8. The Alexander Polynomial

The Alexander module

Def. Let X be a path-connected space that has a universal covering.

The <u>universal abelian covering</u> $\begin{matrix}\hat{X}\\ \downarrow \\ X\end{matrix}$ is the one with

commutator subgroup

$\pi_1(\hat{X}) = [\pi_1(X), \pi_1(X)]$. In particular, there is a canonical isomorphism $\mathrm{Aut}\begin{pmatrix}\hat{X}\\ \downarrow \\ X\end{pmatrix} \cong \pi_1(X)_{ab}$, and for every regular covering Y of X with abelian automorphism group there is a factorization $\hat{X} \to Y \to X$

Let K be an oriented knot.

The universal abelian covering $\begin{matrix}\hat{X}_K\\ \downarrow \\ X_K\end{matrix}$ is also called the <u>infinite cyclic covering</u>, because its automorphism group is

$$\pi_1(X_K)_{ab} \cong H_1(X_K) = \langle [\underbrace{\mu}_{\text{meridian of } K\text{, positively oriented.}}] \rangle_{\mathbb{Z}} \cong \mathbb{Z}$$

We will denote the preferred generator of $\mathrm{Aut}\begin{pmatrix}\hat{X}_K\\ \downarrow \\ X_K\end{pmatrix}$ by τ (depends on the orientation of K).

Ex. Recall that for U the unknot, $X_U \cong S^1 \times D^2$.

In particular, $\pi_1(X_U) \cong \mathbb{Z}$ is abelian, so $\hat{X}_U$ is the universal covering. Hence, $\hat{X}_U \cong \mathbb{R} \times D^2$, with τ given by $(t, p) \mapsto (t+1, p)$.

The action $\mathbb{Z} \cong \langle t \rangle \curvearrowright H_1(\hat{X}_K)$ where t acts by τ_* turns $H_1(\hat{X}_K)$ into a module over the ring of <u>Laurent polynomials</u>

$$\Lambda := \mathbb{Z}[t^{\pm 1}].$$

(polynomials over $\mathbb{Z}$ with possibly negative exponents)

When regarded as a Λ-module, $H_1(\hat{X}_K)$ is called the <u>Alexander module</u> of K.

We will define the Alexander polynomial as a certain invariant of this Λ-module. This requires introducing some algebraic machinery.

<u>Algebraic interlude – The order ideal of a module</u>

Let R be a commutative ring (with 1), and consider a finitely presented R-module

$$M = \langle x_1, \dots, x_m \mid \rho_1, \dots, \rho_n \rangle$$

Write the relators in terms of generators as

$$\rho_j = \sum_{i=1}^{m} r_{ij} x_i \qquad (r_{ij} \in R),$$

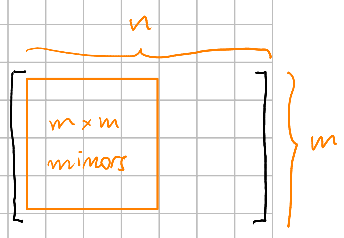
The $m \times n$ matrix over R

$$A := (r_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}} =$$

is called a <u>presentation matrix</u> for M.

In other words, A specifies (the isomorphism type of) R as the cokernel of the R-module map

$$R^n \xrightarrow{A} R^m.$$



We import the following:

<u>Fact</u>. The ideal $O(M) \triangleleft R$ generated by all $m \times m$ minors of A depends only on the R-module M, and not on the chosen presentation of M.

<u>Def</u>. $O(M)$ is called the <u>order ideal</u> of M.

Ex. If $R = \mathbb{Z}$ and M is finitely generated, then M has the form

$$M \cong \mathbb{Z}/a_1 \oplus \dots \oplus \mathbb{Z}/a_k \quad \text{with } a_i \in \mathbb{N} \text{ (possibly zero)}$$

In particular M is finitely presented, with $A = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_k \end{pmatrix}$ a presentation matrix, and

$$\mathcal{O}(M) = \langle \det A \rangle = \langle a_1 \cdot \ldots \cdot a_k \rangle .$$

Hence : $$\mathcal{O}(M) = \begin{cases} 0 & \text{if } M \text{ is infinite} \\ \langle |M| \rangle & \text{if } M \text{ is torsion.} \end{cases}$$

(the order of M, in the usual sense)

More generally, if M is a fin. gen. module over a principal ideal domain, then M has the form

$$M \cong R^n \oplus R/a_1 \oplus \dots \oplus R/a_k \quad (a_1, \dots, a_k \in R \setminus 0),$$

and $$\mathcal{O}(M) = \begin{cases} 0 & \text{if } n = 0 \\ \langle a_1 \cdots a_k \rangle & \text{if } n \geq 1 \end{cases}$$

BACK TO KNOTS — The Alexander polynomial

We will show that the Alexander Λ-module $H_1(\hat{X}_K)$ is finitely presented and has a square presentation matrix A, so its order ideal is principal:

$$\mathcal{O}(H_1(\hat{X}_K)) = \langle \underbrace{\det(A)}_{\Lambda} \rangle .$$

(WARNING: Λ is not a principal ideal domain!)

