An overview of Σ-invariants

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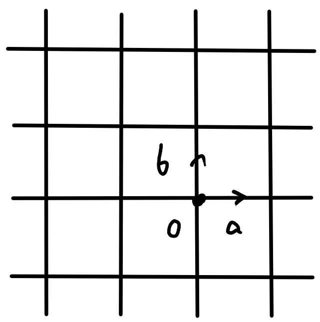
The invariant Σ^1

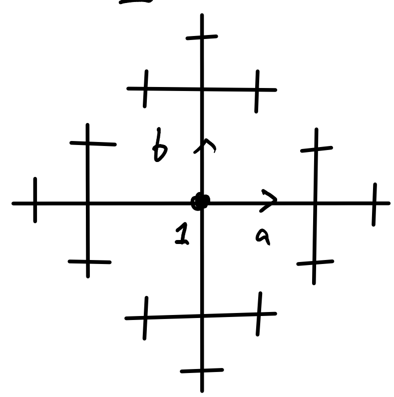
(a.k.a. BNSR - invariants, for Bieri - Neumann - Strebel - Renz)

Let G be a group.

G is finitely generated $\Leftrightarrow$ There is a finite subset $S \subseteq G$ s.th. $\mathrm{Cay}(G, S) =: \Gamma$ is connected.

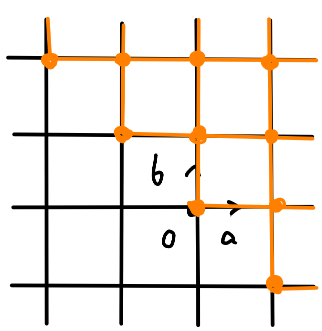
$\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle$ $\qquad$ $F_2 = \langle a, b \rangle$

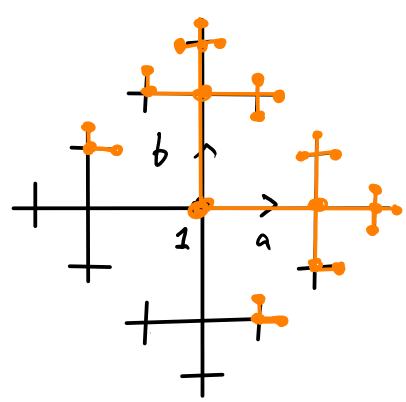




Let $\chi : G \longrightarrow \mathbb{R}$ be a homomorphism, (a "character")
for example $\chi : a \longmapsto 1$, $b \longmapsto 1$

Consider the sub-graph generated by the vertices $g \in G$ with $\chi(g) \geq 0$:





Q. For what characters $\chi: G \to \mathbb{R}$ is this sub-graph Γ_χ connected?

$G = \mathbb{Z}^2$: Γ_χ connected for all χ

$G = F_2$: Γ_χ connected for $\chi = 0$, otherwise disconnected.

Fact/exercise: The question of whether Γ_χ is connected does not depend on the choice of finite generating set.

Def. $\Sigma^1(G)$ is the set of characters $\chi: G \to \mathbb{R}$ given by:

$\chi \in \Sigma^1(G) \Leftrightarrow$ there is a finite subset $S \subseteq G$ such that $\mathrm{Cay}(G,S)_\chi$ is connected.

So: $\Sigma^1(\mathbb{Z}^2) = \mathrm{Hom}(\mathbb{Z}^2, \mathbb{R})$

$\Sigma^1(F_2) = \{0\}$

- Slogan: $\chi \in \Sigma^1(G)$ means G is "finitely generated in the direction of χ".
- $0 \in \Sigma^1(G) \Leftrightarrow G$ is finitely generated.
- $\Sigma^1(G) \subseteq \mathrm{Hom}(G, \mathbb{R})$ is invariant under re-scaling by positive reals.

An asymmetric example:

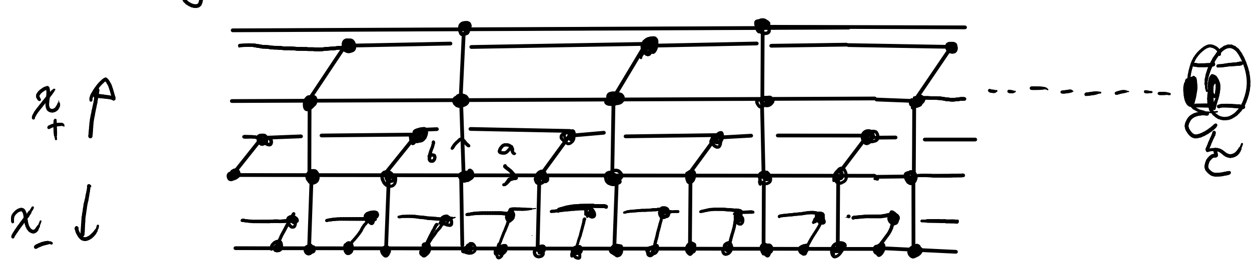
$$G = BS(1,2) := \langle a, b \mid bab^{-1} = a^2 \rangle$$

