7. The fundamental group

Recall. • For every knot k, we have $\pi_1(X_k)_{ab} = H_1(X_k) \cong \mathbb{Z}$.

• For U the unknot, $\pi_1(X_U) \cong \mathbb{Z}$.

This characterizes the unknot:

Prop For every knot k:

$$\pi_1(X_k) \cong \mathbb{Z} \iff k \text{ is the unknot.}$$

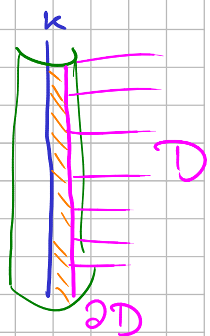
Pf of ($\Rightarrow$) We will need:

The Loop Theorem [Papakyriakopoulos]

don't need "compact" nor "oriented"

Let M be a smooth 3-mfd and C a connected component of ∂M. Suppose the inclusion-induced map $\pi_1(C) \to \pi_1(M)$ is not injective. Then there is an embedding $f: \mathbb{D}^2 \hookrightarrow M$ with $f^{-1}(C) = \mathbb{S}^1$, such that $f|_{\mathbb{S}^1}$ is nontrivial in $\pi_1(C)$.

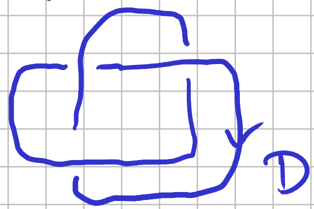
So suppose $\pi_1(X_k) \cong \mathbb{Z}$. In particular, the map $\pi_1(\partial X_k) \to \pi_1(X_k)$ is not injective. So there is an embedded disk $D \subset X_k$ with $D \cap \partial X_k = \partial D$ ("D is properly embedded") and ∂D a nontrivial curve in ∂X_k. Since ∂D is homologically trivial in X_k, it is (up to orientation) a longitude of k. So D can be enlarged to an embedded disk $D_+ \subset \mathbb{S}^3$ with $\partial D_+ = k$. $\square$



Exercise. Show that this generalizes to links: If L is an m-component link, then

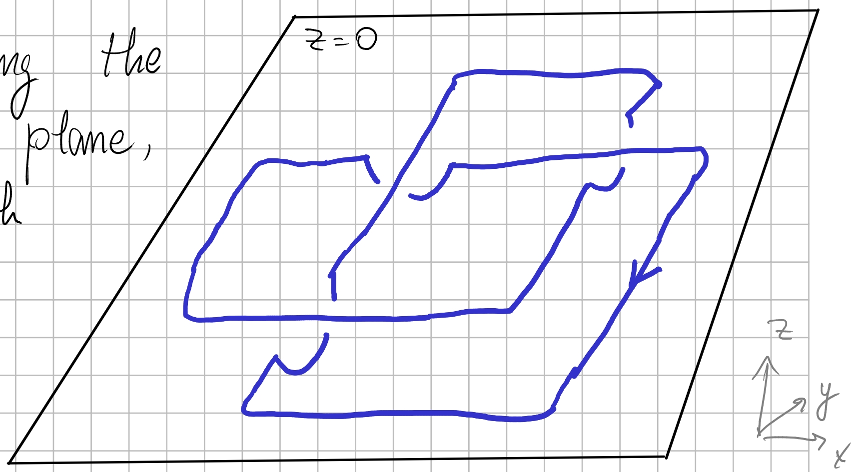
$$\pi_1(X_L) \cong F_m \iff X_L \text{ is an unlink of unknots.}$$

Next goal: Read off a presentation of $\pi_1(X_L)$ from a link diagram D. (We will assume no component of D is a circle)