Def. Any generator Δ_K of $\mathcal{O}(H_1(\hat{X}_K))$ will be called an <u>Alexander polynomial</u> of K.

"The" Alexander polynomial Δ_K is only defined up to multiplication by units of Λ. These are

$$\Lambda^\times = \{ \pm t^k \mid k \in \mathbb{Z} \}.$$

Recap:

comes with a $\mathbb{Z}$-action — a Λ-module

$$K \rightsquigarrow X_K \rightsquigarrow \hat{X}_K \rightsquigarrow H_1(\hat{X}_K)$$

oriented knot — exterior — infinite cyclic cover — Alexander module

$$\rightsquigarrow \mathcal{O}(H_1(\hat{X}_K)) \rightsquigarrow \Delta_K$$

order ideal — Alexander polynomial

principal, because $H_1(\hat{X}_K)$ has a square presentation matrix (we are on our way to proving this)

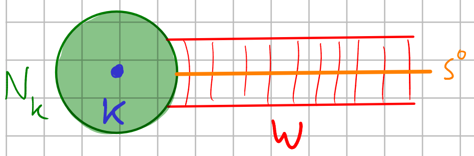
Ex. For U the unknot, $H_1(\hat{X}_U) = 0$.
The 0×0 "empty" matrix is a presentation matrix for 0, and its determinant is 1. So the order ideal is $\langle 1 \rangle = \Lambda$, and

$$\Delta_U(t) = 1$$

(If the empty matrix makes you uncomfortable, find a different presentation matrix for the trivial Λ-module and reach the same conclusion.)

A model of $\widehat{X}_k$

Let $S \subset S^3$ be a Seifert surface of genus g for k, and assume X_k has been constructed by removing a tubular neighborhood N_k of k such that $S \cap N_k$ is an annulus transverse to N_k.

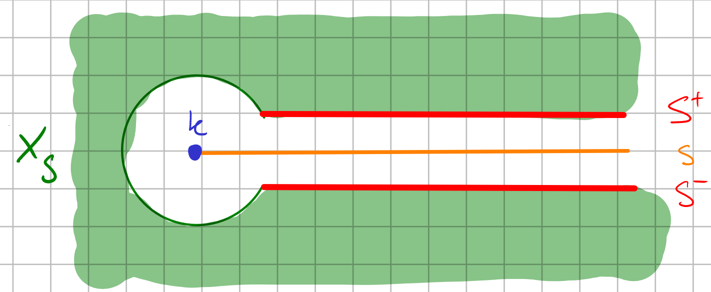


We write $S^0 := S \cap X_k$.

Fact. S^0 has a neighborhood $W \subset X_k$ homeomorphic to $S^0 \times [-1,1]$, with $W \cap \partial X_k$ corresponding to $\partial S^0 \times [-1,1]$

Assume the $[-1,1]$ factor is oriented like the normal bundle of S^0. $S^0 \times \{-1, 1\} \subset \partial W$ consists of two parallel copies of S^0, which we denote by S^-, S^+.

Write $N := N_k \cup W$, $\quad X_S := S^3 \setminus N$



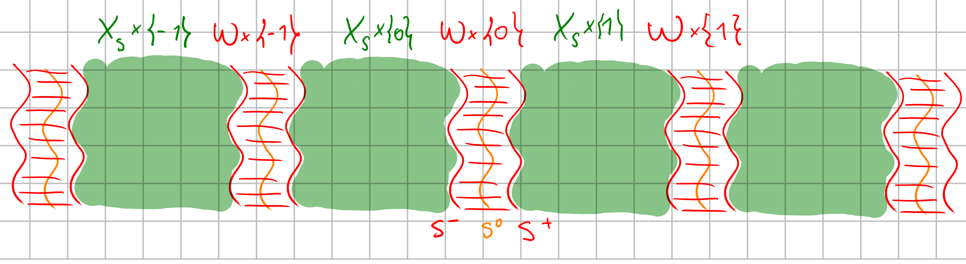
We construct $\widehat{X}_k$ as the quotient

$$\widehat{X}_k = \left(W \times \mathbb{Z} \;\sqcup\; X_S \times \mathbb{Z} \right) \big/ \sim \, ,$$

where we identify for $t \in \mathbb{Z}$, $x \in S^-$, $y \in S^+$:

$$\underset{W \times \mathbb{Z}}{\underset{\cap}{(x,t)}} \sim \underset{X_S \times \mathbb{Z}}{\underset{\cap}{(x,t)}} \qquad \underset{W \times \mathbb{Z}}{\underset{\cap}{(y,t)}} \sim \underset{X_S \times \mathbb{Z}}{\underset{\cap}{(y,t+1)}}$$

$\widehat{X}_k$:



The covering map $\widehat{X}_k \to X_k$ is given by forgetting the $\mathbb{Z}$-factors.

