[Gompf, Stipsicz, 4-Manifolds and Kirby Calculus]

# 9. An introduction to Kirby Calculus

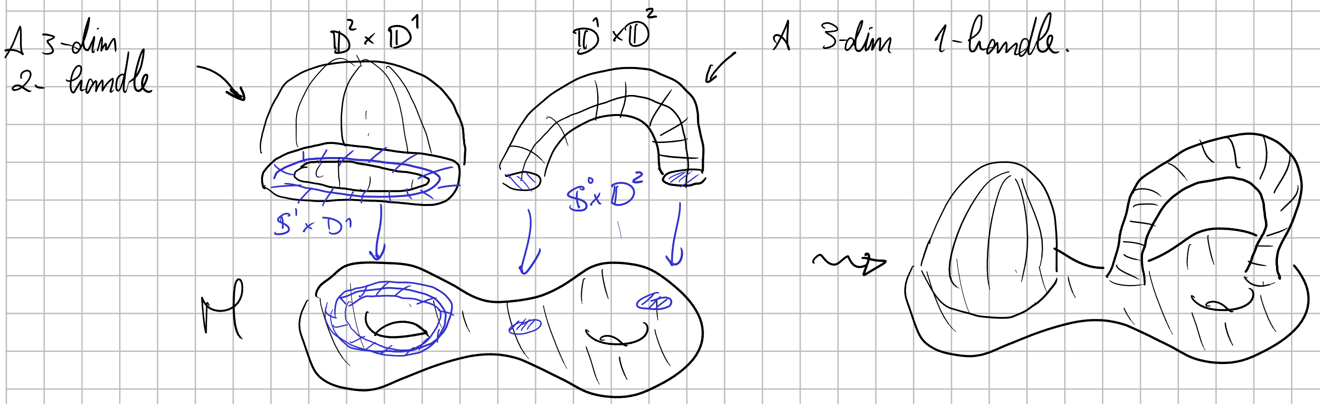
**Goal.** Specify smooth 4-manifolds using links (plus decorations).

## Handle decompositions

(Like CW-decompositions, but "manifoldier")

Intuitively, an ($n$-dimensional) $k$-handle is a "thickened $k$-cell", attached along its (thickened) boundary.

$\underbrace{\text{"thickened } k\text{-cell"}}_{\mathbb{D}^k \times \mathbb{D}^{n-k}}$, $\underbrace{\text{(thickened) boundary}}_{\mathbb{S}^{k-1} \times \mathbb{D}^{n-k}}$




**Def.** Let $M$ be a smooth $n$-manifold (with boundary), let $k \in \{0, \dots, n\}$.

An $n$-manifold $M'$ is obtained from $M$ by <u>attaching a $k$-handle</u> if there is a smooth embedding

$$\varphi : \mathbb{S}^{k-1} \times \mathbb{D}^{n-k} \hookrightarrow \partial M \subset M$$

"attaching map"

such that $M'$ is a pushout

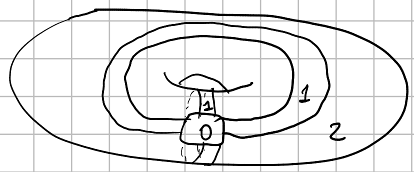
$$\begin{tikzcd} \mathbb{S}^{k-1} \times \mathbb{D}^{n-k} \arrow[r, hook, "\varphi"] \arrow[d, hook] & M \arrow[d] \\ \mathbb{D}^k \times \mathbb{D}^{n-k} \arrow[r] & M' \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

