{array}$$

have a universally closed diagonal. □

Lemma 12.10. *Let $f : \text{Spa}(X', S') \rightarrow \text{Spa}(X, S)$ come from a square*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

of separated, finite type morphisms of quasi-compact and quasi-separated schemes. Assume that f is étale and surjective. Then there exists a finite set of morphisms of finite type $T_i \rightarrow S'$ such that the induced morphism

$$\coprod_i \text{Spa}(X', T_i) \longrightarrow \text{Spa}(X, S)$$

is a surjective (étale) Riemann-Zariski morphism.

Proof. Let \bar{X} be a compactification of X over S (see [Con07], Theorem 4.1). Then f coincides with the induced morphism

$$\text{Spa}(X', \bar{X} \times_X S') \rightarrow \text{Spa}(X, \bar{X}).$$

We may thus assume that $X \rightarrow S$ is an open immersion. By the same arguments we reduce to the case where also $X' \rightarrow S'$ is an open immersion. We thus have a diagram

$$\begin{array}{ccccc} X' & & & & \\ & \searrow & & \searrow & \\ & & X \times_S S' & \longrightarrow & X \\ & \searrow & \downarrow \phi & & \downarrow \phi \\ & & S' & \longrightarrow & S \end{array}$$

The map $X' \rightarrow X \times_S S'$ is thus an open immersion as well. We denote by $Z \subset X \times_S S'$ the complement of X' (with reduced scheme structure). Let I be a set indexing the isomorphism classes of $X \times_S S'$ -modifications of S' , i.e., for each $i \in I$ we have a diagram

$$\begin{array}{ccc} X \times_S S' & \longleftarrow \circ & \longrightarrow S_i \\ & \searrow \circ & \swarrow \text{proper, birational} \\ & & S' \end{array}$$

Denote by Z_i the closure of Z in S_i and set $T_i = S_i \setminus Z_i$. We obtain a diagram

$$\begin{array}{ccccc}
 X' & & & & \\
 \swarrow & & & & \searrow \\
 \circ & & & & \circ \\
 \downarrow & & & & \downarrow \\
 X \times_S T_i & \xleftarrow{\circ} & X \times_S S_i & \xrightarrow{\circ} & X \\
 \downarrow \circ & & \downarrow \circ & & \downarrow \circ \\
 T_i & \xleftarrow{\circ} & S_i & \xrightarrow{\circ} & S
 \end{array}$$

In fact we have

$$X \times_S T_i = T_i \cap (X \times_S S_i) = (X \times_S S') \setminus Z = X',$$

where we have used that $X \times_S S_i = X \times_S S'$. Hence, by Lemma 12.7, $\text{Spa}(X', T_i) \rightarrow \text{Spa}(X, S)$ is Riemann-Zariski. Furthermore, it dominates $\text{Spa}(X', S') \rightarrow \text{Spa}(X, S)$.

What is left to show is that the family $(\text{Spa}(X', T_i) \rightarrow \text{Spa}(X, S))_{i \in I}$ is surjective. Then we know by the quasi-compactness of $\text{Spa}(X, S)$ that finitely many $\text{Spa}(X', T_i)$ cover $\text{Spa}(X, S)$ and we can take their disjoint union. In order to show this surjectivity it suffices to prove that every *closed* point of $\text{Spa}(X, S)$ is contained in the image of one of the $\text{Spa}(X', T_i)$. Let (x, v, ϕ) be a closed point of $\text{Spa}(X, S)$ and pick a closed point (x', v', ϕ') in its preimage in $\text{Spa}(X', S')$.

We can view (x', v', ϕ') as a point of $\text{Spa}(X \times_S S', S')$ and as such it is a Riemann-Zariski point for the following reason. By Corollary 12.4 we have to show that x' is a closed point in

$$\text{Spec } \mathcal{O}_{v'} \times_{S'} (X \times_S S') = \text{Spec } \mathcal{O}_{v'} \times_S X.$$

But this is equivalent to saying that

$$\begin{array}{ccc}
 \text{Spec } k(x') & \longrightarrow & X \\
 \downarrow & & \downarrow \circ \\
 \text{Spec } \mathcal{O}_{v'} & \longrightarrow & S
 \end{array}$$

has universally closed diagonal, which in turn is equivalent by Lemma 12.7 to

$$\text{Spa}(k(x'), \mathcal{O}_{v'}) \rightarrow \text{Spa}(X, S)$$

being Riemann-Zariski. This is true as the image (x, v, ϕ) of (x', v', ϕ') in $\text{Spa}(X, S)$ is a closed point.

For any $i \in I$ we have $\text{Spa}(X \times_S S', S') \cong \text{Spa}(X \times_S S', S_i)$. The point (x', v', ϕ') in $\text{Spa}(X \times_S S', S')$ identifies with the point (x', v', ϕ_i) , where $\phi_i : \text{Spec } \mathcal{O}_{v'} \rightarrow S_i$ is the unique lift of the morphism $\phi' : \text{Spec } \mathcal{O}_{v'} \rightarrow S'$ coming from the valuative criterion of properness. Denote by c_i the center of (x', v', ϕ_i) in S_i , i.e., the image of the closed point of $\text{Spec } \mathcal{O}_{v'}$ under ϕ_i . Now [TT18], Lemma 5.1.3 gives us $i \in I$ such that c_i is not contained in Z_i . In other words, $(x', v', \phi_i) \in \text{Spa}(X', T_i)$ \square

13. CECH COHOMOLOGY OF DISCRETELY RINGED ADIC SPACES

In this section we transfer the results of Section 7 to the realm of adic spaces. Let τ denote one of the topologies ‘et’ (étale), ‘set’ (strongly étale) or ‘t’ (tame). We make the general assumption that all schemes and scheme morphisms are quasi-compact and quasi-separated.

Proposition 13.1. *For every affinoid adic space $\text{Spa}(A, A^+)$ there is a surjective Riemann-Zariski pro- τ morphism*

$$\text{Spa}(\tilde{A}, \tilde{A}^+) \rightarrow \text{Spa}(A, A^+)$$

with (\tilde{A}, \tilde{A}^+) τ -acyclic.