The subspaces $W \times \mathbb{Z}$ and $X_s \times \mathbb{Z}$ each come with a $\mathbb{Z}$-action on the second factor, and clearly:

$$\mathrm{Aut}\begin{pmatrix} \widehat{X}_k \\ \downarrow \\ X_k \end{pmatrix} \cong \mathbb{Z}$$

This implies that $\widehat{X}_k$ is indeed the universal abelian covering (why?).

We will compute $H_1(\widehat{X}_k)$ from the Mayer-Vietoris sequence

$$\begin{array}{ccccc} H_1(\mathbb{Z}\times S^{\pm}) & \to & H_1(\mathbb{Z}\times W) \oplus H_1(\mathbb{Z}\times X_s) & \longrightarrow & H_1(\widehat{X}_k) \\ \Vert\wr & & \Vert\wr \quad\quad\quad \Vert\wr & & \\ \bigoplus_{t\in\mathbb{Z}} H_1(S^{\pm}) & & \bigoplus_{t\in\mathbb{Z}} H_1(S) \quad \bigoplus_{t\in\mathbb{Z}} H_1(X_s) & & \\ \Vert\wr & & \Vert\wr & & \\ \Lambda^{4g} \cong \bigoplus_{t\in\mathbb{Z}} \mathbb{Z}^{4g} & & \bigoplus_{t\in\mathbb{Z}} \mathbb{Z}^{2g} \cong \Lambda^{2g} & & \end{array}$$

All maps are compatible with the $\mathbb{Z}$-action, so this is an exact sequence of Λ-modules.

Exercise: Show that the second map is surjective by inspecting the lower terms of the Mayer-Vietoris sequence.

Next goal: Understand the maps
$H_1(S^+) \to H_1(X_s)$
$H_1(S^-) \to H_1(X_s)$

Computing Δ_K

Prop. Let $(a_1, \dots, a_{2g})$ be any $\mathbb{Z}$-basis of $H_1(S) \cong \mathbb{Z}^{2g}$. Then $H_1(X_S) \cong \mathbb{Z}^{2g}$ and there is a $\mathbb{Z}$-basis $(\alpha_1, \dots, \alpha_{2g})$ of $H_1(X_S)$ such that, for every i,j:

$$lk(a_i, \alpha_j) = \delta_{ij} := \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

"$(\alpha_j)_j$ is dual to $(a_i)_i$ w.r.t. the linking pairing"

In other words, the expression of any $\alpha \in H_1(X_S)$ in this basis is

$$\alpha = \sum_{i=1}^{2g} lk(a_i, \alpha)\, \alpha_i$$

For each $i \in \{1, \dots, 2g\}$, we have $\alpha_i = \sum_j \delta_{ij} \alpha_j = \sum_j lk(a_j, \alpha_i)\alpha_j$, so given $\alpha = \sum_i m_i \alpha_i$, we have

$$\alpha = \sum_i m_i \sum_j lk(a_j, \alpha_i)\alpha_j = \sum_j lk\Big(a_j, \sum_i m_i \alpha_i\Big)\alpha_j = \sum_j lk(a_j, \alpha)\alpha_j .$$

Pf. We first show:

Claim It suffices to prove the statement for <u>some</u> $\mathbb{Z}$-basis $(a_1, \dots, a_{2g})$.

Pf Suppose $(a_1, \dots, a_{2g})$ has a dual basis $(\alpha_1, \dots, \alpha_{2g})$ and let $(b_1, \dots, b_{2g})$ be a second basis of $H_1(S)$.

Then there is $M \in GL(2g, \mathbb{Z})$ such that $\begin{pmatrix} b_1 \\ \vdots \\ b_{2g} \end{pmatrix} = M \begin{pmatrix} a_1 \\ \vdots \\ a_{2g} \end{pmatrix}$.

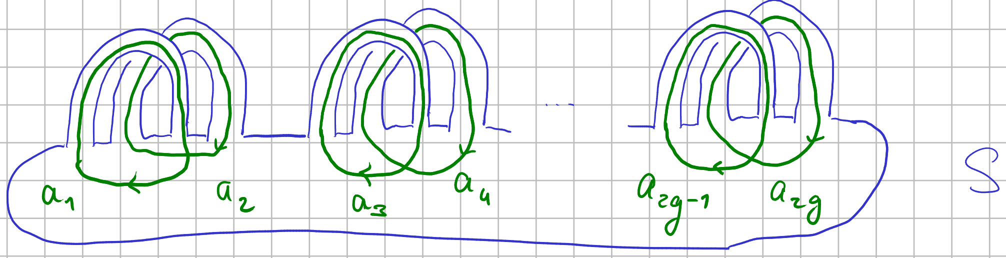
We have $\begin{pmatrix} a_1 \\ \vdots \\ a_{2g} \end{pmatrix} (\alpha_1 \dots \alpha_{2g}) = \mathrm{Id}$,