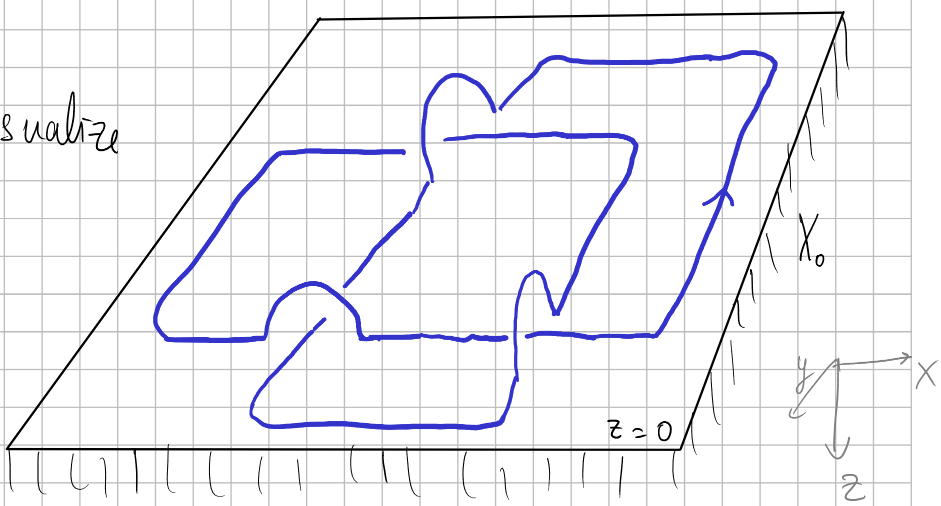
Idea: Build X_L by starting with a 3-ball, and successively attaching spaces to it. Then apply Seifert-van Kampen's theorem.

Step 0. Construct L by placing the arcs of D on the $z=0$ plane, and connecting them with short arcs below:

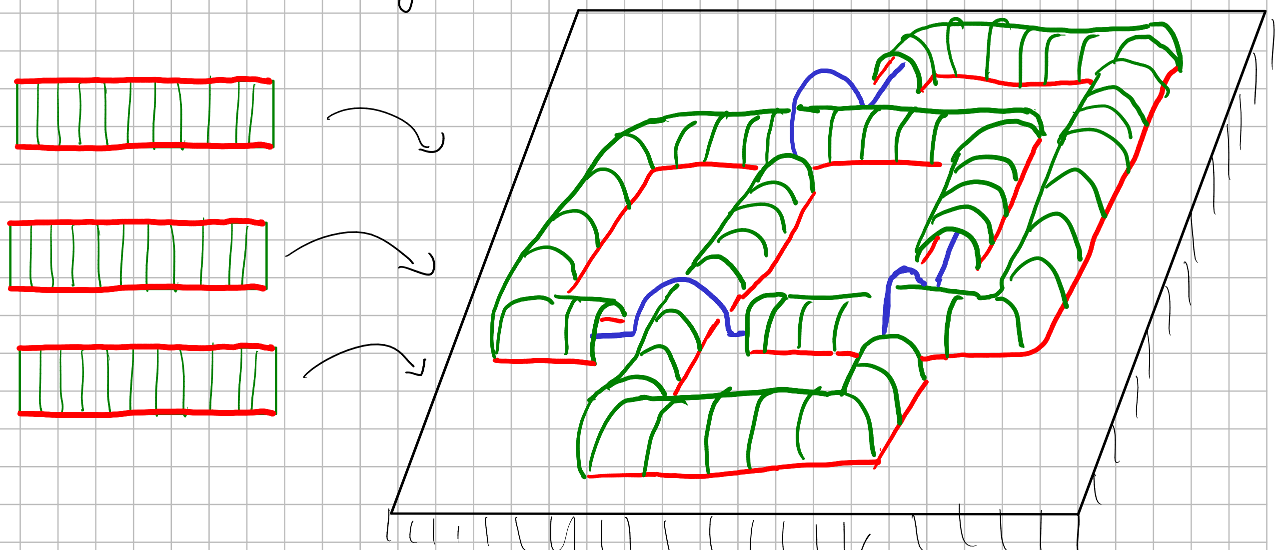


Visualizing S^3 as $\mathbb{R}^3 \cup \{\infty\}$, the North hemisphere corresponds to the upper half-space of $\mathbb{R}^3$, together with ∞. This is the 3-ball X_0 with which we start the construction.

For convenience of illustration, let us visualize this plane from below:

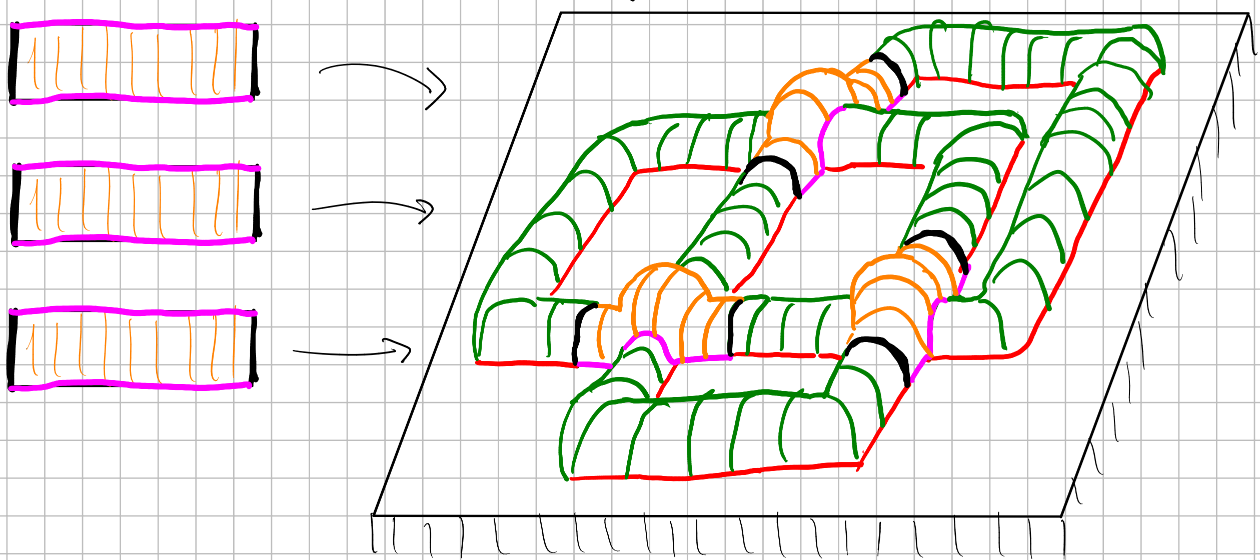


Step 1. For each arc α, attach a band $b_\alpha = [0,1] \times [0,1]$ along $[0,1] \times \{0,1\}$, building a "tunnel" encasing α:



Denote by X_1 the resulting space.

Step 2. Complete the tunnel around L by attaching one band at each crossing. This time, each band is attached along its entire boundary:

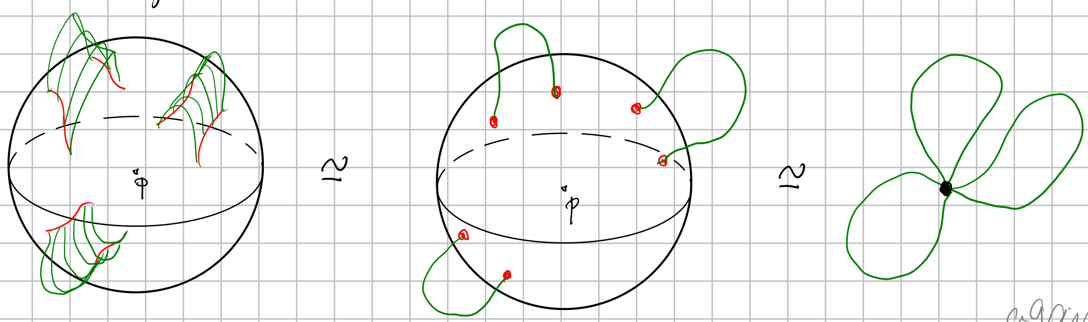


Call this space X_2.

Step 3. Attach a 3-ball along its boundary to complete the space X_L.

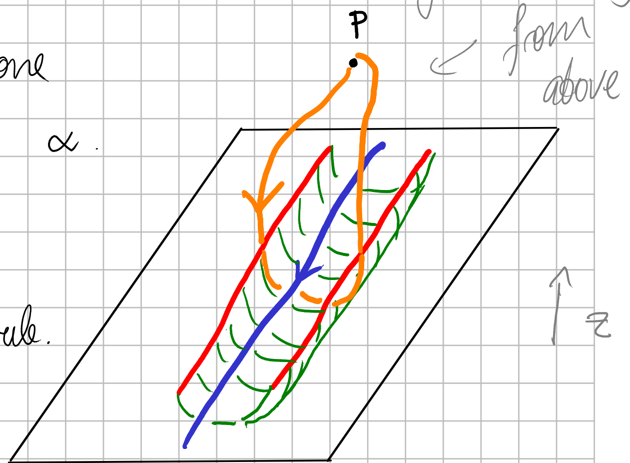
Next: Compute the fundamental group of these spaces, with basepoint any $p \in X_0 \setminus \partial X_0$.

- X_0 is a 3-ball, so $\pi_1(X_0, p) = 1$.
- X_1 is homotopy-equivalent to a bouquet of circles, one for each arc:

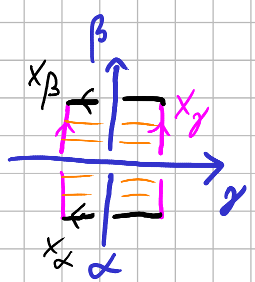


So $\pi_1(X_1, p)$ is free, with one free generator x_α for each arc α.