Proof. As in the proof of Proposition 7.12, we follow the method of Bhatt-Scholze, [BS15, Proof of Lemma 2.2.7]. Let I be the set of isomorphism classes of surjective Riemann-Zariski τ -morphisms $Spa(A', A'^+) \rightarrow Spa(A, A^+)$. For every $i \in I$ pick a representative $Spa(B_i, B_i^+) \rightarrow Spa(A, A^+)$ and set

$$(A_1, A_1^+) := \operatorname{colim}_{J \subset I \text{ finite}} \bigotimes_{j \in J} (B_j, B_j^+),$$

where the tensor product is taken over (A, A^+) and the (filtered) colimit is indexed by the poset of finite subsets of I . There is an obvious ind- τ map $(A, A^+) \rightarrow (A_1, A_1^+)$ which induces a surjective Riemann-Zariski morphism on adic spectra. Moreover, by Lemma 12.10, any surjective τ -morphism $Spa(B, B^+) \rightarrow Spa(A, A^+)$ is dominated by a Riemann-Zariski surjective τ -morphism $Spa(X, S) \rightarrow Spa(A, A^+)$. By Lemma 12.9 we may assume that $Spa(X, S)$ is affinoid, i.e., $Spa(X, S) \cong Spa(A_i, A_i^+)$ for some $i \in I$. We therefore obtain an (A, A^+) -homomorphism $(B, B^+) \rightarrow (A_1, A_1^+)$. This implies that $(A, A^+) \rightarrow (B, B^+)$ splits after base change to (A_1, A_1^+) . Iterating the construction with (A_1, A_1^+) replacing (A, A^+) and proceeding inductively defines a tower

$$(A, A^+) \rightarrow (A_1, A_1^+) \rightarrow (A_2, A_2^+) \rightarrow \dots$$

of faithfully flat Riemann-Zariski (A, A^+) -algebras with ind- τ transition maps. Set $(\tilde{A}, \tilde{A}^+) = \operatorname{colim}(A_n, A_n^+)$. As étale homomorphisms are finitely presented, one checks that any faithfully flat τ - (\tilde{A}, \tilde{A}^+) -algebra has a section, so (\tilde{A}, \tilde{A}^+) is τ -acyclic.

Being a filtered colimit of faithfully flat Riemann-Zariski τ - (A, A^+) -algebras, $(A, A^+) \rightarrow (\tilde{A}, \tilde{A}^+)$ is ind- τ and Riemann-Zariski. Moreover, since the inverse limit of finite non-empty sets is non-empty, every point of $Spa(A, A^+)$ has a preimage in $Spa(\tilde{A}, \tilde{A}^+)$. \square

Theorem 13.2. *Let \mathcal{X} be a discretely ringed adic space with the property that any finite subset of \mathcal{X} is contained in an affinoid open. Let $\tau \in \{\text{et}, \text{set}, t\}$. If $\tau = t$ assume that \mathcal{X} is of pure characteristic $p \geq 0$. Suppose we are given for $i = 1, \dots, n$ pro- τ Riemann-Zariski morphisms $\mathcal{U}_i = Spa(A_i, A_i^+) \rightarrow \mathcal{X}$ with (A_i, A_i^+) τ -acyclic. Then the fiber product*

$$\mathcal{U} := Spa(A_1, A_1^+) \times_{\mathcal{X}} Spa(A_2, A_2^+) \times_{\mathcal{X}} \dots \times_{\mathcal{X}} Spa(A_n, A_n^+)$$

is affinoid and τ -acyclic.

Proof. If $f : \mathcal{Y} \rightarrow \mathcal{X}$ is a Riemann-Zariski morphism which factors through an open subspace $\mathcal{Z} \subset \mathcal{X}$, then $\mathcal{Y} \rightarrow \mathcal{Z}$ is also Riemann-Zariski. Indeed, if $x \in \mathcal{Z}$ is Riemann-Zariski in \mathcal{X} , it is Riemann-Zariski in \mathcal{Z} .

Moreover, a Riemann-Zariski morphism sends closed points to Riemann-Zariski points. Hence, if there is an affinoid open subspace $\mathcal{V} \subset \mathcal{X}$ such that all \mathcal{U}_i map to \mathcal{V} , the result follows from Corollary 11.7.

In the general case, let $p_i \in \mathcal{U}_i$, $i = 1, \dots, n$, be closed points. By assumption, there is an affinoid open $\mathcal{V} = Spa(B, B^+) \subset \mathcal{X}$ containing the images of p_1, \dots, p_n . Since \mathcal{U}_i is quasi-acyclic, every component of \mathcal{U}_i has a unique closed point and it maps to \mathcal{V} if and only if its closed point maps to \mathcal{V} . By Proposition 7.13, $\varphi_i : \mathcal{U}_i^c \rightarrow \pi_0(\mathcal{U}_i)$ is a homeomorphism of profinite spaces. Therefore, we find a closed and open subset $\mathcal{W}_i \subset \mathcal{U}_i^c$ which contains p_i and maps to \mathcal{V} . Then also the preimage $\mathcal{V}_i \subset \mathcal{U}_i$ of $\varphi_i(\mathcal{W}_i) \subset \pi_0(\mathcal{U}_i)$ under $\mathcal{U}_i \rightarrow \pi_0(\mathcal{U}_i)$ maps to \mathcal{V} . Hence \mathcal{V}_i is a closed and open subset of \mathcal{U}_i containing p_i and mapping to \mathcal{V} . Being closed and open in \mathcal{U}_i , \mathcal{V}_i is the adic spectrum of an ind-étale, τ -acyclic (B, B^+) -algebra. Moreover $\mathcal{V}_i \rightarrow \mathcal{V}$ is a Riemann-Zariski morphism for the following reason. The composition $\mathcal{V}_i \rightarrow \mathcal{U}_i \rightarrow \mathcal{X}$ is Riemann-Zariski because the first morphism is closed

and the second is Riemann-Zariski by assumption. Hence $\mathcal{V}_i \rightarrow \mathcal{V}$ is Riemann-Zariski by the remark at the beginning of the proof.

By the first part of the proof, $\mathcal{V}(p_1, \dots, p_n) := \mathcal{V}_1 \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} \mathcal{V}_n$ is a closed and open subspace of \mathcal{U} , hence affinoid ind-étale and τ -acyclic. Any nonempty closed subset in a spectral space contains a closed point. Hence, varying the points p_1, \dots, p_n , the $\mathcal{V}(p_1, \dots, p_n)$ cover \mathcal{U} . Since \mathcal{U} is quasi-compact, we find a finite subcovering. Replacing the $\mathcal{V}(p_1, \dots, p_n)$ by closed and open subspaces, which are ind-étale and τ -acyclic, we may assume that the finite union is disjoint. Hence \mathcal{U} is affinoid and τ -acyclic. \square

Using Proposition 13.1 and Theorem 13.2 instead of Proposition 7.12 and Theorem 7.10, the proof of Theorem 13.3 below is word-by-word the same as the proof of Theorem 7.16.