interpret the product of homology classes as taking linking numbers

Defining $\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{2g} \end{pmatrix} = (M^{-1})^T \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{2g} \end{pmatrix}$, we obtain:

$$\begin{pmatrix} b_1 \\ \vdots \\ b_{2g} \end{pmatrix} (\beta_1 \dots \beta_{2g}) = M \begin{pmatrix} a_1 \\ \vdots \\ a_{2g} \end{pmatrix} (\alpha_1 \dots \alpha_{2g}) M^{-1} = MM^{-1} = \mathrm{Id} \quad \square_{\text{Claim}}$$

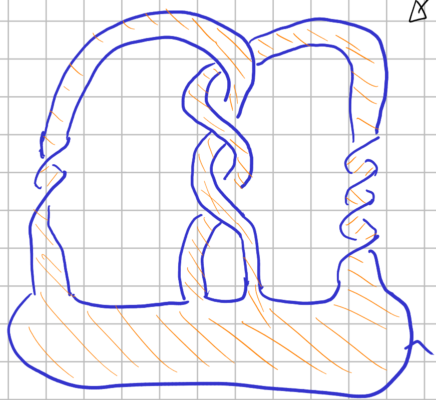
Fact. A surface of genus g with 1 boundary component (namely S) is abstractly homeomorphic to a disk with $2g$ bands attached:



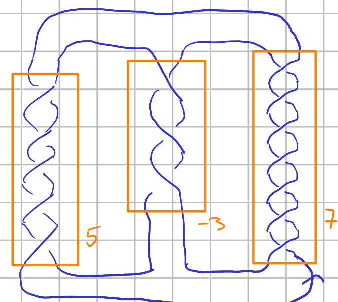
We will prove the proposition for a basis $(a_1, \dots, a_{2g})$ represented by the cores of these bands.

As an embedded subsurface, $S \subset \mathbb{S}^3$ is still a disk with bands attached in the same manner, but away from the disk, they may be twisted, linked and knotted.

Ex.



This is a Seifert surface for the $(5,-3,7)$-pretzel knot



Idea: Apply Mayer-Vietoris to the decomposition

$$\mathbb{S}^3 = N \cup_{\partial N} X_S$$

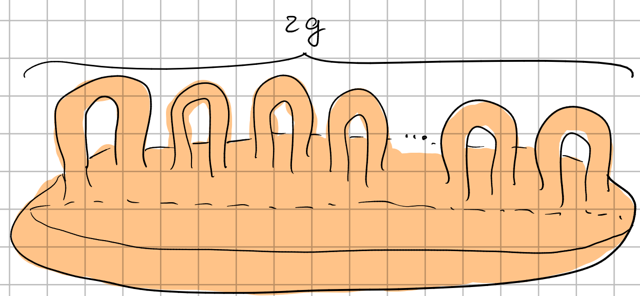
$$\underbrace{H_2(\mathbb{S}^3)}_{0} \longrightarrow H_1(\partial N) \xrightarrow{\cong} \underbrace{H_1(N)}_{\substack{\cong \\ H_1(S) \cong \mathbb{Z}^{2g}}} \oplus H_1(X_S) \longrightarrow \underbrace{H_1(\mathbb{S}^3)}_{0}$$

Now, N is a "thickening" of S inside $\mathbb{S}^3$.

So N is an oriented 3-manifold obtained by starting with a 3-ball (a neighborhood of the disk) and attaching $2g$ 1-handles (neighborhoods of the bands).

↳ a 3-dimensional 1-handle is a copy of $\mathbb{D}^1 \times \mathbb{D}^2$ attached along $\mathbb{S}^0 \times \mathbb{D}^2$

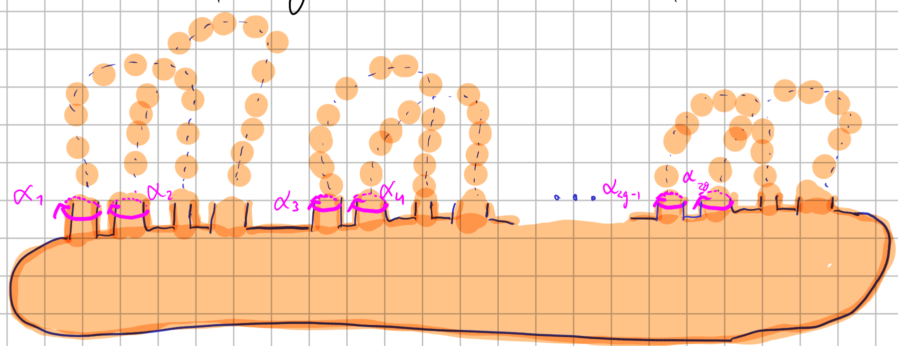
So N is a handlebody of genus $2g$.

