Computing generalized Seifert matrices for closures of colored braids

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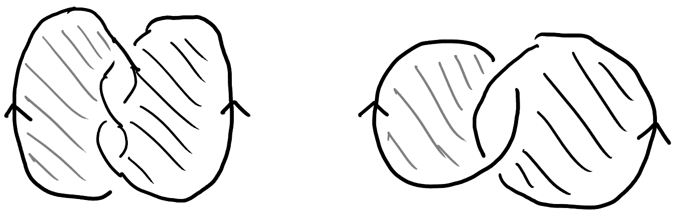
joint w/ Stefan Friedl & Chinmaya Kausik

An <u>n-component link</u> $L \subset S^3$ is a union of n oriented disjoint smoothly embedded circles ($n=1$: "knot").

A <u>Seifert surface</u> $S \subset S^3$ for L is an oriented compact connected smoothly embedded surface with $\partial S = L$.
(as oriented manifolds)

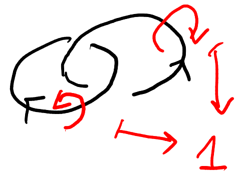


- Every link has (many) Seifert surfaces.
- Seifert surfaces are of independent interest. For example, what is the minimal possible genus $\underbrace{g(L)}_{\text{"genus of the link"}}$ of a Seifert surface for a given link?
- Seifert surfaces can be used for computing the Alexander polynomial $\Delta_L \in \mathbb{Z}[t^{\pm 1}]$ of L.
- To a Seifert surface S, we associate its <u>Seifert pairing</u>

$$H_1(S) \times H_1(S) \longrightarrow \mathbb{Z}$$
$$([\gamma], [\delta]) \longmapsto \mathrm{lk}(\underbrace{\gamma^-}_{\text{push-off of } \gamma \text{ toward the negative side of } S}, \delta)$$

- Choosing a $\mathbb{Z}$-basis for $H_1(S)$ determines a matrix A representing this form.
  A is called a <u>Seifert matrix</u> for L.

- The *infinite cyclic covering* $\begin{array}{c} X_\infty \\ \downarrow \\ S^3 \setminus L \end{array}$ is the one induced by the epimorphism $\pi_1(S^3 \setminus L) \twoheadrightarrow \mathbb{Z}$ sending each meridian to 1.



$\rightsquigarrow$ $H_1(X_\infty)$ becomes a $\mathbb{Z}[t^{\pm 1}]$-module

- Thm: $At - A^T$ is a presentation matrix for $H_1(X_\infty)$

- The Alexander polynomial

$$\Delta_L(t) = \det(At - A^T)$$

is a link invariant (up to multiplication by units $\pm t^k$)

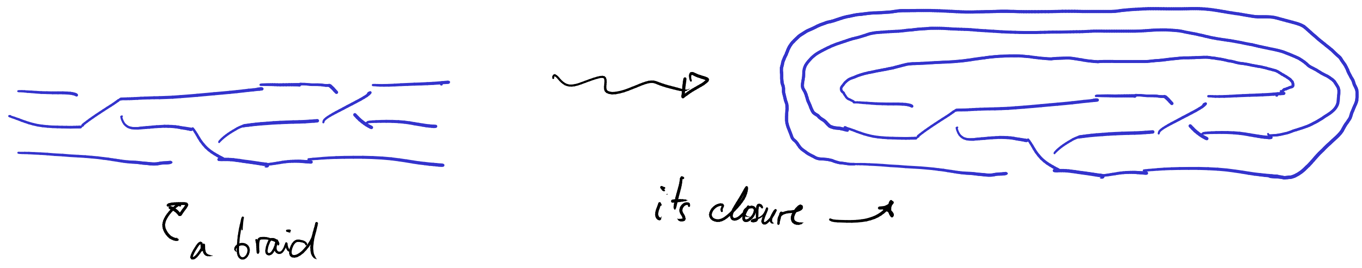
Obs: $\deg(\Delta_L) \leq 2g(L) + |\pi_0(L)| - 1$.

UPSHOT. • Seifert matrices are useful

• To compute a Seifert matrix:

1. Find a Seifert surface S
2. Choose a basis of $H_1(S)$
3. For every two basis elements $[\alpha], [\beta]$, compute $lk(\alpha^-, \beta)$

In 2016, Collins gave an algorithm to produce a Seifert matrix for a link given as a braid closure.



Braids are a convenient input. Need only specify sequence of crossings.

Thm [Alexander]. Every link is isotopic to the closure of a braid.

Collins, Köpke, Lewark implemented it as a computer program:

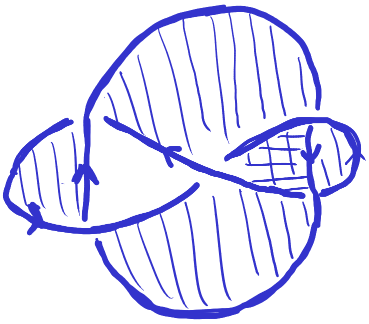
"Seifert Matrix Computations" https://www.maths.ed.ac.uk/~v1ranick/julia/index.htm

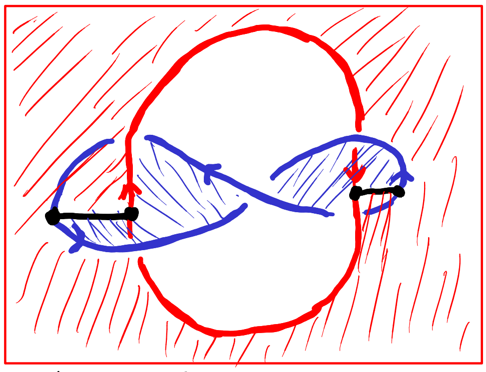
The COLORED setting

Def. Let $\mu \in \mathbb{N}_{\geq 1}$.
A μ-coloring on a link L is a decomposition of L into μ links: $L = \bigsqcup_{i=1}^{\mu} L_i$

- To a μ-colored link L one associates its *multivariable Alexander polynomial* $\Delta_L \in \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \dots, t_\mu^{\pm 1}]$
  (well-defined up to multiplication by units $\pm t_1^{k_1} \cdots t_\mu^{k_\mu}$)

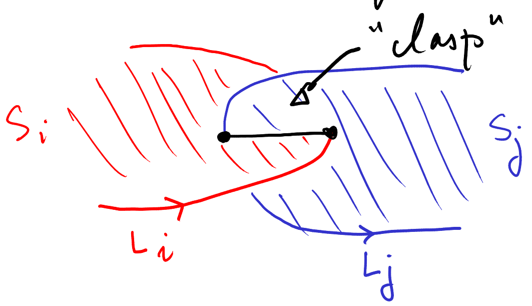
- In computing Δ_L, the role of Seifert surfaces is played by *clasp-complexes*.