**Thm** Every compact smooth $n$-manifold $M$ is an ascending union

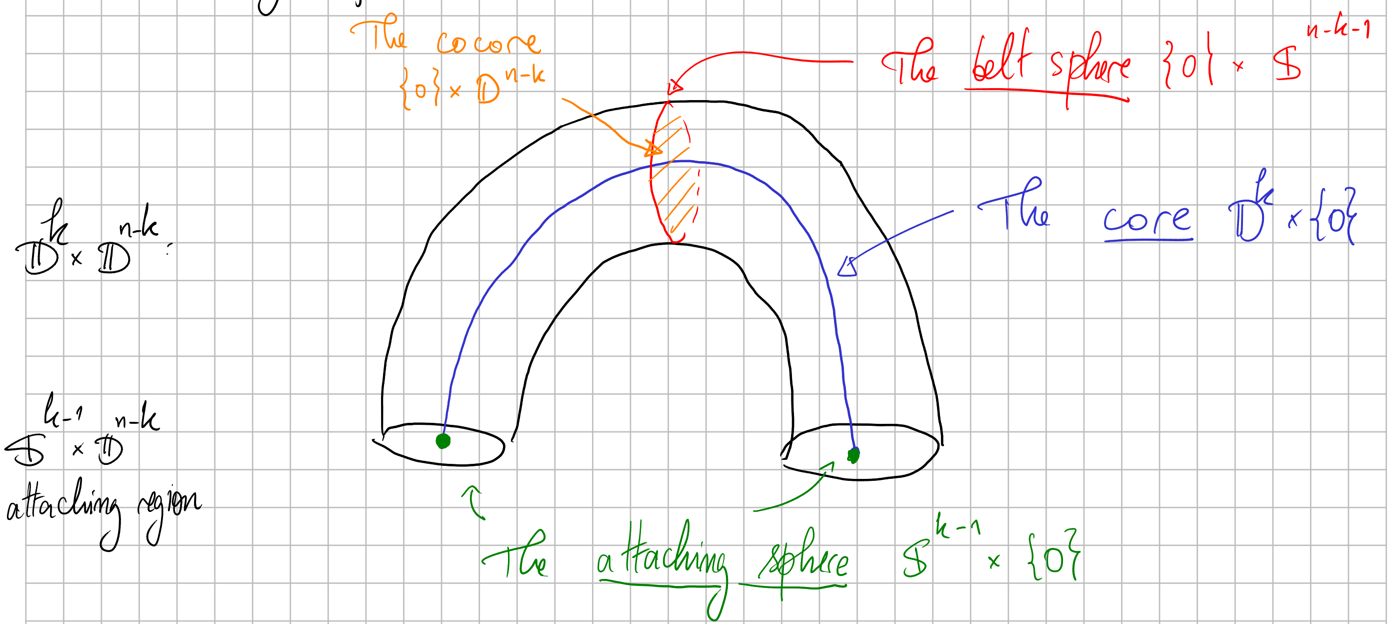
$$\emptyset = M_{-1} \subseteq M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots \subseteq M_n = M,$$

where each $M_k$ is obtained from $M_{k-1}$ by attaching (finitely-many) $k$-handles.

**Ex** Say $M = S^1 \times S^1$. A possible decomposition is:




The anatomy of a $k$-handle:




A $k$-handle attachment is specified (up to isotopy) by:

- An embedding $\varphi_0 : S^{k-1} \hookrightarrow \partial M$ of the attaching sphere (up to isotopy and self-diffeo of $S^{k-1}$)
- An identification of the normal bundle of $\varphi_0(S^{k-1})$ with the normal bundle $S^{k-1} \times \mathbb{R}^{n-k}$ of $S^{k-1} \times \{0\} \subseteq S^{k-1} \times D^{n-k}$.

"A framing"

<u>Fact</u>. • Up to a self-diffeo of the $k$-handle, we may assume the framing is fixed at some basepoint of $S^{n-k}$.

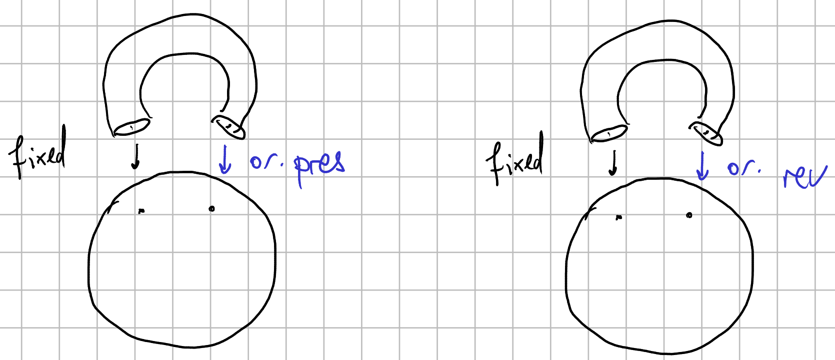
• Up to isotopy, every two such framings differ by an element of
$$\pi_{k-1}(O(n-k)).$$

If this difference is the trivial element, then the framings are isotopic.