Theorem 13.3 (Comparison with Čech cohomology). *Let \mathcal{X} be a quasi-compact discretely ringed adic space with the property that any finite subset of \mathcal{X} is contained in an affinoid open. Let $\tau \in \{\text{et}, \text{set}, t\}$. If $\tau = t$ assume that \mathcal{X} is of pure characteristic $p \geq 0$.*

Then, for every presheaf P of abelian groups on \mathcal{X}_τ with sheafification $P^{\#\tau}$, the natural maps

$$\check{H}_\tau^n(\mathcal{X}, P) \rightarrow H_\tau^n(\mathcal{X}, P^{\#\tau})$$

are isomorphisms for all $n \geq 0$.

14. COMPARISON BETWEEN ALGEBRAIC AND ADIC TAME COHOMOLOGY

For X/S we consider the tame site $(X/S)_t$ and the adic tame site $\text{Spa}(X, S)_t$ and want to compare their cohomology. Since there is no obvious site morphism between the two, we consider an additional site $\text{Spa}(X, S)_{t, \text{Nis}}$ which admits morphisms

$$\text{Spa}(X, S)_t \xleftarrow{\psi} \text{Spa}(X, S)_{t, \text{Nis}} \xrightarrow{\phi} (X/S)_t$$

and we will prove that there are isomorphisms for all $q \geq 0$:

$$\begin{aligned} H_t^q(X/S, F) &\cong H_{t, \text{Nis}}^q(\text{Spa}(X, S), \phi^* F), \quad F \in \text{Sh}_t(X/S) \\ H_t^q(\text{Spa}(X, S), \psi_* G) &\cong H_{t, \text{Nis}}^q(\text{Spa}(X, S), G), \quad G \in \text{Sh}_{t, \text{Nis}}(\text{Spa}(X, S)). \end{aligned}$$

Definition 14.1. Let \mathcal{X} be a discretely ringed adic space. The *Nisnevich-tame site* $\mathcal{X}_{\text{Nis}, t}$ is defined by the following data:

The category $\text{Cat}(\mathcal{X}_{\text{Nis}, t})$ is the category of étale morphisms of adic spaces $\mathcal{U} \rightarrow \mathcal{X}$.

A family $(p_i : \mathcal{U}_i \rightarrow \mathcal{U})$ of morphisms in $\text{Cat}(\mathcal{X}_{\text{Nis}, t})$ is a covering if for every point $u \in \mathcal{U}$ there exists an index i and a point $u_i \in \mathcal{U}_i$ mapping to u such that p_i is tame at u_i .

The category $\text{Cat}(\mathcal{X}_t)$ is a full subcategory of $\text{Cat}(\mathcal{X}_{\text{Nis}, t})$ and obviously every covering in \mathcal{X}_t is also a covering in $\mathcal{X}_{\text{Nis}, t}$. This explains the existence of the morphism $\psi : \mathcal{X}_{\text{Nis}, t} \rightarrow \mathcal{X}_t$ (in particular for $\mathcal{X} = \text{Spa}(X, S)$). The morphism ϕ is defined by mapping an étale X -scheme U to $\text{Spa}(U, S)$, which lies étale over $\text{Spa}(X, S)$.

The comparison result for ψ will follow from the openness of the tame locus (Proposition 4.4).

Lemma 14.2. *Let \mathcal{X} be a discretely ringed adic space and \mathcal{U} an object in \mathcal{X}_t . The coverings of \mathcal{U} in \mathcal{X}_t are cofinal among the coverings in $\mathcal{X}_{t, \text{Nis}}$.*

Proof. Consider a covering $(\mathcal{U}_i \rightarrow \mathcal{U})$ in $\mathcal{X}_{t, \text{Nis}}$. For every i we denote by $\mathcal{V}_i \subseteq \mathcal{U}_i$ the subset of points where $\mathcal{U}_i \rightarrow \mathcal{U}$ is tame. This is an open subset by [Hüb18, Corollary 4.4]. Moreover, $(\mathcal{V}_i \rightarrow \mathcal{U})$ is a surjective family by the definition of coverings in $\mathcal{X}_{t, \text{Nis}}$. Hence, $(\mathcal{V}_i \rightarrow \mathcal{U})$ is a covering dominating $(\mathcal{U}_i \rightarrow \mathcal{U})$. \square

Lemma 14.3. *The morphism of sites*

$$\psi : \mathcal{X}_{t, \text{Nis}} \rightarrow \mathcal{X}_t$$

induces isomorphisms

$$H_t^q(\mathcal{X}, \psi_* G) \cong H_{t, \text{Nis}}^q(\mathcal{X}, G)$$

for all $q \geq 0$ and for every $G \in \text{Sh}_{t, \text{Nis}}(\mathcal{X})$.

Proof. We have to show that the higher direct images $R^q \psi_* G$ vanish for $q > 0$. Unravelling the definitions this comes down to showing that every covering $(\mathcal{V}_j \rightarrow \mathcal{U})$ in $\mathcal{X}_{t, \text{Nis}}$ of an object \mathcal{U} in \mathcal{X}_t is dominated by a covering in \mathcal{X}_t . This is the assertion of Lemma 14.2. \square

Lemma 14.4. *Let \mathcal{X} be a discretely ringed adic space and F a presheaf on $\mathcal{X}_{t, \text{Nis}}$. Then for any \mathcal{U} in \mathcal{X}_t the natural homomorphism*

$$\check{H}_t^q(\mathcal{U}, \psi_* F) \rightarrow \check{H}_{t, \text{Nis}}^q(\mathcal{U}, F)$$

is an isomorphism for all $q \geq 0$.