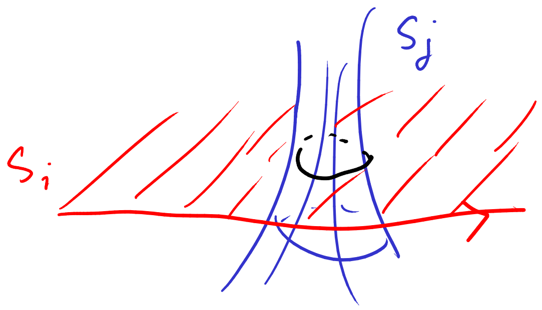
Def. [Cooper, Cimasoni] A clasp-complex for a μ-colored link $L = L_1 \sqcup \cdots \sqcup L_\mu$ is a collection of oriented surfaces $S_1, \dots, S_\mu \subseteq S^3$ s.th.:

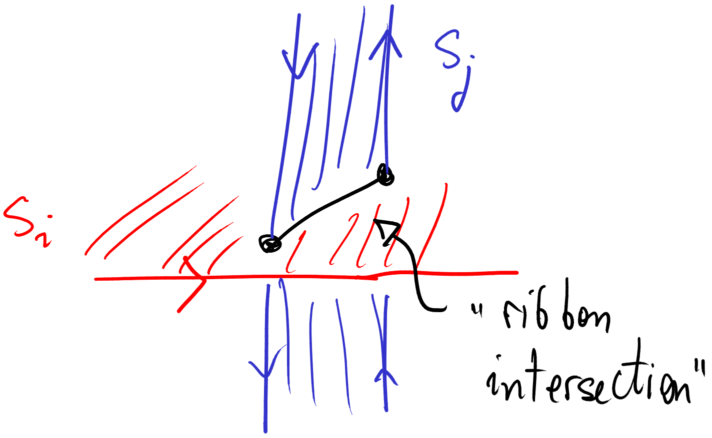
- Each S_i is a Seifert surface for L_i
- The union $S_1 \cup \cdots \cup S_\mu$ is connected
- There are no triple points: $S_i \cap S_j \cap S_k = \emptyset$ for i, j, k distinct
- Distinct S_i, S_j intersect along clasps



Component of $S_i \cap S_j$ homeomorphic to an interval connecting L_i to L_j, and otherwise disjoint from L.

NOT ALLOWED:





<u>Thm.</u> [Cimasoni] Every colored link has a clasp-complex.

(We will see a different proof!)

- To a clasp complex $S = S_1 \cup \dots \cup S_\mu$, associate a family of 2^μ Seifert forms:

↳ Let $\varepsilon \in \{\pm 1\}^\mu$ be a μ-tuple of signs $(\varepsilon_1, \dots, \varepsilon_\mu)$.

↳ ε specifies push-off directions for curves on S.

⇝ ε determines the <u>generalized Seifert pairing</u>.

at the surface S_i, γ is pushed in the direction given by the sign ε_i

$$H_1(S) \times H_1(S) \longrightarrow \mathbb{Z}$$
$$([\gamma], [\delta]) \longmapsto lk(\gamma^{\varepsilon}, \delta)$$

- Choosing a basis for $H_1(S)$, get a family of <u>generalized Seifert matrices</u> A^ε. [Cimasoni]

The A^ε determine Δ_L.

Defined without the multiplicative indeterminacy of Δ_L, so a finer invariant.

The <u>Conway potential function</u> $\nabla_L \in \mathbb{Q}(t_1, \dots, t_\mu)$ is:

$$\operatorname{sgn}(S) \cdot \prod_{i=1}^{\mu} (t_i - t_i^{-1})^{-1+\chi\left(\bigcup_{j\neq i} S_j\right)} \cdot \det\left(- \sum_{\epsilon \in \{\pm 1\}^\mu} A^\epsilon \cdot \epsilon_1 \cdots \epsilon_\mu \cdot t_1^{\epsilon_1} \cdots t_\mu^{\epsilon_\mu}\right)$$

product of signs of all clasps

∇_L determines Δ_L by:

$$\nabla_L(t_1, \dots, t_\mu) = \begin{cases} \dfrac{1}{t_1 - t_1^{-1}} \cdot \Delta_L(t_1^2), & \text{if } \mu = 1, \\ \Delta_L(t_1^2, \dots, t_\mu^2), & \text{if } \mu \geq 2. \end{cases}$$

Other applications

- The A^ε determine the Cimasoni-Florens signatures and nullities of L

$$\sigma_L, \eta_L : (S^1 \setminus \{1\})^M \longrightarrow \mathbb{Z}$$

Values on a certain subset $\mathbb{T}_!^M \subseteq (S^1\setminus\{1\})^M$ are top. concordance invariants.