<u>Example</u>. 1) Say we want to attach a $\underbrace{1\text{-handle}}_{D^1\times D^2}$ to $D^3$:

• There is only one embedding $S^0 \hookrightarrow \partial D^3 = S^2$ (up to isotopy)

• There are two possible framings to consider, because
$$\pi_0(O(2)) \cong \mathbb{Z}/2$$




(This boundary is the klein bottle)

twisted $D^2$-bundle over $S^1$.

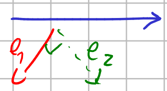
This produces the manifolds $S^1\times D^2$ and $S^1\tilde{\times} D^2$.

The situation is analogous in all dimensions $n\geq 2$ since $\pi_0(O(n-1))\cong\mathbb{Z}/2$

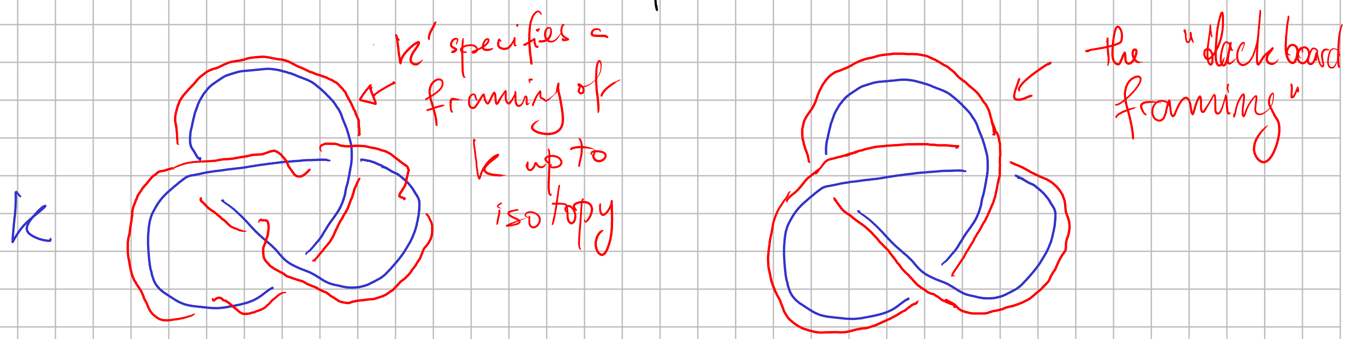
( Ex $n=2$:


)

2) Say we want to attach a 2-handle to $\mathbb{D}^4$.
We visualize $\partial \mathbb{D}^4 = \mathbb{S}^3$ as $\mathbb{R}^3 \cup \{\infty\}$

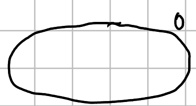
- An embedding $\varphi: \mathbb{S}^1 \hookrightarrow \mathbb{S}^3$ up to self-diffeo of $\mathbb{S}^1$ and isotopy is precisely a knot!
- A framing of a knot $K \subset \mathbb{S}^3$ can be specified by a unit section of its normal bundle $\nu(K)$
(This specifies the first basis vector of $\nu(K)$; the second one is given by the right-hand rule  )

This traces out a "companion knot" $K'$:




Every two framings differ (up to isotopy) by an element of $\pi_1(O(2)) \cong \mathbb{Z}$. The generators act by adding a twist to $K'$ around $K$.

$\rightsquigarrow$ We indicate a framing of $K$ by specifying the integer $lk(K, K')$.

Ex. 

Attaching a 2-handle along a 0-framed unknot produces $\mathbb{S}^2 \times \mathbb{D}^2$

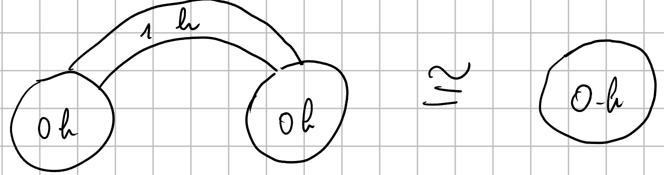
(Lower-dimensional analogy: 2-handle $\mathbb{D}^2 \times \mathbb{D}^1$  $\mathbb{S}^2 \times \mathbb{D}^1$ )