$N \cong$ 

(The fact that the bands were interlocked in pairs disappears after thickening, but they may still be knotted)

· ∂N is a closed oriented surface of genus $2g$.

So: $H_1(\partial N) \cong \mathbb{Z}^{4g}$ and has a $\mathbb{Z}$-basis $(\gamma_1, \ldots, \gamma_{2g}, \alpha_1, \ldots, \alpha_{2g})$, where $\gamma_i \longmapsto a_i$ under $\partial N \hookrightarrow N \simeq S$, and each α_i is represented by a curve bounding a disk in the i-th 1-handle, and oriented compatibly with a_i. In particular, $lk(a_i, \alpha_j) = \delta_{ij}$.



Contemplate the isomorphism

$$\underset{\mathbb{Z}^{2g}}{\langle \gamma_i \rangle_{\mathbb{Z}}} \oplus \underset{\mathbb{Z}^{2g}}{\langle \alpha_i \rangle_{\mathbb{Z}}} = H_1(\partial N) \xrightarrow{\cong} \underbrace{H_1(S)}_{\mathbb{Z}^{2g} \cong \langle a_i \rangle_{\mathbb{Z}}} \oplus H_1(X_S).$$

We must have $H_1(X_S) \cong \mathbb{Z}^{2g}$ and since this map has

Moreover, this isomorphism has the form

$$\begin{array}{cc} & \begin{array}{cc} \langle \gamma_i \rangle_{\mathbb{Z}} & \langle \alpha_i \rangle_{\mathbb{Z}} \end{array} \\ \begin{array}{c} \langle a_i \rangle_{\mathbb{Z}} \\ H_1(X_S) \end{array} & \left[\begin{array}{c|c} \begin{smallmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{smallmatrix} & 0 \\ \hline * & * \end{array}\right] \end{array}$$

So the map $\langle \alpha_i \rangle_{\mathbb{Z}} \to H_1(X_S)$ is an isomorphism.

We identify the α_i with their images in $H_1(X_S)$. $\square$

Def. Let $a_1, \dots, a_{2g}$ be a $\mathbb{Z}$-basis for $H_1(S)$.

The $2g \times 2g$ matrix

$$A := \left(lk\,(a_i, a_j^+) \right)_{ij}$$

copy of a_j in the surface S^+

is called a <u>Seifert matrix</u> associated to the given basis.

Thm. If A is any Seifert matrix for a knot k, then the matrix $A - tA^T$ is a presentation matrix for the Alexander module $H_1(\widehat{X}_k)$.

In particular, the Alexander polynomial is given by

$$\Delta_k(t) = \det(A - tA^T).$$

(only defined up to mult. by $\pm t^k$)

Def. The matrix $A - tA^T$ is called an <u>Alexander matrix</u>.

Pf. Recall the exact sequence of Λ-modules

$$\bigoplus_{t\in\mathbb{Z}} H_1(S^-) \oplus \bigoplus_{t\in\mathbb{Z}} H_1(S^+) \longrightarrow \bigoplus_{t\in\mathbb{Z}} H_1(S) \oplus \bigoplus_{t\in\mathbb{Z}} H_1(X_S) \twoheadrightarrow H_1(\widehat{X}_k) \qquad (*)$$

Given any $\mathbb{Z}$-basis $(a_1, \dots, a_{2g})$ of $H_1(S) \cong H_1(S^-) \cong H_1(S^+)$, let $(\alpha_1, \dots, \alpha_{2g})$ be a dual basis of $H_1(X_S)$ with respect to the linking pairing (using the earlier proposition).

These are also Λ-bases for the corresponding terms in ⊛. In particular, since the modules in the middle and left are free over Λ, ⊛ gives a presentation for $H_1(\hat{X}_K)$

$$\overset{\langle a_i^-\rangle_\Lambda}{\bigoplus_{t\in\mathbb{Z}} H_1(S^-)} \oplus \overset{\langle a_i^+\rangle_\Lambda}{\bigoplus_{t\in\mathbb{Z}} H_1(S^+)} \longrightarrow \overset{\langle a_i\rangle_\Lambda}{\bigoplus_{t\in\mathbb{Z}} H_1(S)} \oplus \overset{\langle \alpha_i\rangle_\Lambda}{\bigoplus_{t\in\mathbb{Z}} H_1(X_S)} \twoheadrightarrow H_1(\hat{X}_K)$$

$$a_i^- \longmapsto \left(a_i\ ,\ a_i^- = \sum_j lk(a_j, a_i^-)\,\alpha_j\right)$$

$$a_i^+ \longmapsto \left(a_i\ ,\ t a_i^+ = t\sum_j lk(a_j, a_i^+)\,\alpha_j\right)$$