- Determine the isometry type of generalized Blanchfield pairing $TH_1(S^3 \setminus L; \Lambda_\mu) \times TH_1(S^3 \setminus L; \Lambda_\mu) \to \mathbb{Q}(t_1, \ldots, t_\mu)$

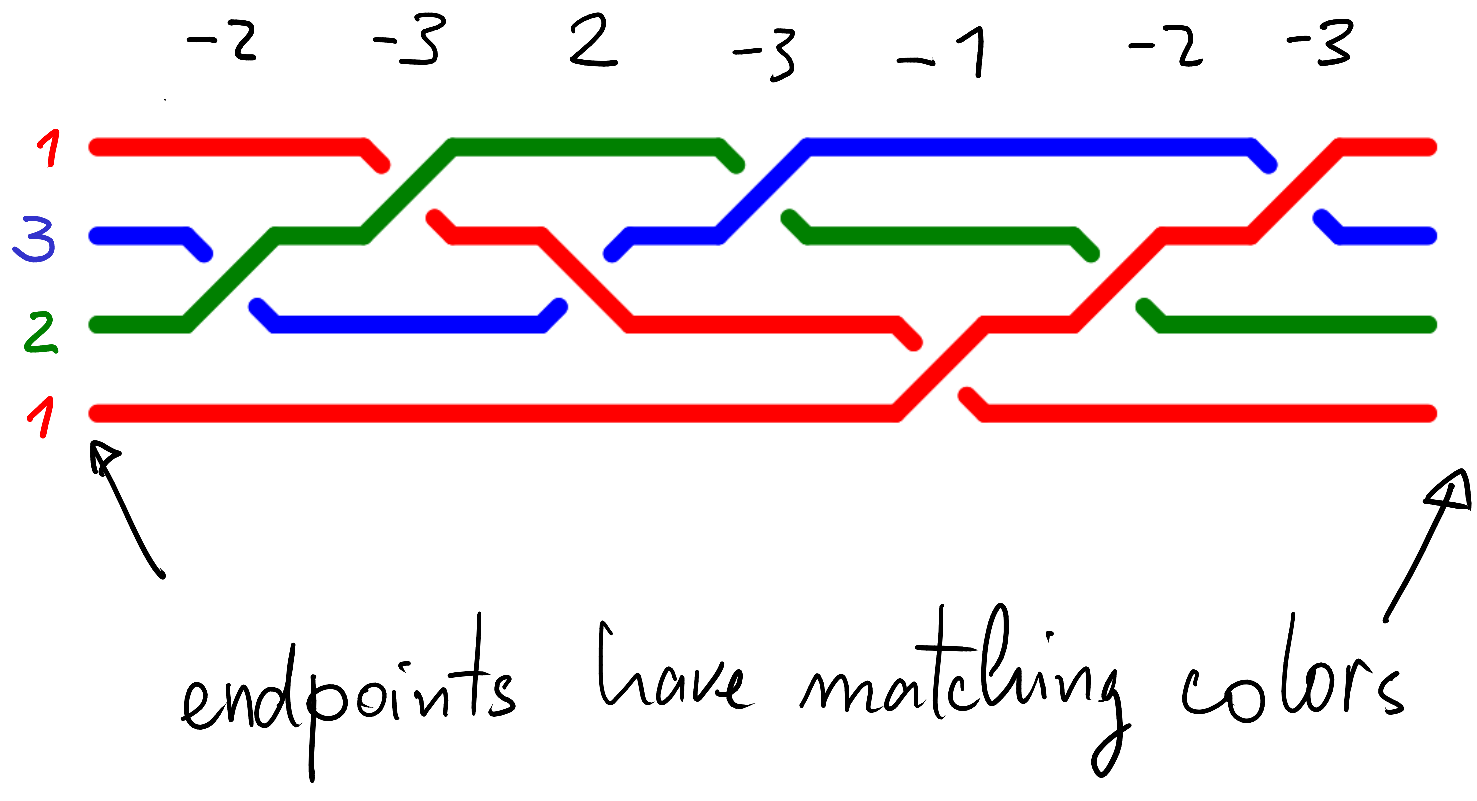
- We have given an algorithm for producing a family of generalized Seifert matrices for a link given as the closure of a colored braid.
  - Construct clasp-complex
  - choose homology basis
  - compute linking numbers

- Chinmaya Kausik has a computer implementation "CLASPER".

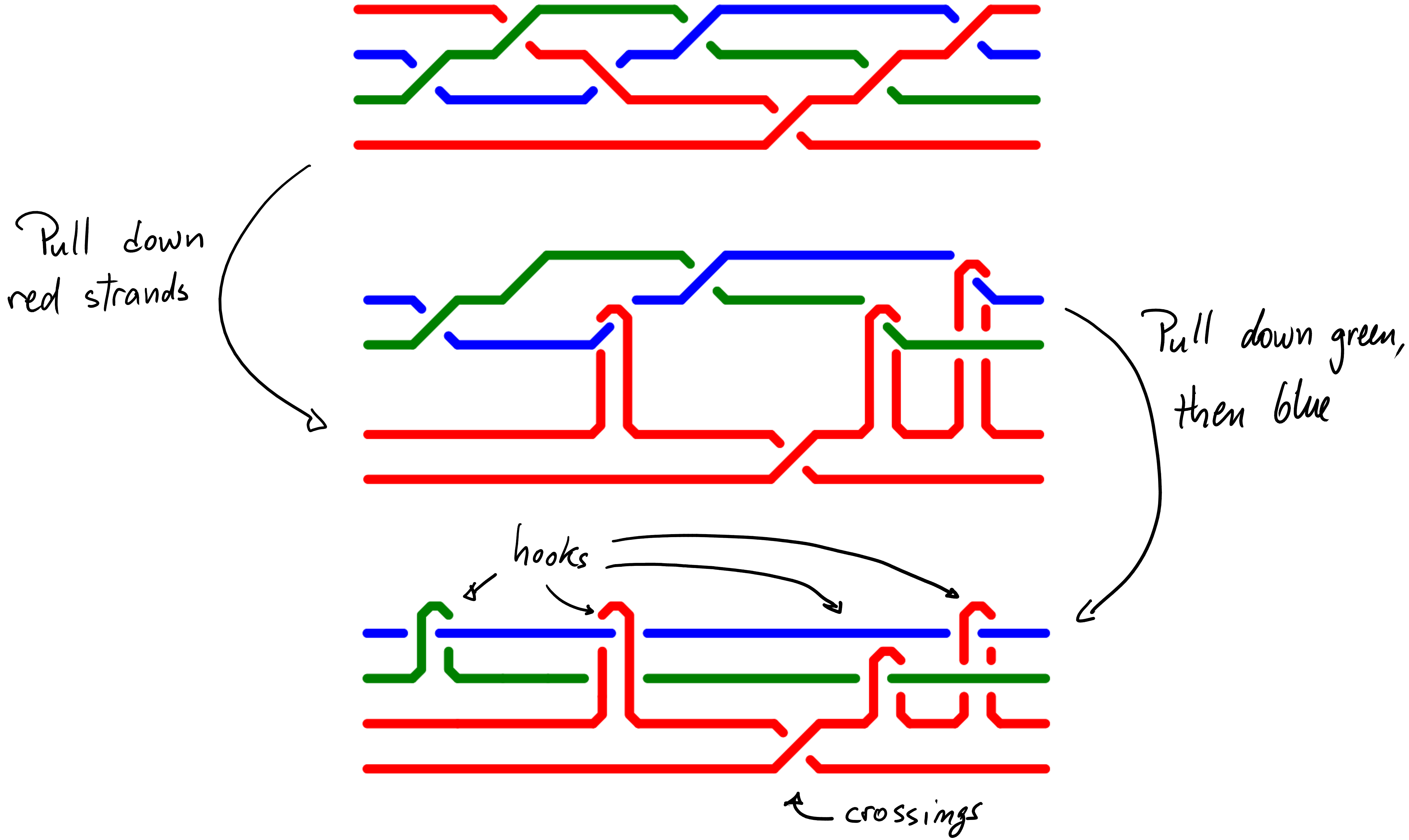
  https://github.com/Chinmaya-Kausik/py_knots/