Proof. This is a direct consequence of Lemma 14.2. \square

Proposition 14.5. *Let \mathcal{X} be a quasi-compact discretely ringed adic space with the property that any finite subset of \mathcal{X} is contained in an affinoid open. Moreover assume that \mathcal{X} is of pure characteristic $p \geq 0$. Let F be a presheaf on $\mathcal{X}_{t, \text{Nis}}$ with sheafification $F^\#$. Then there are natural isomorphisms*

$$\check{H}_{t, \text{Nis}}^q(\mathcal{X}, F) \cong H_{t, \text{Nis}}^q(\mathcal{X}, F^\#)$$

for all $q \geq 0$.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \check{H}_t^q(\mathcal{X}, \psi_* F) & \longrightarrow & H_t^q(\mathcal{X}, (\psi_* F)^\#) \\ \downarrow & & \downarrow \\ & & H_t^q(\mathcal{X}, \psi_*(F^\#)) \\ \downarrow & & \downarrow \\ \check{H}_{t, \text{Nis}}^q(\mathcal{X}, F) & \longrightarrow & H_{t, \text{Nis}}^q(\mathcal{X}, F^\#). \end{array}$$

The upper horizontal map is an isomorphism by Theorem 13.3, the left vertical map by Lemma 14.4, and the lower right vertical map by Lemma 14.3. In order to see that the upper right vertical map is an isomorphism we note that sheafification is given by applying \check{H}^0 twice. Hence, $(\psi_* F)^\# \rightarrow \psi_*(F^\#)$ is an isomorphism by Lemma 14.4 for $q = 0$. \square

Lemma 14.6. *Let $X \rightarrow S$ be a morphism of schemes and $F \in \text{PrSh}_t(X/S)$ a presheaf. We consider the morphism of sites*

$$\phi : \text{Spa}(X, S)_{t, \text{Nis}} \rightarrow (X/S)_t$$

and let $G = \phi_{\text{pre}}^*(F) \in \text{PrSh}_{t, \text{Nis}}(\text{Spa}(X, S))$ be the presheaf pull-back of F . Then the natural homomorphism on Čech cohomology

$$\check{H}_t^n(X/S, F) \longrightarrow \check{H}_{t, \text{Nis}}^n(\text{Spa}(X, S), G)$$

is an isomorphism for all $n \geq 0$.

Proof. We consider all diagrams

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & S, \end{array}$$

with $U \rightarrow X$ étale and $T \rightarrow S$ of finite type, i.e., $Spa(U, T) \in Spa(X, S)_{t, \text{Nis}}$. By definition of ϕ_{pre}^* , we have $G(Spa(U, T)) = \text{colim}_V F(V)$, where the colimit is taken over all diagrams

$$\begin{array}{ccccc} U & \longrightarrow & V & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ T & \longrightarrow & S & \xlongequal{\quad} & S \end{array}$$

with $V \rightarrow X$ étale. Since U is a final object, we obtain $G(Spa(U, T)) = F(U)$.

For a tame covering $\mathcal{U} = (U_\alpha \rightarrow X)$ in $(X/S)_t$ we consider the pull-back covering $\phi^*\mathcal{U} = (Spa(U_\alpha, S) \rightarrow Spa(X, S))$ in $Spa(X, S)_{t, \text{Nis}}$. We have a natural isomorphism of Čech complexes $\check{C}^\bullet(\mathcal{U}, F) \cong \check{C}^\bullet(\phi^*\mathcal{U}, G)$. It therefore suffices to show that the colimit of the Čech cohomology groups of G with respect to all pull-back coverings coincides with the colimit with respect to all coverings of $Spa(X, S)$ in the (t, Nis) -topology. Among these, coverings of the form $\mathcal{V} = (Spa(U_\alpha, T_\alpha) \rightarrow Spa(X, S))$ with all $U_\alpha \rightarrow X$ étale and all $T_\alpha \rightarrow S$ of finite type are cofinal, so we can restrict to coverings of this type. The observation at the beginning of this proof shows that the Čech cohomology of G for \mathcal{V} coincides with the Čech cohomology of F for the covering $(U_\alpha \rightarrow X)$ and hence with the Čech cohomology of G for the pull-back covering $r(\mathcal{V}) := (Spa(U_\alpha, S) \rightarrow Spa(X, S))$, of which \mathcal{V} is a refinement. Hence our result follows from Lemma 14.7 below with J the subset of pull-back coverings and H the Čech cohomology for G . \square

Lemma 14.7. *Let I be a partially ordered directed set and $J \subset I$ a directed subset. Let $(H_i)_{i \in I}$ be a system of abelian groups indexed by I . Assume there is an order preserving retraction r to the inclusion $J \subset I$ such that $r(i) \leq i$ for all $i \in I$ and the natural map $H_{r(i)} \rightarrow H_i$ is an isomorphism for all $i \in I$. Then the natural map*

$$\text{colim}_J H_j \rightarrow \text{colim}_I H_i$$

is a isomorphism.

Proof. Surjectivity follows from the surjectivity of $H_{r(i)} \rightarrow H_i$ for all $i \in I$. Let $j \in J$ and $a \in H_j$ with trivial image in H_i for some $i \in I$, $i \geq j$. We have to find a $j' \in J$, $j' \geq j$, such that a has trivial image in $H_{j'}$. By our assumptions, we have $j = r(j) \leq r(i) \leq i$ and $H_{r(i)} \rightarrow H_i$ is injective. Hence $j' = r(i)$ has the required property. \square

Now we are ready to prove our comparison theorem.

Theorem 14.8. *Let S be an affine \mathbb{F}_p -scheme for some prime number p and let X be an S -scheme. Then for any sheaf $F \in Sh_t(X/S)$ the homomorphism*

$$H_t^n(X/S, F) \longrightarrow H_{t, \text{Nis}}^n(Spa(X, S), \phi^*F)$$

is an isomorphism for all $n \geq 0$.

Proof. Let $X = \bigcup U_i$ be an affine open covering. Then $Spa(X, S) = \bigcup Spa(U_i, S)$ is also an open covering. By the Čech-to-derived spectral sequences on both sides,

we may therefore assume that X itself is affine. Let $G = \phi_{pre}^* F$, i.e., $\phi^* F = G^\#$. Then the statement follows from the commutative diagram

$$\begin{array}{ccc} \check{H}_t^n(X/S, F) & \xrightarrow{a} & \check{H}_{t, \text{Nis}}^n(\text{Spa}(X, S), G) \\ \downarrow b & & \downarrow c \\ H_t^n(X/S, F) & \longrightarrow & H_{t, \text{Nis}}^n(\text{Spa}(X, S), \phi^* F), \end{array}$$

in which a , b and c are isomorphisms by Lemma 14.6, Theorem 7.16 and Theorem 13.3, respectively. \square

15. PURITY AND HOMOTOPY INVARIANCE

With the help of our comparison results between algebraic and adic tame cohomology, we can now exploit the results of [Hüb18].