More generally: the $\mathbb{D}^2$-bundles over $\mathbb{S}^2$ are precisely the manifolds obtained by attaching a 2-handle to $\mathbb{D}^4$ along a framed unknot:


$(n \in \mathbb{Z})$

(and these are all non-homeomorphic)

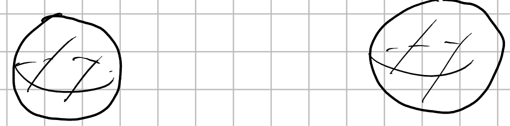
<u>Obs</u>. If a compact manifold $M$ is connected, then it has a handle decomposition with only one 0-handle:




By flipping the handle decomposition upside-down, $M$ has at most one $n$-handle (exactly one, if $M$ is closed)

# Kirby diagrams – Representing 4-manifolds

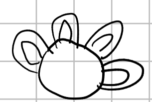
- Assuming our 4-mfd $M$ is connected, we may use only one 0-handle $D^4$. We visualize its boundary as $\mathbb{R}^3 \cup \{\infty\}$.
- We indicate the attaching regions for 1-handles as pairs of 3-balls:



We think of them as identified by
- an orientation-reversing map, or
- an orientation-preserving map

If $M$ is oriented, only this occurs.

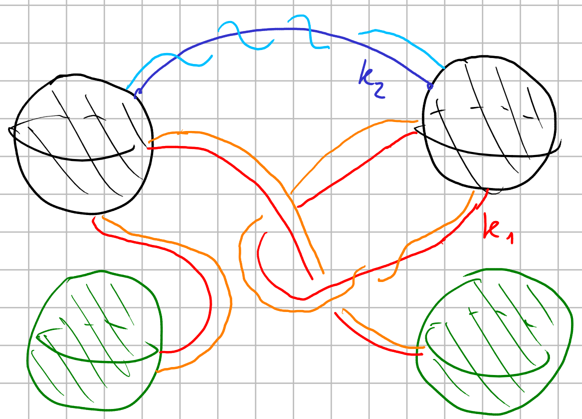

If $M$ is oriented and has $\ell$ 1-handles, then

$M_1 \cong \natural_\ell \, S^1 \times D^3$ 

boundary-connected sum

- For attaching 2-handles, we give embedded framed circles in this space.

Ex.

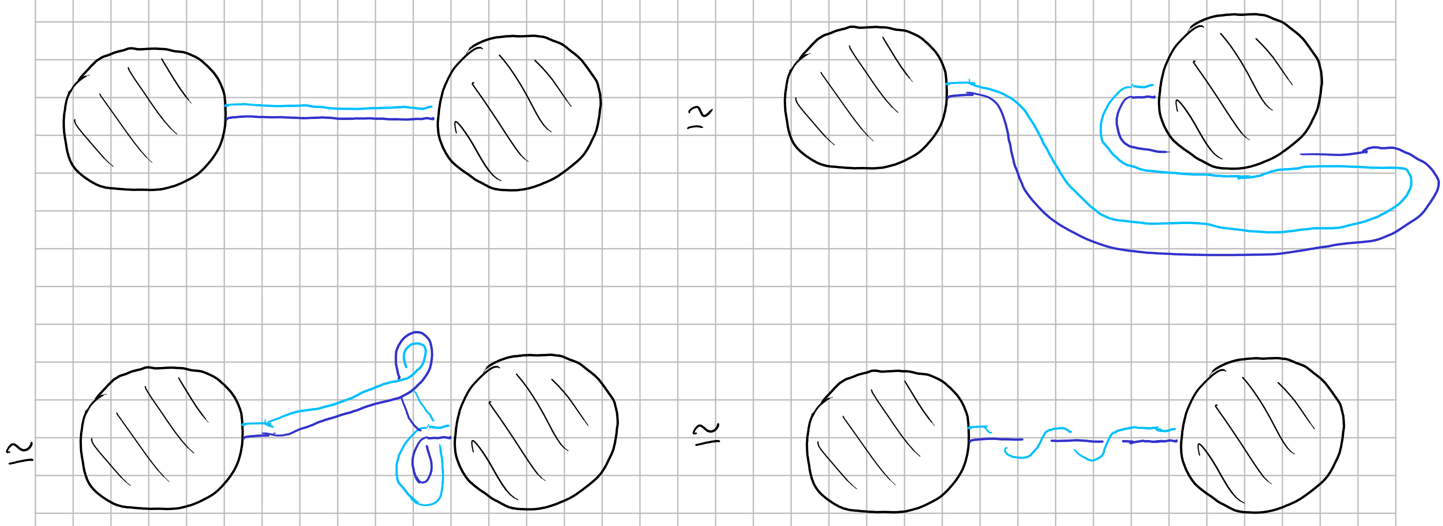



(companion curves indicate framing)