More explicitly:

$$H_1(\hat{X}_K) \overset{\text{as } \Lambda\text{-module}}{\cong} \langle \cancel{a_1}, \dots, \cancel{a_{2g}}, \alpha_1, \dots, \alpha_{2g} \mid \cancel{a_i} = a_i^- = t a_i^+ \rangle$$

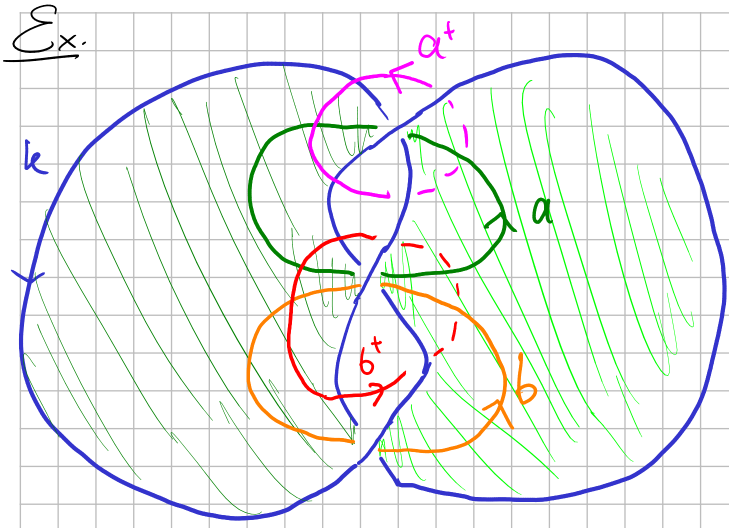
$$= \langle \alpha_1, \dots, \alpha_{2g} \mid a_i^- - t a_i^+ \rangle$$

$$= \Big\langle \alpha_1, \dots, \alpha_{2g} \ \Big|\ \sum_j \Big(\underbrace{lk(a_j, a_i^-)}_{\substack{= lk(a_j^+, a_i) \\ = lk(a_i, a_j^+)}} - t\, lk(a_j, a_i^+)\Big)\alpha_j \ \text{ for } i \in \{1, \dots, 2g\} \Big\rangle$$

Hence $H_1(\hat{X}_K)$ has presentation matrix

$$\big(lk(a_i, a_j^+) - t\, lk(a_j, a_i^+) \big)_{ij} = A - tA^T \qquad \square$$

Ex.



The Seifert matrix for the basis (a, b) is

$$A = \begin{pmatrix} lk(a, a^+) & lk(a, b^+) \\ lk(b, a^+) & lk(b, b^+) \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

So the Alexander polynomial of the (right-handed) trefoil is

$$\Delta_K(t) = \det(A - tA^T) = \det\begin{pmatrix} -1+t & 1 \\ -t & -1+t \end{pmatrix} = t^2 - t + 1$$

Some consequences

Prop The Alexander polynomial is symmetric:

$$\Delta_K(t) = \Delta_K(t^{-1})$$

Equality up to mult. by $\pm t^k$

Pf. Suppose A is a Seifert matrix of size $2g$.
Then up to mult. by $\pm t^k$:

$$\begin{aligned} \Delta_K(t) &= \det(A - tA^T) \\ &= (-t)^{-2g} \det(A - tA^T) \\ &= \det(-t^{-1}(A - tA^T)) \\ &= \det(A^T - t^{-1}A) \\ &= \det(A - t^{-1}A^T) = \Delta_K(t^{-1}) \end{aligned}$$

Cor. $\Delta_K = \Delta_{K^{rev}}$, where K^{rev} is K with reversed orientation.

Pf. Changing the orientation of K amounts to changing the covering automorphism $\tau : \hat{X}_K \to \hat{X}_K$ by which t acts, to τ^{-1}. That is, replacing the role of t by that of t^{-1}. $\square$

Cor. Δ_K is insensitive to mirroring: $\Delta_K = \Delta_{\overline{K}}$

Pf. The mirror of a Seifert surface S for K is a Seifert surface S' for $\overline{K}^{rev}$, so we can use a Seifert matrix associated to it to compute $\Delta_{\overline{K}^{rev}} = \Delta_{\overline{K}}$.

If A is a Seifert matrix associated to the basis $(a_i)_i$ of $H_1(S)$, then for the mirrored basis (a_i') of $H_1(S')$, we have
$$lk(a_i', (a_j')^+) = -lk(a_i, a_j^+),$$
so the resulting Seifert matrix is $A' = -A$.

Thus
$$\Delta_{\overline{K}}(t) = \Delta_{\overline{K}^{rev}}(t) = \det(-A - t(-A)^T) = \pm\det(A - tA^T) = \Delta_K(t). \quad \square$$

Def. The <u>degree</u> of an Alexander polynomial Δ_K is the exponent difference between the highest and lowest degree monomials. (This is clearly unchanged when we multiply $\Delta_K(t)$ by $\pm t^{\pm k}$).