We say that *resolution of singularities holds over S* if

- (1) for any reduced scheme X of finite type over S there is a proper birational morphism $X' \rightarrow X$ such that:
 - X' is regular,
 - $X' \rightarrow X$ is an isomorphism over the regular locus of X , and
 - $X' \rightarrow X$ factors into a chain of blow-ups in regular centers, and
- (2) every projective birational morphism $X' \rightarrow X$ of regular S -schemes factors into a chain of blow-ups in regular centers.

A sheaf $F \in \text{Sh}_t(X/S)$ is *locally constant* if there is a tame covering $(U_i \rightarrow X)_i$ such that the restriction $F|_{U_i}$ is isomorphic to a constant sheaf \underline{A}_i on U_i for all i .

Theorem 15.1 (Purity). *Let S be an affine quasi-excellent scheme of characteristic $p > 0$ and X a regular scheme which is separated and essentially of finite type over S . Assume that resolution of singularities holds over S . Then for any open dense subscheme $U \subset X$ and every locally constant p -torsion sheaf $F \in \text{Sh}_t(X/S)$ the natural map*

$$H_t^q(X/S, F) \longrightarrow H_t^q(U/S, F|_U)$$

is an isomorphism for all $q \geq 0$.

Proof. Let $(U_i \rightarrow X)_i$ be a trivializing tame covering for F . Then we have compatible Čech-to-derived spectral sequences

$$E_2^{rs} = \check{H}^r((U_i \rightarrow X)_i, \mathcal{H}_t^s(-/S, F)) \Rightarrow H_t^{r+s}(X/S, F), \text{ and}$$

$$E_2^{rs} = \check{H}^r((U_i \times_X U \rightarrow U)_i, \mathcal{H}_t^s(-/S, F|_U)) \Rightarrow H_t^{r+s}(U/S, F|_U).$$

We therefore may assume that F is constant. By Theorem 4.5, we may assume that $F = \underline{A}$ for a finite abelian group A . For this it suffices to consider the cyclic case $A = \mathbb{Z}/p^n\mathbb{Z}$ and by dévissage, we may assume $n = 1$. Then the statement follows from its adic version [Hüb18, Corollary 14.5], and the comparison results Theorem 14.8 and Lemma 14.3. \square

Remark 15.2. Being an étale sheaf over a tame covering, a locally constant tame sheaf is already an étale sheaf. In view of the comparison result Proposition 8.1, purity for locally constant prime-to- p torsion sheaves is identical to the known purity result for étale cohomology.

Theorem 15.3 (Homotopy invariance). *Let S be an affine quasi-excellent scheme of characteristic $p > 0$ and X a regular scheme which is essentially of finite type over S . Assume that resolution of singularities holds over S . Then for every locally constant torsion sheaf $F \in \text{Sh}_t(X/S)$ the natural map*

$$H_t^q(X/S, F) \longrightarrow H_t^q(\mathbb{A}_X^1/S, pr^* F),$$

where $pr : \mathbb{A}_X^1 \rightarrow X$ is the natural projection, is an isomorphism for all $q \geq 0$.

Proof. As in the proof of 15.1, we may reduce to the case where $F = \underline{\mathbb{Z}/q\mathbb{Z}}$ for a prime number q . If $q \neq p$, then the result follows from Proposition 8.1 and the homotopy invariance of étale cohomology with invertible coefficients [Mil80, VI, Corollary 4.20]. For $q = p$ the result follows from its adic version [Hüb18, Corollary 14.6], and the comparison results Theorem 14.8 and Lemma 14.3. \square

16. CONNECTION TO SUSLIN HOMOLOGY

Let k be an algebraically closed field, X/k a connected scheme of finite type and $m \geq 1$ an integer. In [SV96], A. Suslin and V. Voevodsky defined the Suslin (or singular) homology groups $H_n^S(X, \mathbb{Z}/m\mathbb{Z})$ and constructed for $(m, \text{char}(k)) = 1$ an isomorphism between étale and Suslin cohomology

$$H_{\text{et}}^n(X, \mathbb{Z}/m\mathbb{Z}) \cong H_S^n(X, \mathbb{Z}/m\mathbb{Z}) = \text{Hom}(H_n^S(X, \mathbb{Z}/m\mathbb{Z}), \mathbb{Z}/m\mathbb{Z}).$$

Originally, [SV96] had to assume resolution of singularities but later this could be avoided by using deJong's alterations.

In [GS16], T. Geisser and the second author extended this result in degree 1 to general m not necessarily prime to $\text{char}(k)$ by replacing the group $H_{\text{et}}^1(X, \mathbb{Z}/m\mathbb{Z})$ by the tame cohomology group $H_t^1(X, \mathbb{Z}/m\mathbb{Z})$, which was defined in an ad hoc manner as the dual of the curve-tame fundamental group (cf. [KS10]), i.e.,

$$H_t^1(X, \mathbb{Z}/m\mathbb{Z}) := \text{Hom}_{\text{cts}}(\pi_1^{\text{ct}}(X), \mathbb{Z}/m\mathbb{Z}).$$

Here the assumption on resolution of singularities could not be avoided. A generalization of this result to higher degrees was not possible since a definition of (higher) tame cohomology groups was missing.

Having tame cohomology at hand now, we construct for all $n \geq 0$ a natural pairing

$$H_n^S(X, \mathbb{Z}/m\mathbb{Z}) \times H_t^n(X/k, \mathbb{Z}/m\mathbb{Z}) \longrightarrow \mathbb{Z}/m\mathbb{Z},$$

which defines maps

$$H_t^n(X/k, \mathbb{Z}/m\mathbb{Z}) \longrightarrow H_S^n(X, \mathbb{Z}/m\mathbb{Z})$$

in all degrees n . The construction is the following.