Outline of the algorithm

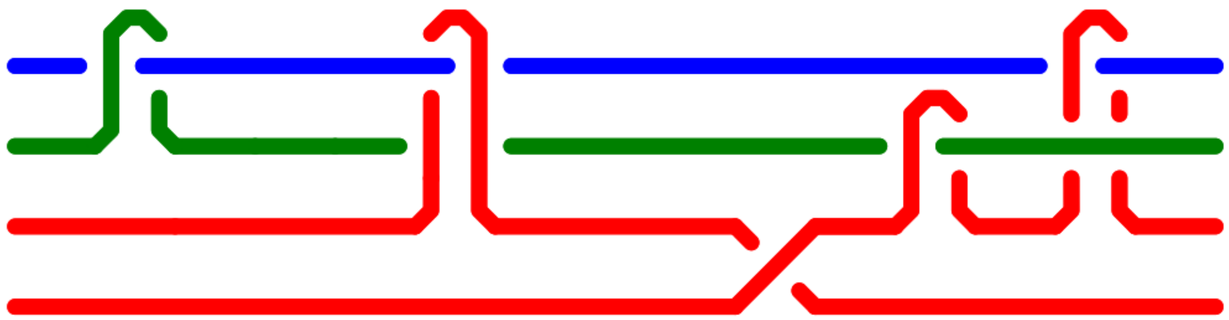
0. Start with a braid whose closure is colored

