Profinite completions of 3-manifold groups

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October 22, 2020

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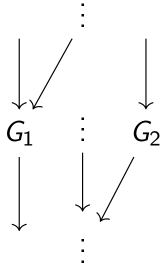

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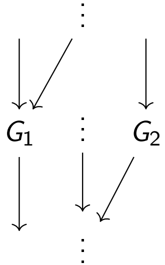

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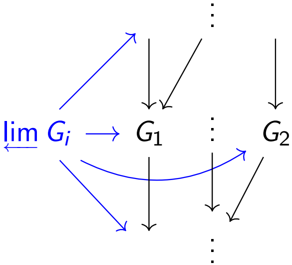

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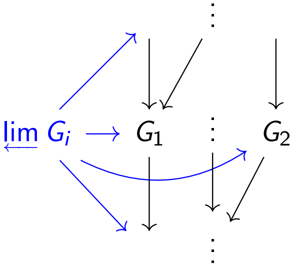

Definition

A topological group obtained as the limit $\varprojlim G_i$ *over such a diagram is called a* **profinite group**.

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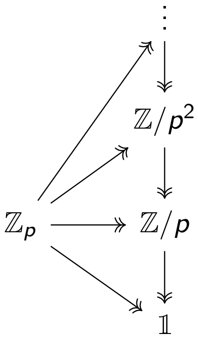

The limit $\mathbb{Z}_p$ over this tower is the group of p-**adic integers**.

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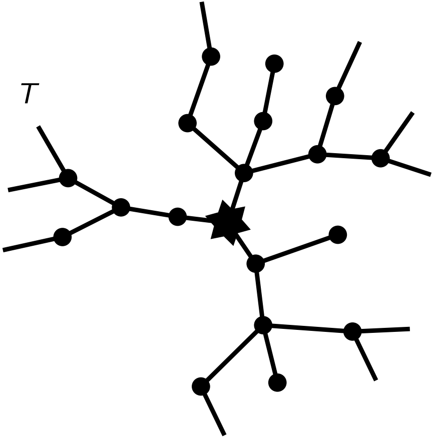
Consider a locally compact rooted tree T.



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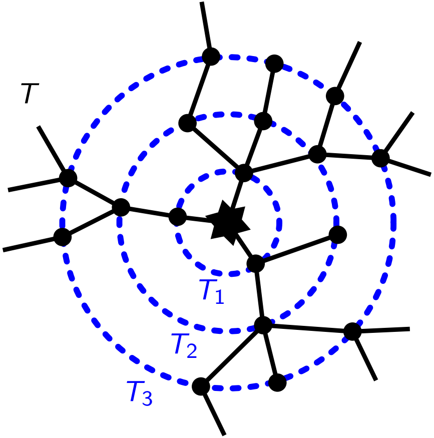

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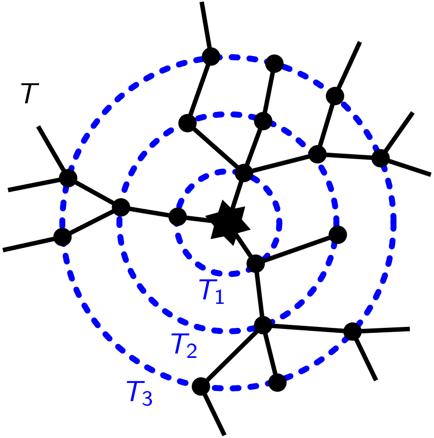

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The (finite) automorphism groups of the T_n form an inverse system:

$$\begin{array}{c} \vdots \\ \downarrow \\ \mathrm{Aut}_*(T_2) \\ \downarrow \\ \mathrm{Aut}_*(T_1) \\ \downarrow \\ \mathrm{Aut}_*(T_0) \cong \mathbb{1} \end{array}$$

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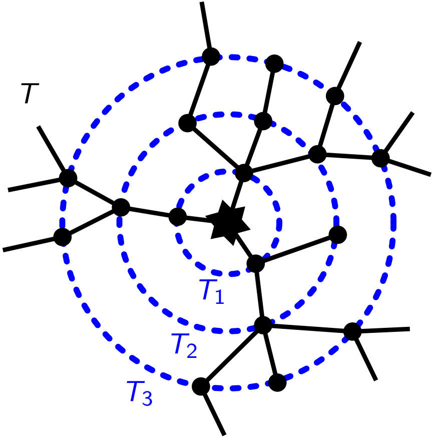

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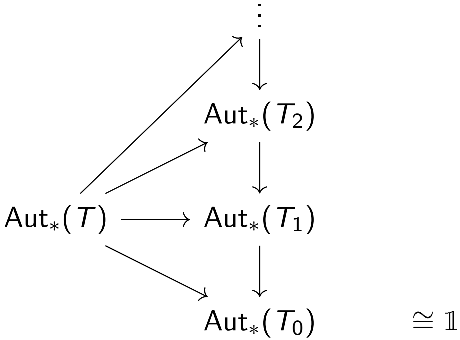

$\mathrm{Aut}_*(T)$ is its limit.

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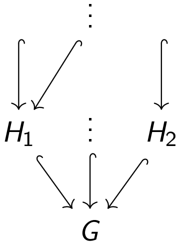

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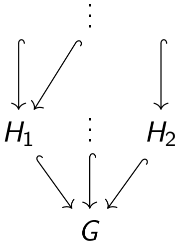

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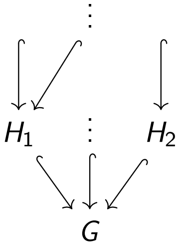

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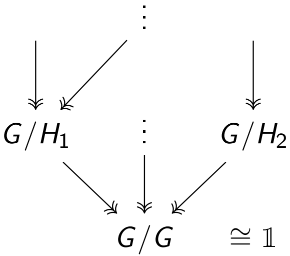
We obtain the inverse system of finite quotients of G:



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Theorem (Dixon, Formanek, Poland, Ribes, 1982)

Let G, H be finitely generated groups. Then G and H have the same set of isomorphism classes of finite quotients if and only if $\widehat{G} \cong \widehat{H}$.

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(Not true for 4-manifolds!)

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Having determined r, the maximal p-subgrop of G^{ab} is the maximal abelian p-group H such that for all $n \in \mathbb{N}$, G has a quotient isomorphic to $H \oplus (\mathbb{Z}/n)^r$. □

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(The analogous statement for 4-manifolds is false!)

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Let M, N be closed, orientable 3-manifolds with prime decompositions

$$M = M_1 \# \ldots \# M_m \# r(\mathbb{S}^1 \times \mathbb{S}^2), \qquad N = N_1 \# \ldots \# N_n \# s(\mathbb{S}^1 \times \mathbb{S}^2).$$

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- $\widehat{\pi_1(E_K)}$ also determines the Alexander polynomial of K (Ueki, 2018).

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- Is there an infinite family of pairwise non-homeomorphic 3-manifolds with isomorphic $\widehat{\pi_1}$?
- Are knot complements profinitely rigid among knot complements? This would imply that a prime knot K is determined by $\widehat{\pi_1(E_K)}$ (Whiten, 1987).