Let Sch/k be the category of separated schemes of finite type over k and Cor/k the category with the same objects as Sch/k and finite correspondences as morphisms. Recall that the group $\text{Cor}_k(X, Y)$ of *finite correspondences* from X to Y is the group of relative cycles in $X \times_k Y/X$ which are finite, equidimensional and universally integral, see [MVW06, A.1] or [CD19, §9]. The graph functor

$$\text{Sch}/k \longrightarrow \text{Cor}/k, \quad f \in \text{Mor}_k(X, Y) \longmapsto \Gamma_f \in \text{Cor}_k(X, Y)$$

is a faithful embedding. A *presheaf with transfers* on Sch/k is a contravariant additive functor $F : \text{Cor}/k \rightarrow \mathcal{A}b$. The category of all presheaves with transfers is denoted by $\text{PrSh}(\text{Cor}/k)$. Composition with the graph functor yields a faithful embedding

$$\text{PrSh}(\text{Cor}/k) \longrightarrow \text{PrSh}(\text{Sch}/k)$$

from the category of presheaves with transfers to the category of presheaves on Sch/k . We will consider presheaves with transfers as presheaves with the additional structure of transfer maps for finite correspondences.

If X/k is smooth and connected, then $\text{Cor}_k(X, Y)$ is the free abelian group generated by integral subschemes of $X \times_k Y$ whose projection to X is finite and surjective, see [SV00, Proposition 3.3.5]. For any $Y \in \text{Sch}/k$ and any abelian group A the *Suslin*

homology $H_{\bullet}^S(Y, A)$ of Y with values in A (defined in [SV96]) is the homotopy of the simplicial abelian group

$$\cdots \rightrightarrows \text{Cor}_k(\Delta_k^2, Y) \otimes A \rightrightarrows \text{Cor}_k(\Delta_k^1, Y) \otimes A \longrightarrow \text{Cor}_k(\Delta_k^0, Y) \otimes A,$$

i.e., the homology of the complex

$$\cdots \text{Cor}_k(\Delta_k^2, Y) \otimes A \rightarrow \text{Cor}_k(\Delta_k^1, Y) \otimes A \rightarrow \text{Cor}_k(\Delta_k^0, Y) \otimes A \rightarrow 0,$$

where the differentials are induced by the alternating sums of the face maps. Likewise the *Suslin cohomology* $H_S^{\bullet}(Y, A)$ of Y with values in an abelian group A is the cohomology of the complex

$$\text{Hom}_{\mathbb{Z}}(\text{Cor}_k(\Delta_k^{\bullet}, Y), A).$$

We consider the *big tame site* $(\text{Sch}/k)_t$ which consists of the category Sch/k with tame coverings. For $X \in \text{Sch}/k$ and $F \in \text{Sh}_t(\text{Sch}/k)$, the cohomology of X with values in F coincides with the cohomology of the restriction of F to the (small) tame site $(X/k)_t$ (cf. [Mil80, III, 3.1]). A *tame sheaf with transfers* is a presheaf with transfers which is a tame sheaf on Sch/k when considered as a presheaf on Sch/k via the graph functor. We denote the category of these by $\text{Sh}_t(\text{Cor}/k)$. For an abelian group A , the constant Zariski-sheaf \underline{A} on Sch/k is already an étale sheaf, in particular a tame sheaf. For $X \in \text{Sch}/k$ we have

$$\underline{A}(X) = \text{Cor}_k(X, \text{Spec } k) \otimes A,$$

in particular, \underline{A} has transfers in a natural way. Similar arguments as for the Nisnevich and étale cohomology show the following

Lemma 16.1. *The category $\text{Sh}_t(\text{Cor}/k)$ has sufficiently many injectives. An injective object $I \in \text{Sh}_t(\text{Cor}/k)$ is flabby as a sheaf in $\text{Sh}_t(\text{Sch}/k)$. In particular, we can calculate the tame cohomology of a tame sheaf with transfers by using an injective resolution in $\text{Sh}_t(\text{Cor}/k)$. For any $F \in \text{Sh}_t(\text{Cor}/k)$ the tame cohomology presheaves*

$$U \longmapsto H_t^n(U, F)$$

are presheaves with transfers in a natural way.

Proof. As the tame site sits between the étale and the Nisnevich site, the arguments of [Voe00, §3.1] (given there for sheaves with transfers on the category of smooth k -schemes) apply without change. See [CD19, §10.3] for more details and a treatment in the setting of Sch/k . \square

We make the following general observation. Let $(M^{\bullet\bullet}, d', d'')$ be a commutative (or anti-commutative) double complex of abelian groups. For $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ we put

$$Z^{ij} = \ker(d' : M^{ij} \rightarrow M^{i+1, j}) \cap \ker(d'' : M^{ij} \rightarrow M^{i, j+1})$$

$$B^{ij} = \text{im}(d' \circ d'' : M^{i-1, j-1} \rightarrow M^{ij}) \quad \text{and} \quad H^{ij} = Z^{ij}/B^{ij}.$$

If all lines (i.e., the single complexes $M^{\bullet k}$ for all k) are exact, we obtain for all (i, j) a natural homomorphism

$$h_{i, j} : H^{i, j} \longrightarrow H^{i-1, j+1}$$

as follows: Let $x \in H^{i, j}$ and let $m \in Z^{i, j}$ be a representing element. We choose $m' \in M^{i-1, j}$ with $d'(m') = m$ and then we define $h_{i, j}(x) \in H^{i-1, j+1}$ as the class of $d''(m')$. Our assumptions guarantee that $h_{i, j}$ is well-defined.

Fix an injective resolution $\mathbb{Z}/m\mathbb{Z} \rightarrow I^\bullet$ in the category of tame sheaves of $\mathbb{Z}/m\mathbb{Z}$ -modules with transfers on Sch/k . The finite correspondences

$$\partial = \sum_{i=0}^n (-1)^i \partial_i : \Delta^{n-1} \rightarrow \Delta^n$$

induce maps $\partial^* : I^\bullet(\Delta^n) \rightarrow I^\bullet(\Delta^{n-1})$. Consider the double complex given by

$$M^{ij} = \begin{cases} I^i(\Delta^{-j}), & i \geq 0, j \leq 0 \\ \mathbb{Z}/m\mathbb{Z}, & i = -1, j \leq 0 \\ 0, & \text{else} \end{cases}$$