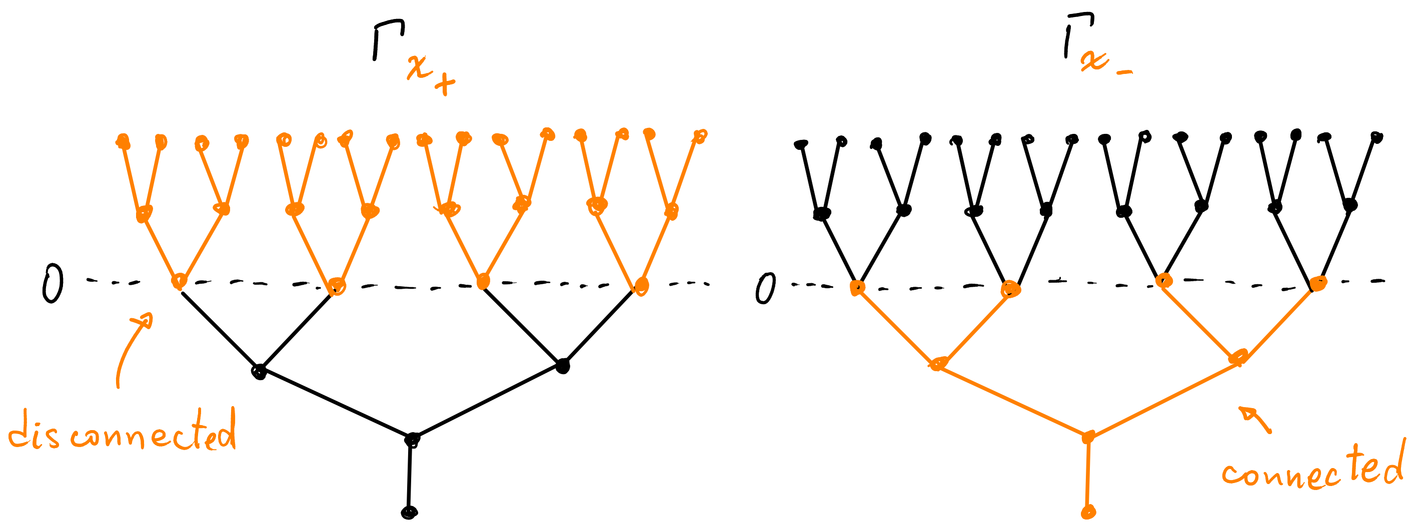
Baumslag-Solitar

- G is finitely generated, so $0 \in \Sigma^1(G)$
- Every $\chi : G \to \mathbb{R}$ sends $a \mapsto 0$
- Since $\Sigma^1(G)$ is invariant under positive re-scaling, need only consider

$$\chi_+ : b \mapsto 1$$
$$\chi_- : b \mapsto -1$$

$\Gamma := Cay(G, \{a, b\})$:





$\chi_+ \notin \Sigma^1(G)$ $\qquad$ $\chi_- \in \Sigma^1(G)$

Motivation: When are finiteness properties of G inherited by subgroups?

- Thm [BNS, 1987]. Let G be fin. gen, $\chi: G \twoheadrightarrow \mathbb{Z} \subset \mathbb{R}$ with $\chi(t) = 1$. Then:
(closely related result)

$\chi \in \Sigma^1(G) \iff$ G is an $\boxed{\text{ascending}}$ HNN-extension $B*_t$ whose base $B \subseteq \ker(\chi)$ and associated subgroups S, $tSt^{-1} \subseteq B$ are finitely generated.

$S \subseteq tSt^{-1}$

- Thm [BNS, 1987]. Let G be finitely generated, $N \trianglelefteq G$ with G/N abelian.
Then N is finitely generated iff every character $\chi: G \longrightarrow \mathbb{R}$ with $\chi(N) = 0$ is in $\Sigma^1(G)$.

- Cor. Let M be a compact oriented connected 3-mfd, let $\phi: G := \pi_1(M) \twoheadrightarrow \mathbb{Z}$.
Then ϕ is fibered iff $\phi, -\phi \in \Sigma^1(G)$

fibered: induced by a fiber bundle map $M \downarrow S^1$

Pf. ϕ fibered

$\Updownarrow$ Stallings's Theorem

$N := \ker(\phi)$ is fin. gen

$\Updownarrow$ BNS

All maps $\chi: G \to \mathbb{R}$ with $\chi(N) = 0$ are in $\Sigma^1(G)$

maps induced from $G/N \cong \mathbb{Z} \to \mathbb{R}$, i.e. scalar multiples of ϕ.

Σ^1 is invariant under positive re-scaling

$\Updownarrow$

$\phi, -\phi, 0 \in \Sigma^1(G)$

Automatically in $\Sigma^1(G)$ because G is finitely generated. $\square$

[Strebel, Notes on the Sigma invariants, arXiv: 1204.0214] gives a comprehensive account on Σ^1.