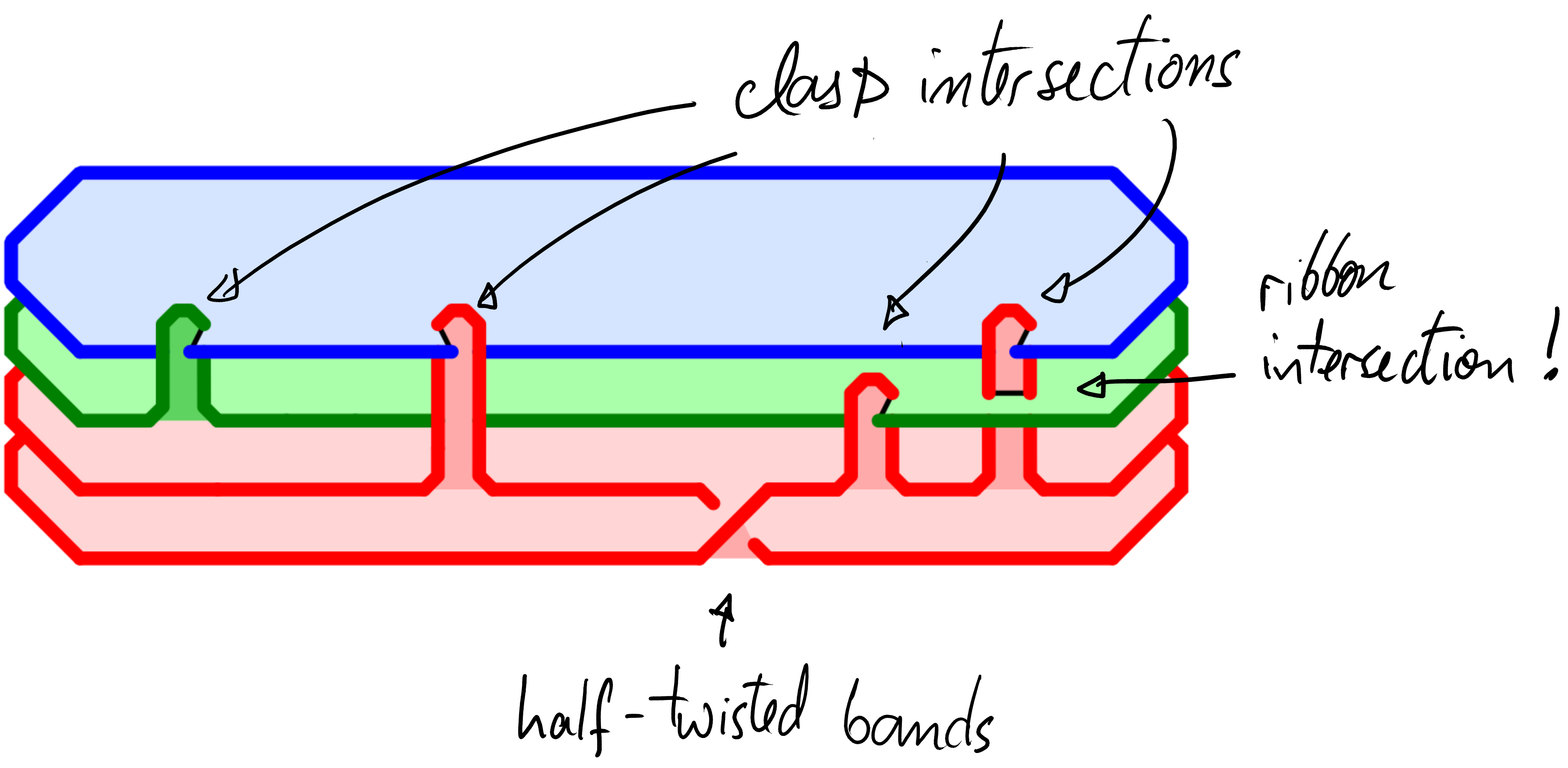


1. Pull down strands in each color

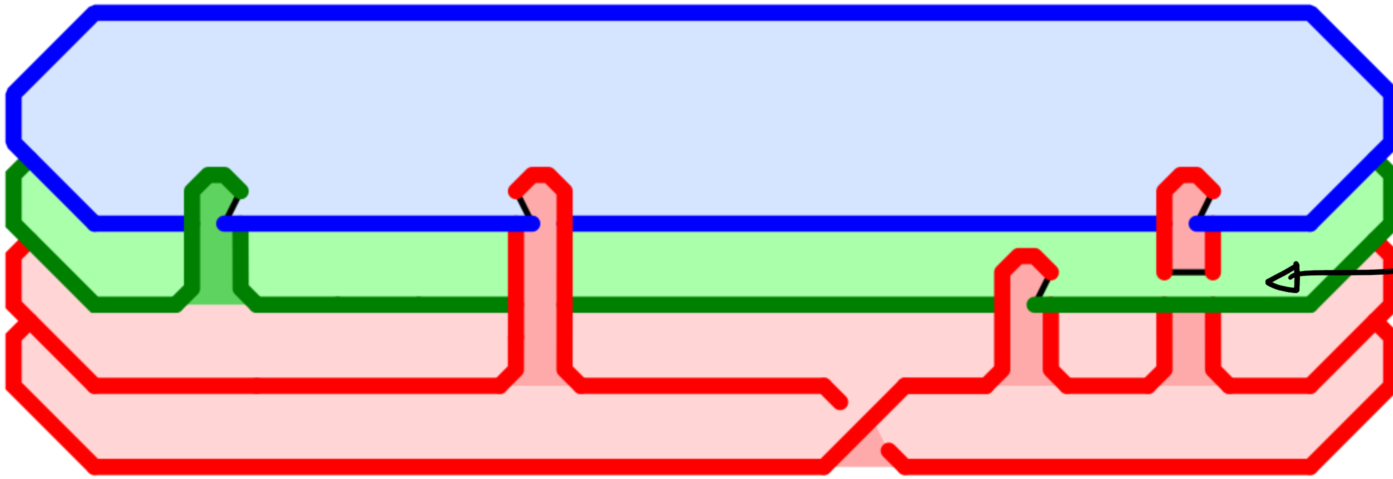


2. Close up braids, fill in surfaces

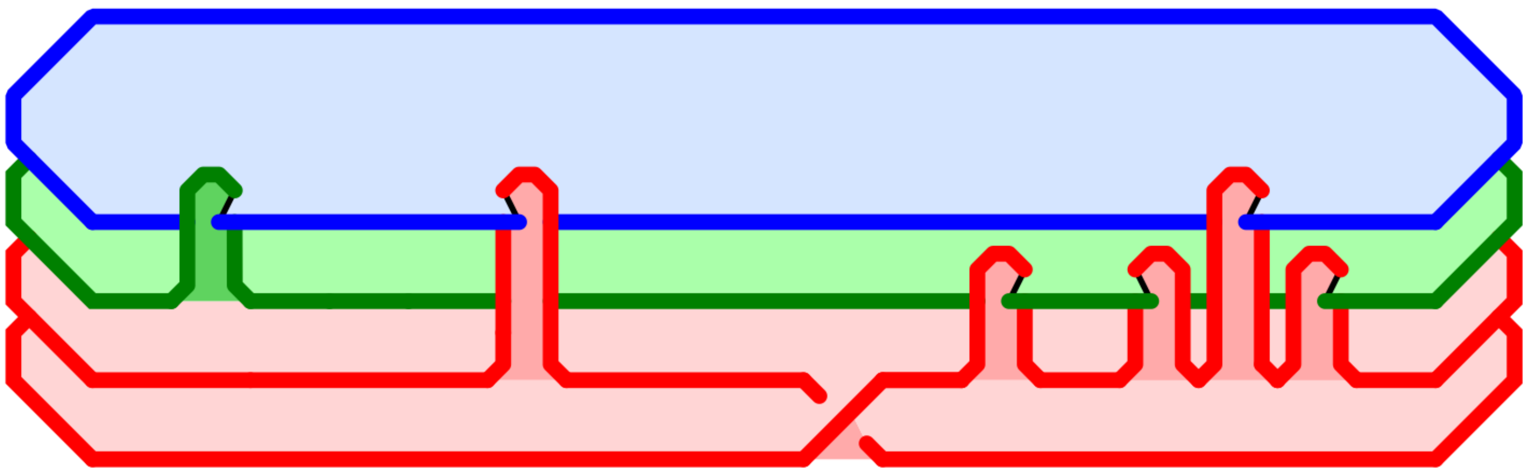




3. Exchange ribbon intersections for clasps

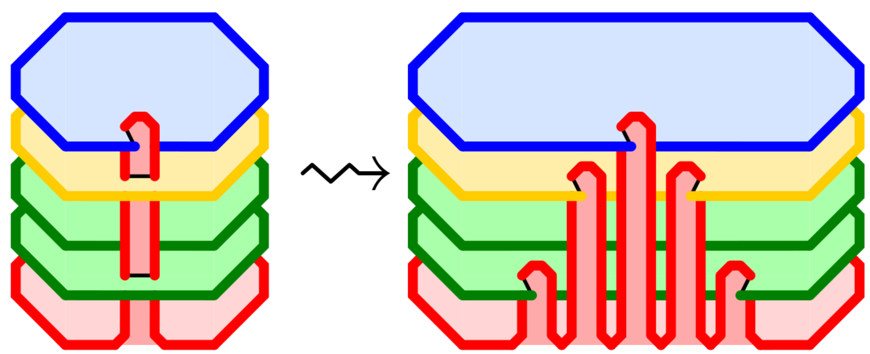


ribbon intersection!