Prop. The Alexander polynomial gives a lower (!) bound on the knot genus:
$$g(K) \geq \frac{1}{2}\deg(\Delta_K)$$

Pf. If A is a Seifert matrix produced from a minimal-genus Seifert surface, then A has size $2g(K)$
$$\deg(\Delta_K) = \deg(\det(\underbrace{A - tA^T}_{\text{all entries have degree } 1})) \leq 2g(K)$$

<u>Exercise</u> • Given two knots K, K', show that
$$\Delta_{K\#K'} = \Delta_K \cdot \Delta_{K'}$$

- Use this to give a new proof that the knot genus is additive under $\#$.

A group-theoretical approach

Let G be any group, suppose $G_{ab} = G/G' \cong \mathbb{Z}$, and fix a generator t of G_{ab}.

commutator subgroup

Let X be a connected space with $\pi_1(X) = G$ and $\hat{X}$ is its universal abelian cover. Then:

- $H_1(X) = G/G'$
- $\pi_1(\hat{X}) = G'$, so $H_1(\hat{X}) = G'/G''$.
- The canonical deck transformation action
$$H_1(X) = G/G' \circlearrowright \hat{X}$$
yields an action
$$\underset{G/G'}{\underset{\shortparallel}{H_1(X)}} \circlearrowright \underset{G'/G''}{\underset{\shortparallel}{H_1(\hat{X})}},$$
which is given by group conjugation.

(Exercise: Brush up on your covering theory and convince yourself of this!)

This makes $H_1(\hat{X})$ a Λ-module (where $\Lambda = \mathbb{Z}[t^{\pm 1}]$).
This Λ-module structure depends only on G and the chosen $t \in G_{ab}$, not on the space X.
In particular, the Alexander module $H_1(\hat{X}_k)$ of a knot k depends only on the group $\pi_1(X_k)$ and the generator $t \in H_1(X_k) \cong \mathbb{Z}$.

So one can compute the Alexander polynomial from any space X with $\pi_1(X) = \pi_1(X_k)$!

Ex. For the torus knot $K := T(p,q)$ $(p,q \geq 1$ coprime$)$:

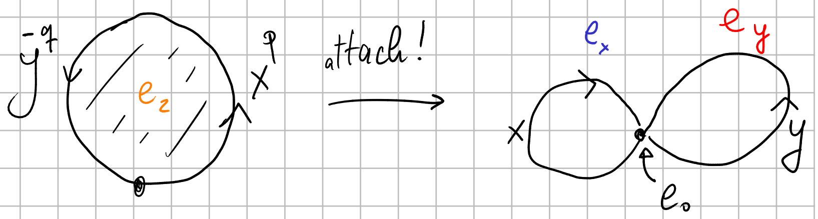
$$\pi_1(X_K) \cong \langle x, y \mid x^p = y^q \rangle =: G$$

The abelianization map $G \twoheadrightarrow \langle t \rangle$ is given by

$$x \mapsto t^q, \qquad y \mapsto t^p$$

Let X be the presentation complex for the above presentation:

X consists of a 2-cell attached to a wedge of two circles.



Each cell of X has "$\mathbb{Z}$-many" lifts in $\hat{X}$.

We wish to compute the first homology of the complex of Λ-modules

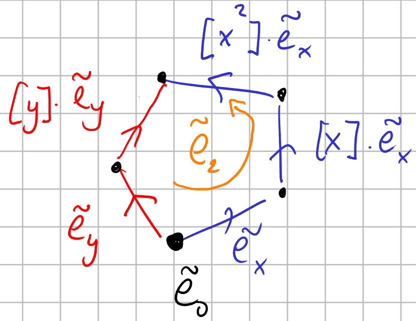
$$0 \to C_2(\hat{X}) \xrightarrow{\partial_2} C_1(\hat{X}) \xrightarrow{\partial_1} C_0(\hat{X})$$

$$C_2(\hat{X}) = \Lambda \tilde{e}_2, \qquad C_1(\hat{X}) = \Lambda \tilde{e}_x \oplus \Lambda \tilde{e}_y, \qquad C_0(\hat{X}) = \Lambda \tilde{e}_0$$

$\tilde{e}_0, \tilde{e}_x, \tilde{e}_y, \tilde{e}_2$ lifts at a common basepoint of $\hat{X}$.

$$\partial \tilde{e}_x = \underbrace{[x]}_{\text{class of } x \text{ in } G_{ab}} \cdot \tilde{e}_0 - \tilde{e}_0 = (t^q - 1)\tilde{e}_0$$

$$\partial \tilde{e}_y = (t^p - 1)\tilde{e}_0$$



$$\partial \tilde{e}_2 = \tilde{e}_x + [x]\tilde{e}_x + \dots + [x^{p-1}]\tilde{e}_x - \left(\tilde{e}_y + [y]\tilde{e}_y + \dots + [y^{q-1}]\tilde{e}_y\right)$$

$$= \left(1 + t^q + \dots + t^{(p-1)q}\right)\tilde{e}_x - \left(1 + t^p + \dots + t^{p(q-1)}\right)\tilde{e}_y$$

$$= \frac{1 - t^{pq}}{1 - t^q}\tilde{e}_x - \frac{1 - t^{pq}}{1 - t^p}\tilde{e}_y$$