with differentials as indicated below

$$\begin{array}{ccccccc} \mathbb{Z}/m\mathbb{Z} & \longrightarrow & I^0(\Delta^0) & \xrightarrow{d} & I^1(\Delta^0) & \xrightarrow{d} & I^2(\Delta^0) \\ 0 \uparrow & & \partial^* \uparrow & & \partial^* \uparrow & & \partial^* \uparrow \\ \mathbb{Z}/m\mathbb{Z} & \longrightarrow & I^0(\Delta^1) & \xrightarrow{d} & I^1(\Delta^1) & \xrightarrow{d} & I^2(\Delta^1) \\ \text{id} \uparrow & & \partial^* \uparrow & & \partial^* \uparrow & & \partial^* \uparrow \\ \mathbb{Z}/m\mathbb{Z} & \longrightarrow & I^0(\Delta^2) & \xrightarrow{d} & I^1(\Delta^2) & \xrightarrow{d} & I^2(\Delta^2) \\ 0 \uparrow & & \partial^* \uparrow & & \partial^* \uparrow & & \partial^* \uparrow \\ \mathbb{Z}/m\mathbb{Z} & \longrightarrow & I^0(\Delta^3) & \xrightarrow{d} & I^1(\Delta^3) & \xrightarrow{d} & I^2(\Delta^3). \end{array} \quad (*)$$

Since $\Delta^n \cong \mathbb{A}_k^n$ and by homotopy invariance (Theorem 15.3), we have

$$H_t^i(\Delta^n, \mathbb{Z}/m\mathbb{Z}) \cong H_t^i(\text{Spec } k, \mathbb{Z}/m\mathbb{Z}) = \begin{cases} \mathbb{Z}/m\mathbb{Z}, & i = 0 \\ 0, & i \geq 1. \end{cases}$$

Hence the lines of (*) are exact and we obtain a map

$$h_n : H^{n,-n} \rightarrow H^{n-1,-n+1} \rightarrow \dots \rightarrow H^{0,0} = \mathbb{Z}/m\mathbb{Z}.$$

Returning to our original task, let $\alpha \in H_n^S(X, \mathbb{Z}/m\mathbb{Z})$ and $\beta \in H_t^n(X, \mathbb{Z}/m\mathbb{Z})$ be given. Let $a \in \text{Cor}(\Delta^n, X)$ represent α and let $b \in I^n(X)$ represent β . By definition we have $db = 0$ and $a \circ \partial = m\chi$ for some $\chi \in \text{Cor}(\Delta^{n+1}, X)$. We consider the pull back

$$a^*(b) \in I^n(\Delta^n) = M^{n,-n}.$$

We have

$$\begin{aligned} d(a^*(b)) &= a^*(db) = a^*(0) = 0, \text{ and} \\ \partial^*(a^*(b)) &= (a \circ \partial)^*(b) = (m\chi)^*(b) = \chi^*(mb) = \chi^*(0) = 0. \end{aligned}$$

Hence $a^*(b) \in Z^{n,-n}$, and we denote its class in $H^{n,-n}$ by $\overline{a^*(b)}$

Definition 16.2. We define

$$\langle \alpha, \beta \rangle = h_n(\overline{a^*(b)}) \in \mathbb{Z}/m\mathbb{Z}.$$

Lemma 16.3. $\langle \alpha, \beta \rangle$ is independent of the choices made, hence we obtain a bilinear pairing

$$H_n^S(X, \mathbb{Z}/m\mathbb{Z}) \times H_t^n(X, \mathbb{Z}/m\mathbb{Z}) \longrightarrow \mathbb{Z}/m\mathbb{Z}.$$

Proof. Let $b' \in I^n(X)$ be another representative of β . If $n = 0$, then $b = b'$ and there is nothing to prove. For $n \geq 1$, $b - b' = dc$, $c \in I^{n-1}(X)$, and we have to show that

$$h_n(\overline{a^*(dc)}) = 0 \text{ for any } c \in I^{n-1}(X).$$

By definition of the map $h_{n,-n} : H^{n,-n} \rightarrow H^{n-1,-n+1}$ above, we have

$$h_{n,-n}(\overline{a^*(dc)}) = h_{n,-n}(\overline{d(a^*(c))}) = \overline{\partial^*(a^*(c))} = \overline{(a \circ \partial)^*(c)} = \overline{(m\chi)^*(c)} = \overline{\chi^*(mc)} = \overline{\chi^*(0)} = 0.$$

Since h_n factors through $h_{n,-n}$, this shows the independence from the choice of b . If $a' \in \text{Cor}(\Delta^n, X)$ is another representative of α , then there exist $x \in \text{Cor}(\Delta^{n+1}, X)$ and $y \in \text{Cor}(\Delta^n, X)$ with

$$a' - a = x \circ \partial + my.$$

It therefore remains to show that, for any $x \in \text{Cor}(\Delta^{n+1}, X)$, the map $(x \circ \partial)^* : H_t^n(X, \mathbb{Z}/m\mathbb{Z}) \rightarrow H^{n,-n}$ is zero. We have

$$(x \circ \partial)^*(b) = \partial^*(x^*(b)).$$

For $n = 0$, $x^*(b)$ lies in $\ker(I^0(\Delta^1) \rightarrow I^1(\Delta^1)) = H_t^0(\Delta^1, \mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}$ and

$$\partial^* : H_t^0(\Delta^1, \mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z} \longrightarrow H^{0,0} = \mathbb{Z}/m\mathbb{Z}$$

is the zero map because $\partial^* = \partial_0^* - \partial_1^* = \text{id}_{\mathbb{Z}/m\mathbb{Z}} - \text{id}_{\mathbb{Z}/m\mathbb{Z}} = 0$.

For $n \geq 1$, we have $H_t^n(\Delta^{n+1}, \mathbb{Z}/m\mathbb{Z}) = 0$, hence $x^*(b) = dc$ for some $c \in I^{n-1}(\Delta^{n+1})$ and we obtain

$$\partial^*(x^*(b)) = \partial^*(dc) \in B^{n,-n}$$

showing the result. \square

Using the pairing and noting that $\mathbb{Z}/m\mathbb{Z}$ is an injective $\mathbb{Z}/m\mathbb{Z}$ -module, we obtain homomorphisms for all $n \geq 0$:

$$\beta_n : H_t^n(X, \mathbb{Z}/m\mathbb{Z}) \longrightarrow \text{Hom}(H_n^S(X, \mathbb{Z}/m\mathbb{Z}), \mathbb{Z}/m\mathbb{Z}) = H_S^n(X, \mathbb{Z}/m\mathbb{Z}).$$

Conjecture 16.4. *For any $m \geq 1$, the homomorphism*

$$\beta_n : H_t^n(X, \mathbb{Z}/m\mathbb{Z}) \longrightarrow H_S^n(X, \mathbb{Z}/m\mathbb{Z})$$

is an isomorphism of finite abelian groups for all $n \geq 0$.