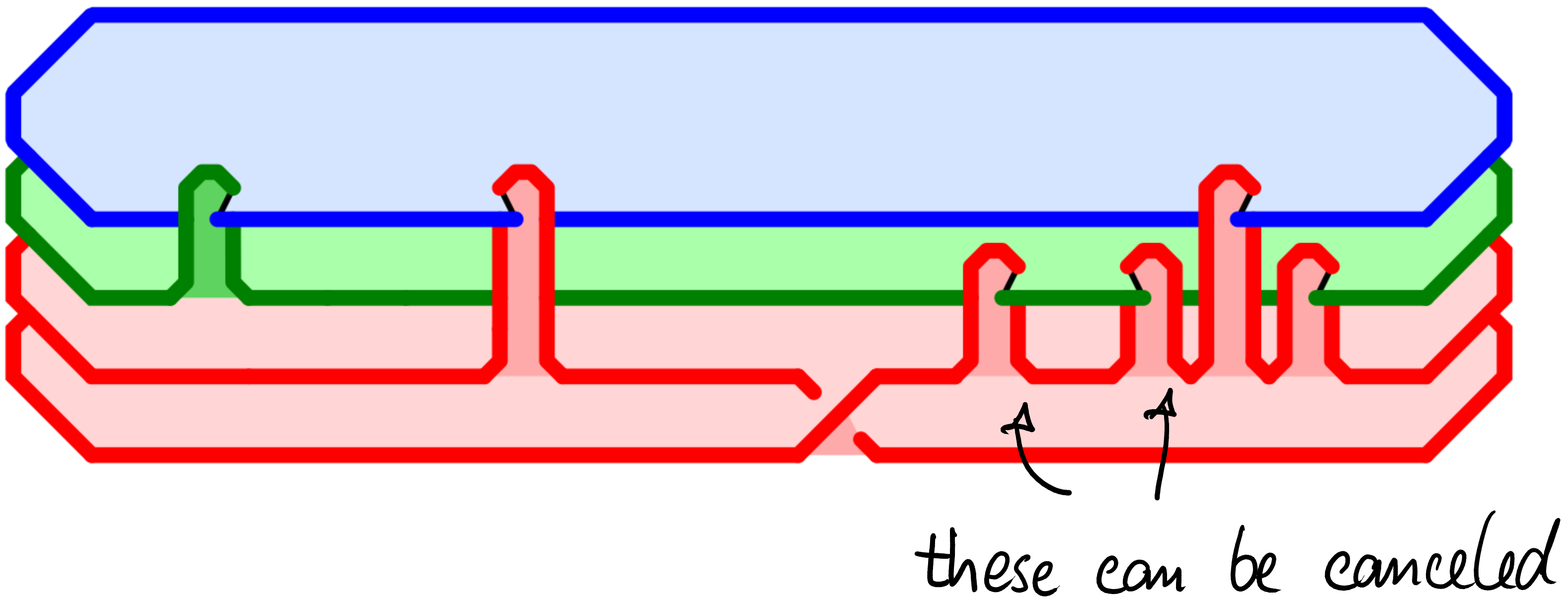


No ribbons!

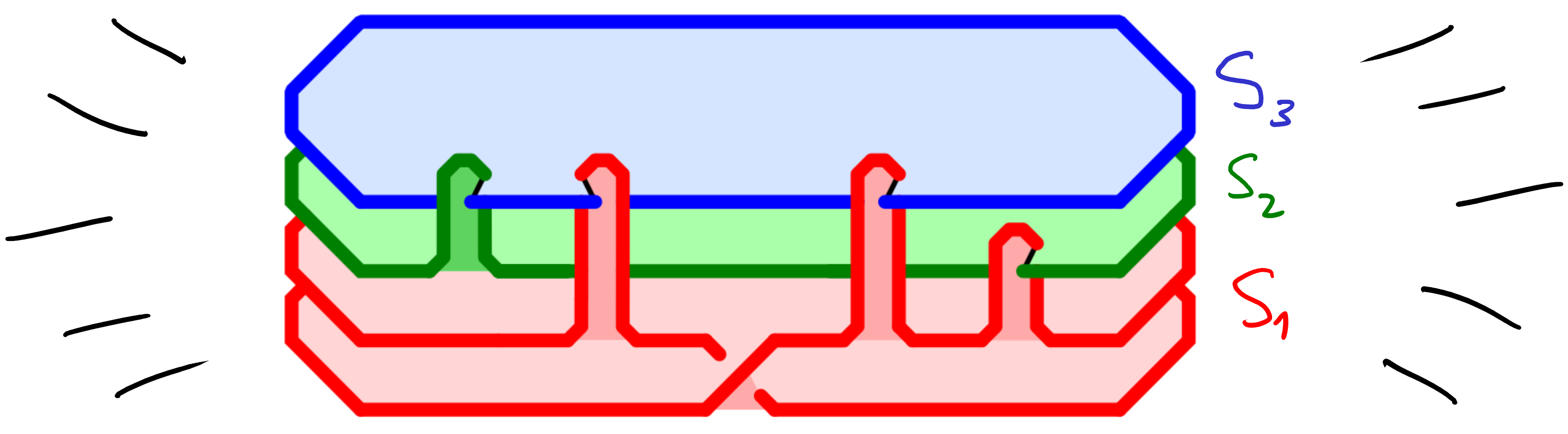
Might need to iterate this trick:



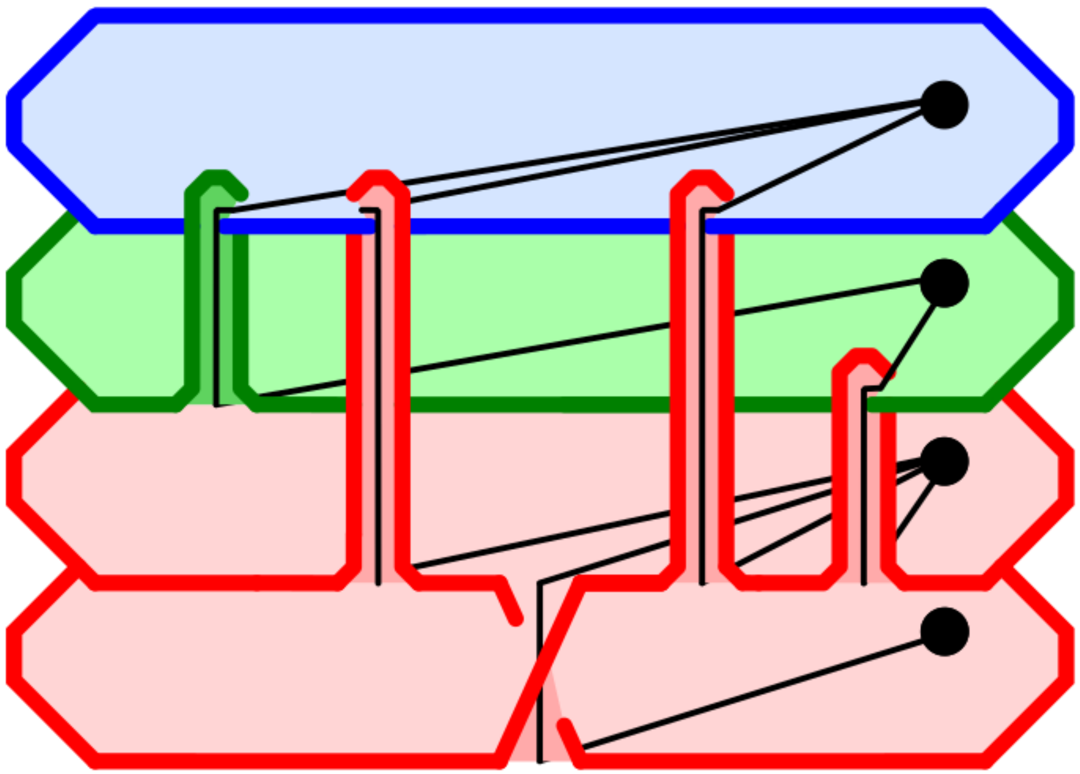
4. Clean up, make sure surfaces are connected,
make sure the complex is connected...

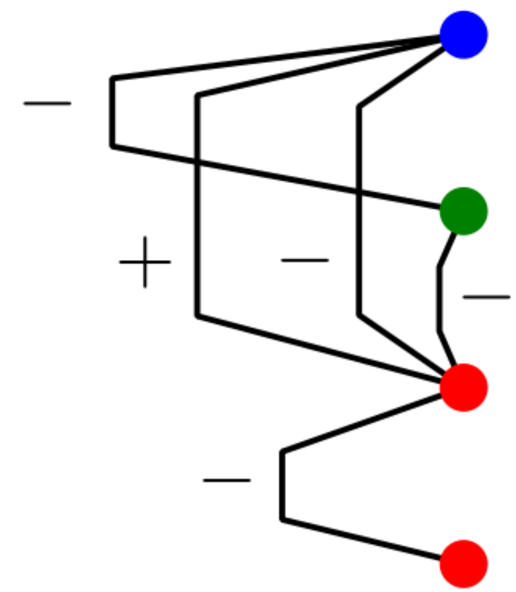


$\rightsquigarrow$ Get a clasp complex S.



5. Encode S as a decorated graph G

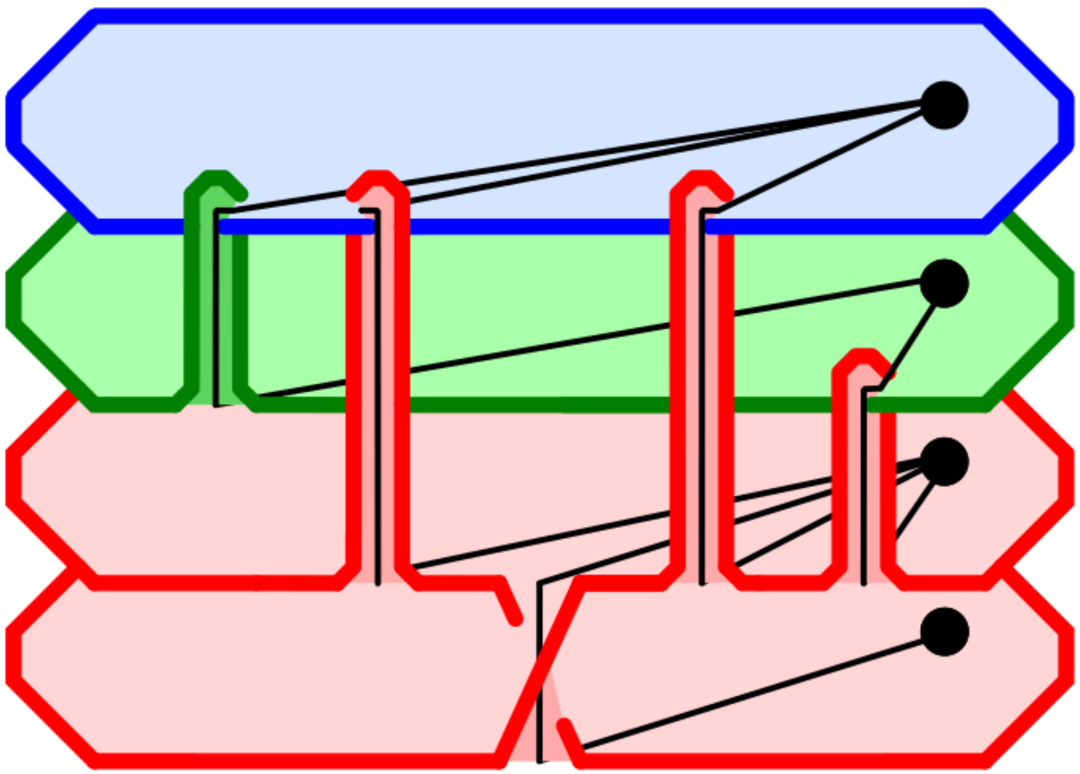


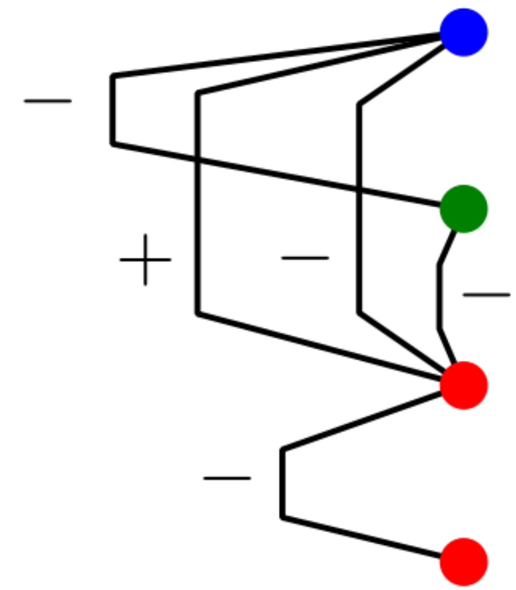


Decoration includes:
- total order on vertices
- total order on edges
- colors of vertices
- signs on edges (to encode handedness)