(suppose the pairs of 3-balls are identified via a reflection)

This is called a *Kirby diagram*.

Warning: When there are 1-handles, the linking numbers of the attaching 1-spheres and their companions are not well-defined:



Suppose now that $M$ is an oriented closed connected 4-manifold.

Then the union of the 3- and 4-handles is diffeomorphic to a union of 0- and 1-handles, i.e. to $\natural_\ell S^1 \times D^3$ for some $\ell$

Fact: Every self-diffeomorphism of

$$\partial\left(\natural_\ell S^1 \times D^3\right) = \#_\ell S^1 \times S^2$$

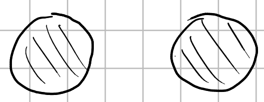
extends to $\natural_\ell S^1 \times D^3$.

So the data of the 0-, 1-, and 2-handles determines $M$!
Hence every closed oriented connected 4-mfd $M$ may be represented by a Kirby diagram for

$M_2$ = 0-handle $\cup$ 1-handles $\cup$ 2-handles

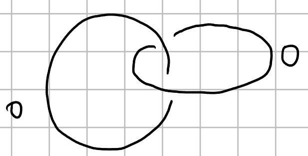
(Though not every such $M_2$ can be completed to a closed 4-mfd by attaching 3- and 4-handles.)

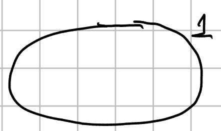
Ex.

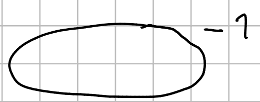
- The "empty" Kirby diagram represents $\mathbb{D}^4$, which closes to $\mathbb{S}^4$.
-  (one 1-handle, no 2-handles)

  $$M_2 = M_1 \cong \mathbb{S}^1 \times \mathbb{D}^3$$

Doubling $M_2$ along its boundary adds a 3- and a 4-handle, producing $\mathbb{S}^1 \times \mathbb{S}^3$.

- 

$\cup$ 4-h $\cong$ $\mathbb{S}^2 \times \mathbb{S}^2$
- 

$\cup$ 4-h $\cong$ $\mathbb{CP}^2$
- 

$\cup$ 4-h $\cong$ $\overline{\mathbb{CP}^2}$

  ($\mathbb{CP}^2$ with opposite orientation)

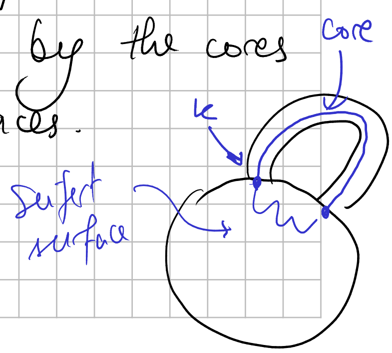
If $M$ is closed 4-mfd of the form

$$M = \text{0-handle} \cup \text{2-handles} \cup \text{4 handle}$$

and $L \subset \mathbb{S}^3$ is a framed link with components $K_1, \dots, K_m$ representing $M_2$, define the $m \times m$ linking matrix

$$A = (a_{ij}) \quad \text{with} \quad a_{ij} = \begin{cases} \text{framing of } K_i & \text{if } i = j \\ lk(K_i, K_j) & \text{if } i \neq j \end{cases}$$

Then $H_2(M) \cong \mathbb{Z}^m$ with a basis given by the cores of the two-handles capped by Seifert surfaces.