(Warning: Here we write fractions but the denominators are not invertible.)

$$\ker(\partial_1) = \{ a\,\tilde{e}_x + b\,\tilde{e}_y \mid a(t^q - 1) + b(t^p - 1) = 0,\ a,b \in \Lambda \}$$

<u>Obs.</u> $\gcd(t^q - 1, t^p - 1) = t - 1$ (up to mult by $\pm t^k$)

Clearly $t-1$ divides both polynomials.

Let $m, n \in \mathbb{Z}$ be such that $mq + np = 1$.

Assuming WLOG that $mq \geq 1$, $np \leq 0$:

$$\underbrace{\frac{t^{mq}-1}{t^q - 1}(t^q - 1)}_{t^{mq}-1} - \underbrace{t\left(\frac{t^{-np}-1}{t^p - 1}\right)(t^p - 1)}_{t^{-np+1}-t} = -1 + t$$

The solutions to $a(t^q-1) + b(t^p - 1) = 0$ are the pairs (a,b) that are Λ-multiples of $\left(\frac{t^p-1}{t-1}, \frac{t^q-1}{t-1}\right)$.

So $\ker(\partial_1) = \Lambda \underbrace{\left(\frac{1-t^p}{1-t}\tilde{e}_x + \frac{1-t^q}{1-t}\tilde{e}_y \right)}_{=:z}$, a rank-1 free Λ-module.

This is necessarily a polynomial!

We have $\partial\tilde{e}_2 = \frac{(1-t^{pq})(1-t)}{(1-t^p)(1-t^q)} z$, so this polynomial is a 1×1 presentation matrix for the Alexander module.

So, we have:

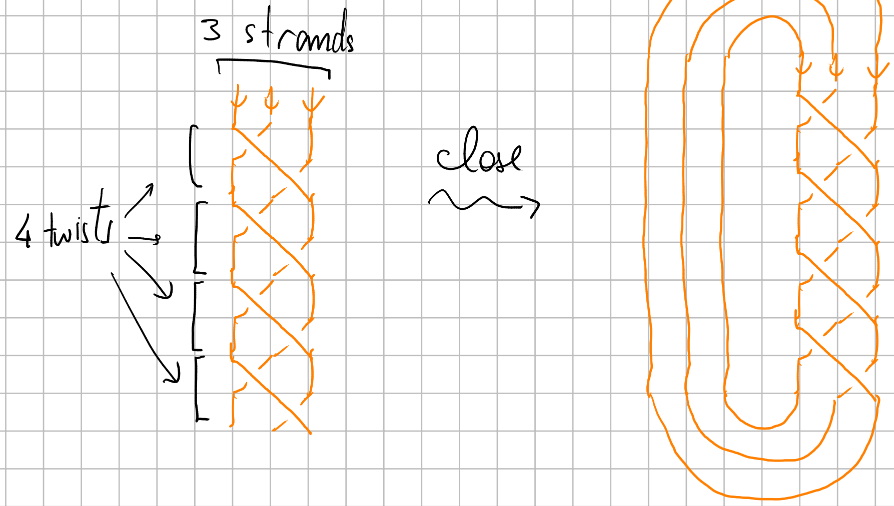
<u>Prop.</u>
$$\Delta_{T(p,q)}(t) = \frac{(1-t^{pq})(1-t)}{(1-t^p)(1-t^q)}.$$

<u>Cor.</u> The genus of a torus knot is
$$g(T(p,q)) = \tfrac{1}{2}(p-1)(q-1).$$

<u>Pf.</u> $(\geq)$ $g(T(p,q)) \geq \frac{1}{2}\deg \Delta_{T(p,q)} = \frac{1}{2}(pq + 1 - p - q)$
$$= \tfrac{1}{2}(p-1)(q-1)$$

($\leq$) $T(p,q)$ is the closure of the braid obtained by "twisting p strands q times":

Ex. $T(3,4)$



Using Seifert's algorithm on this diagram, we have
- p Seifert disks (one for each strand)
- $(p-1)$ crossings for each twist

$\rightsquigarrow q(p-1)$ crossings.

The genus of the resulting Seifert surface is thus:

$$g = \frac{1}{2}(c - s + 1) = \frac{1}{2}\left(q(p-1) - p + 1\right) = \frac{1}{2}(q-1)(p-1). \quad \square$$

Cor. $T(p,q)$ is trivial if and only if $p=1$ or $q=1$.