(edge is a $\begin{cases}\text{half-twisted band} \\ \text{finger}\end{cases}$ if endpoints are $\begin{cases}\text{same} \\ \text{different}\end{cases}$ color)

Obs. G is a deformation retract of S.





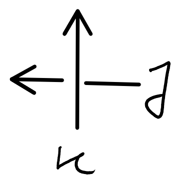
So:

6. Choose a basis of $\mathrm{H}_1(S) \cong \mathrm{H}_1(G)$.

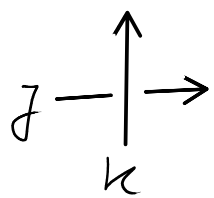
(as circuits in G)

7. Compute linking numbers edge-by-edge

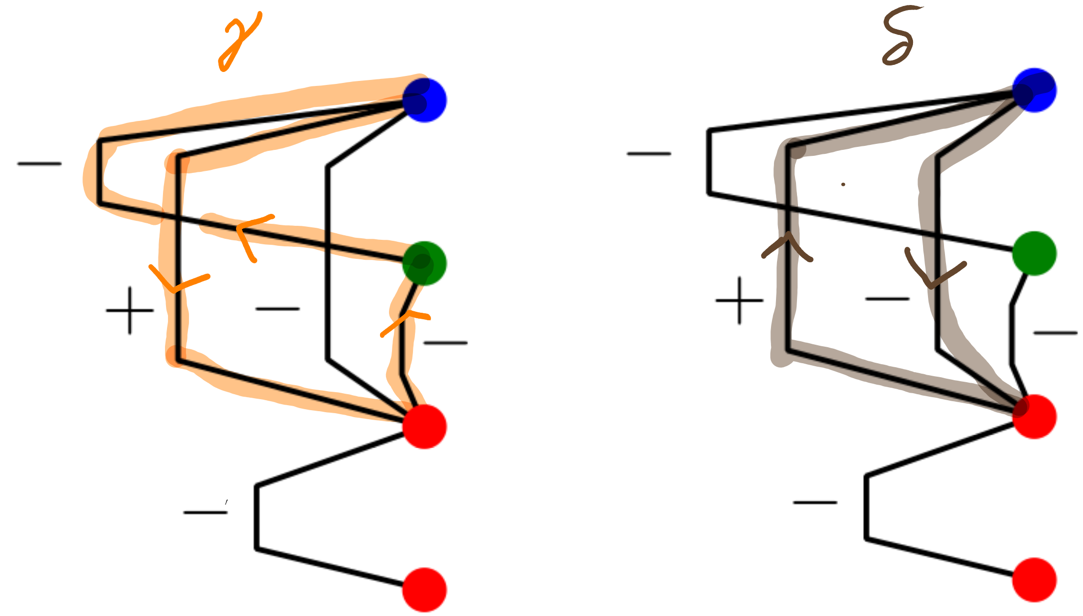
Recall. To compute $lk(\gamma, k)$:



$+1$

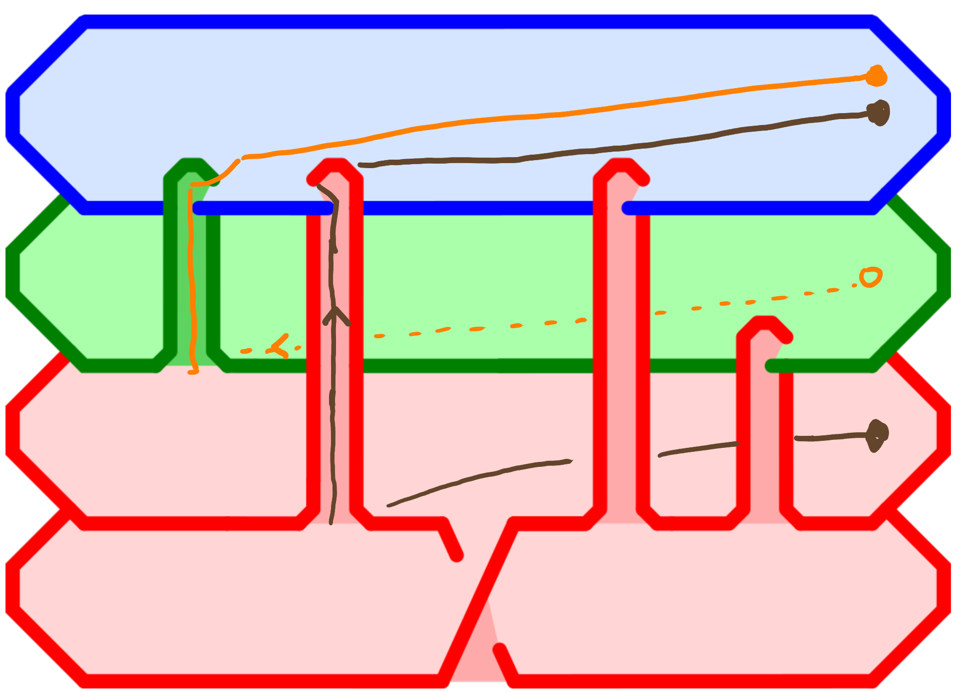


-1



Ex. To compute

$lk(\gamma^{+-+}, \delta)$ for:



- These two edges contribute with $+1$.
  ↳ This can be read off from the decorations.
- Sum contributions of all edges.