The intersection form

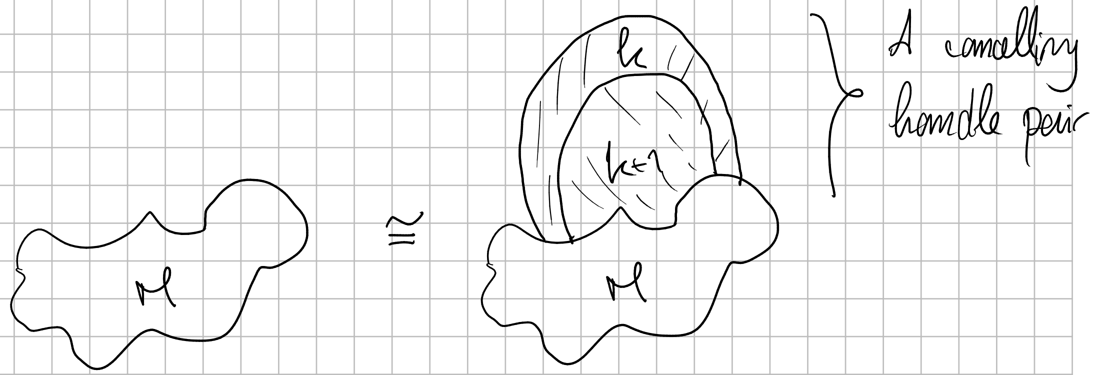
$$H^2(M) \otimes H^2(M) \longrightarrow \mathbb{Z}$$

has matrix $A$ in this basis!

# Handle moves

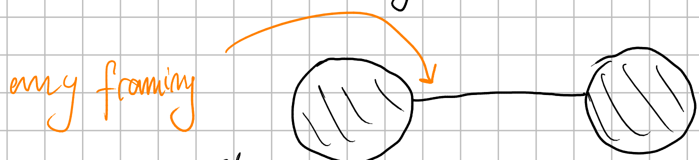
Given a smooth $n$-manifold $M$, any two handle decompositions of $M$ differ by a sequence of the following two moves:

1) Handle cancellation / creation




**Prop.** A $k$-handle and a $(k+1)$-handle may be cancelled if the attaching sphere of the $(k+1)$-handle intersects the belt sphere of the $k$-handle transversely, at exactly one point)

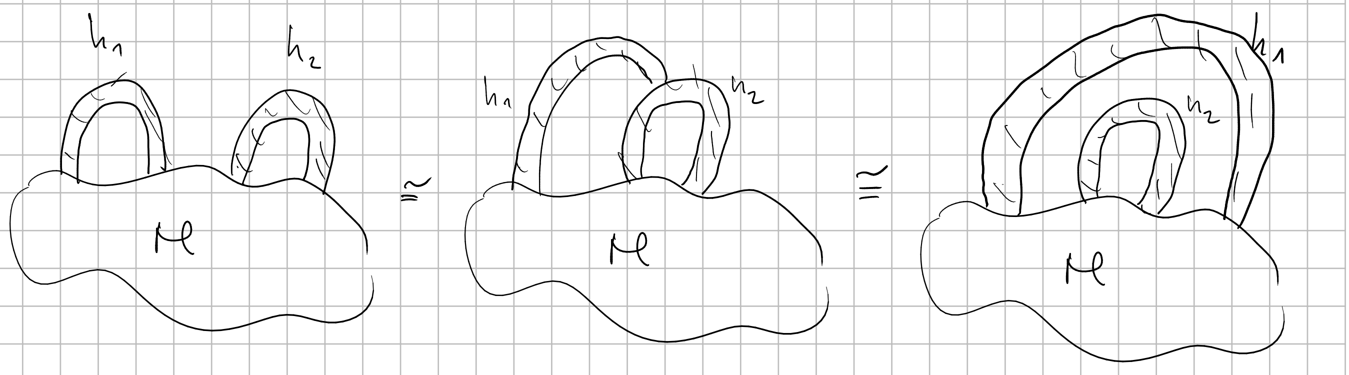
**Ex.** One can always add or delete




to a Kirby diagram.

2) Handle slides:

Given two $k$-handles $h_1$, $h_2$ attached to $M$, isotope the attaching sphere of $h_1$ within $\partial(M \cup h_2)$, pushing it past the belt sphere of $h_2$:




One can work out the effect on a Kirby diagram of sliding 1- and 2-handles.

Kirby calculus is the manipulation of Kirby diagrams using these three moves:

- cancelling 1- and 2-handles
- sliding 1-handles
- sliding 2-handles