The higher Sigma invariants

Recall: A group G is of type F_n $(n \geq 1)$ if there exists a $K(G,1)$ K with finite n-skeleton.

This means:
- K is a connected CW-complex
- $\pi_1(K) \cong G$
- $\pi_i(K) = 0$ for $i \geq 2$

- $F_1 \Leftrightarrow$ finitely generated
- $F_2 \Leftrightarrow$ finitely presented

[Renz, 1988] extends the properties $F_1 \Leftarrow F_2 \Leftarrow F_3 \ldots$ to invariants $\Sigma^1 \supseteq \Sigma^2 \supseteq \Sigma^3 \ldots$

Here is how:

G is of type F_n $\Leftrightarrow$ G acts on a CW-cpx C s.th.
- $G \curvearrowright C$ is free
- C is $\boxed{(n-1)\text{-connected}}$
- The n-skeleton is $\boxed{G\text{-finite}}$

$\pi_i(C)$ trivial for $0 \leq i \leq n-1$

Only finitely-many G-orbits of cells

Obs. If such C exists, may assume it has only one orbit of 0-cells.
$\rightsquigarrow$ choosing a base-point identifies $\underbrace{C_0}_{0\text{-skeleton}} \cong G$
(So $\underbrace{C_1}_{1\text{-skeleton}}$ is a Cayley graph!)

Let $C_\chi :=$ The sub-complex spanned by vertices $g \in G$ with $\chi(g) \geq 0$.

Def. $\Sigma^n(G)$ is the set of characters $\chi: G \to \mathbb{R}$ given by:

$\chi \in \Sigma^n(G) \Leftrightarrow$ There is a free G-CW-complex C with G-finite n-skeleton s.th. C_χ is $(n-1)$-connected.

Thm [Renz, 1988; Bieri, Renz, 1988]. Let G be of type F_n, $N \trianglelefteq G$ with G/N abelian. Then N is of type F_n iff every $\chi: G \to \mathbb{R}$ with $\chi(N)=0$ is in $\Sigma^k(G)$.
(Criterion for finite presentability of normal subgroups with abelian quotient)

Note: There are criteria that might help one determine whether $\chi \in \Sigma^1(G), \Sigma^2(G)$.

The homological story

Given a $\mathbb{Z}G$-module A, we say <u>A is of type FP_n</u> if A has a free resolution

$$\cdots \to F_2 \to F_1 \to F_0 \twoheadrightarrow A$$

with F_i finitely generated up to degree n.

If $\mathbb{Z}$ is of type FP_n, we say <u>G is of type FP_n</u>

$\hookrightarrow$ as $\mathbb{Z}G$-module with trivial G action

[Bieri, Strebel, 1981; Bieri, Renz, 1988] define the "homological invariants"

$$\Sigma^0(G;A) \supseteq \Sigma^1(G;A) \supseteq \Sigma^2(G;A) \supseteq \cdots,$$

which are sets of characters $\chi : G \to \mathbb{R}$.

<u>Fact:</u> G is F_1 $\Leftrightarrow$ G is FP_1

For $n \geq 2$: G is F_n $\Leftrightarrow$ G is FP_n and F_2

<u>Analogously:</u> $\Sigma^1(G) = \Sigma^1(G;\mathbb{Z})$ [Bieri, Renz, 1988]

For $n \geq 2$: $\Sigma^n(G) = \Sigma^n(G;\mathbb{Z}) \cap \Sigma^2(G)$

[Renz, 1988]

Bringing in the topology

For G a locally compact topological group, [Abels, Tiemeyer; 1997] define "compactness properties"

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \dots$$

"compactly generated" (C_1), "compactly presented" (C_2)

for discrete groups: $C_n = F_n$

[Kochloukova; 2003] defined $\Sigma^1_{top}(G)$, $\Sigma^2_{top}(G)$ (consisting of continuous characters $\chi : G \to \mathbb{R}$)

Kai-Uwe Bux, Ilaria Castellano, Elisa Hartmann & myself are developing uniform definitions of $\Sigma^n_{top}(G)$, $\Sigma^n_{top}(G;\mathbb{Z})$, and generalizing classical results from the discrete setting.

Namely (Work in progress results):

Thm [BCHQ]. If G is of type C_n, $\chi : G \to \mathbb{R}$ a character:

$$\chi \notin \Sigma^n_{top}(G) \Rightarrow \chi(\underbrace{Z(G)}_{\text{center}}) = 0$$

Thm [BCHQ]. $\Sigma^1_{top}(G) = \Sigma^1_{top}(G;\mathbb{Z})$

For $n \geq 2$: $\Sigma^n_{top}(G) = \Sigma^n_{top}(G;\mathbb{Z}) \cap \Sigma^2_{top}(G)$