Up to comparing the maps, Conjecture 16.4 is known if $(m, p) = 1$ and all n by [SV96] and for general m and $n = 1$ by [GS16]. We will address Conjecture 16.4 in a forthcoming paper.

REFERENCES

- [Ach17] Piotr Achinger. Wild ramification and $K(\pi, 1)$ spaces. *Invent. Math.*, 210(2):453–499, 2017.
- [AGV72] Michael Artin, Alexander Grothendieck, and Jean-Louis Verdier. *Théorie de topos et cohomologie étale des schémas (SGA 4)*. Séminaire de géométrie algébrique du Bois-Marie - 1963-64. Springer-Verlag Berlin; New York, 1972.
- [AM69] Michael Artin and Barry Mazur. *Étale homotopy*. Springer-Verlag, Berlin, 1969.
- [Art62] Michael Artin. *Grothendieck Topologies*. Harvard University, 1962.
- [Art71] Michael Artin. On the joins of Hensel rings. *Advances in Mathematics*, 7:282–296, 1971.
- [BS15] Bhargav Bhatt and Peter Scholze. The pro-étale topology for schemes. *Astérisque*, 369:99–201, 2015.
- [CD19] Denis-Charles Cisinski and Frédéric Déglise. *Triangulated categories of mixed motives*. Springer Monographs in Mathematics. Springer, Cham, 2019.
- [Con07] Brian Conrad. Deligne’s notes on Nagata compactifications. *Journal of the Ramanujan Mathematical Society*, 22(3):205–257, 2007.
- [DG70] Michel Demazure and Alexander Grothendieck. *Schémas en groupes (SGA 3)*. Séminaire de géométrie algébrique du Bois Marie - 1962-64. Springer-Verlag Berlin; New York, 1970.
- [EP05] Antonio Engler and Alexander Prestel. *Valued fields*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2005.
- [GM71] Alexander Grothendieck and Jacob P. Murre. *The tame fundamental group of a formal neighbourhood of a divisor with normal crossings on a scheme*. Lecture Notes in Mathematics, Vol. 208. Springer-Verlag, Berlin-New York, 1971.

- [Gro67] Alexander Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. *Inst. Hautes Études Sci. Publ. Math.*, 32:361, 1967.
- [GS16] Thomas Geisser and Alexander Schmidt. Tame class field theory for singular varieties over algebraically closed fields. *Doc. Math.*, 21:91–123, 2016.
- [HK94] Roland Huber and Manfred Knebusch. On valuation spectra. *Contemporary Mathematics*, 155:167–206, 1994.
- [Hoc69] Melvin Hochster. Prime ideal structure in commutative rings. *Transactions of the American Mathematical Society*, 142:43–60, 1969.
- [Hub93] Roland Huber. Continuous valuations. *Mathematische Zeitschrift*, 212:455–477, 1993.
- [Hub96] Roland Huber. *Étale cohomology of rigid analytic varieties and adic spaces*. Aspects of Mathematics. Vieweg, 1996.
- [Hüb18] Katharina Hübner. The adic tame site. *arXiv:1801.04776 [math.AG]*, 2018.
- [Hüb20] Katharina Hübner. Tame and strongly étale cohomology of curves. *arXiv:1911.05595 [math.AG]*, 2020.
- [KS09] Moritz Kerz and Alexander Schmidt. Covering data and higher dimensional global class field theory. *J. Number Theory*, 129(10):2569–2599, 2009.
- [KS10] Moritz Kerz and Alexander Schmidt. On different notions of tameness in arithmetic geometry. *Mathematische Annalen*, 346:641–668, 2010.
- [Maz73] Barry Mazur. Notes on étale cohomology of number fields. *Ann. Sci. École Norm. Sup. (4)*, 6:521–552 (1974), 1973.
- [Mil80] James S. Milne. *Étale cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, New Jersey, 1980.
- [MVW06] Carlo Mazza, Vladimir Voevodsky, and Charles Weibel. *Lecture notes on motivic cohomology*, volume 2 of *Clay Mathematics Monographs*. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2006.
- [NSW08] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of number fields*, volume 323 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, second edition, 2008.
- [Sch07] Alexander Schmidt. Singular homology of arithmetic schemes. *Algebra Number Theory*, 1(2):183–222, 2007.
- [Sch20] Alexander Schmidt. On quasi-purity of the branch locus. *Manuscripta Math.*, 161(3-4):325–331, 2020.
- [Sta19] The Stacks Project Authors. The stacks project. <http://stacks.math.columbia.edu>, 2019.
- [SV96] Andrei Suslin and Vladimir Voevodsky. Singular homology of abstract algebraic varieties. *Invent. Math.*, 123(1):61–94, 1996.
- [SV00] Andrei Suslin and Vladimir Voevodsky. Relative cycles and Chow sheaves. In *Cycles, transfers, and motivic homology theories*, volume 143 of *Ann. of Math. Stud.*, pages 10–86. Princeton Univ. Press, Princeton, NJ, 2000.
- [Tem11] Michael Temkin. Relative Riemann-Zariski spaces. *Israel Journal of Mathematics*, pages 1–42, 2011.
- [TT18] Michael Temkin and Ilya Tyomkin. On relative birational geometry and Nagata’s compactification for algebraic spaces. *International Mathematics Research Notices. IMRN*, 11:3342–3387, 2018.
- [Voe00] Vladimir Voevodsky. Triangulated categories of motives over a field. In *Cycles, transfers, and motivic homology theories*, volume 143 of *Ann. of Math. Stud.*, pages 188–238. Princeton Univ. Press, Princeton, NJ, 2000.

THE HEBREW UNIVERSITY OF JERUSALEM, EINSTEIN INSTITUTE OF MATHEMATICS,
GIV’AT RAM, JERUSALEM, 91904, ISRAEL

E-mail address: katharin.hubner@mail.huji.ac.il

UNIVERSITÄT HEIDELBERG, MATHEMATISCHES INSTITUT, IM NEUENHEIMER FELD
205, D-69120 HEIDELBERG, DEUTSCHLAND

E-mail address: schmidt@mathi.uni-heidelberg.de