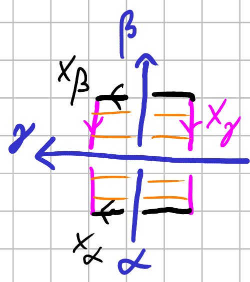
We may orient x_α compatibly with the orientation of α via right-hand rule. So x_α is represented by a meridian.



- With each crossing c, we attach a disk along its boundary, so we introduce a relation r_c:



or



$r_c = x_\gamma^{-1} x_\alpha x_\gamma x_\beta^{-1}$ $\qquad$ $r_c = x_\gamma x_\alpha x_\gamma^{-1} x_\beta^{-1}$

Thus, $\pi_1(X_2, p) \cong \langle x_\alpha, \ \alpha \text{ arc} \mid r_c, \ c \text{ crossing} \rangle$

- Attaching the final 3-ball does not change π_1.

So, we have:

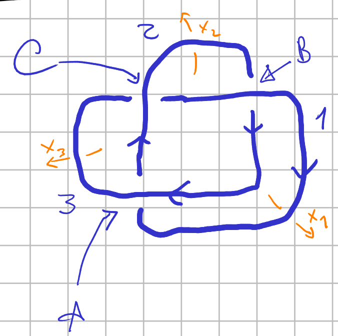
Thm. For every oriented link L with diagram D,

$$\pi_1(X_L) \cong \langle x_\alpha, \ \alpha \text{ arc of } D \mid r_c, \ c \text{ crossing of } D \rangle,$$

where:
- The relations have the form
$$r_c = x_\gamma^{\pm 1} x_\alpha x_\gamma^{\mp 1} x_\beta^{-1}$$
with γ the over-crossing arc at c, and α, β the two consecutive arcs of D at c.
- Each generator x_α is represented by a meridian.

This is called the <u>Wirtinger presentation</u>

Ex. Consider the following diagram of the trefoil:



We number the arcs and crossings.
The Wirtinger presentation has:
Generators: x_1, x_2, x_3
Relations:
$$r_A = x_3 x_1 x_3^{-1} x_2^{-1}$$
$$r_B = x_1 x_2 x_1^{-1} x_3^{-1}$$
$$r_C = x_2 x_3 x_2^{-1} x_1^{-1}$$

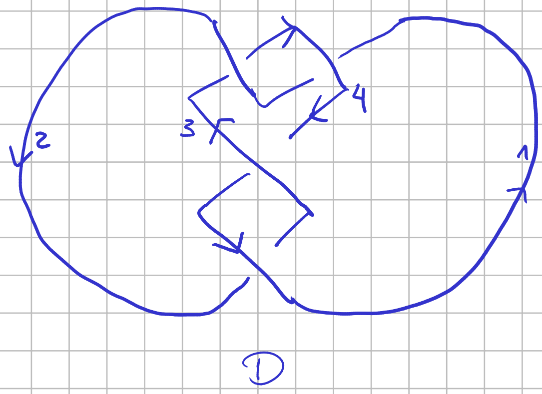
Exercise.
- Write down a Wirtinger presentation for $\pi_1(X_K)$, where K is the figure-8 knot.
- Use the Wirtinger presentation to ~~show~~ re-prove the fact that for every link L, the $\mathbb{Z}$-module $H_1(X_L)$ is free, with basis
$$\{[\mu_K] \mid K \text{ comp of } L\}.$$

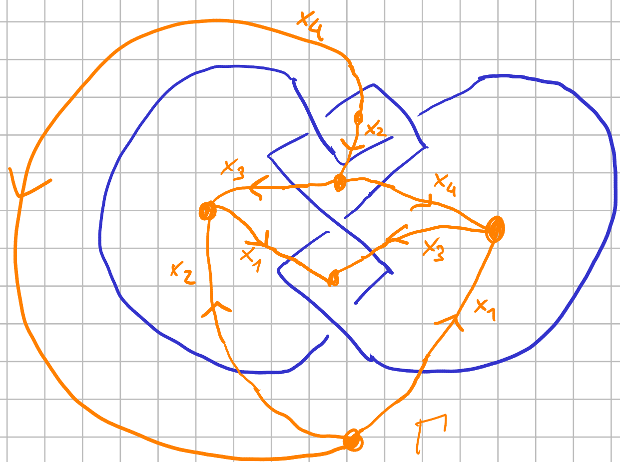
meaning: D has crossings.

Prop. Let D be a nontrivial oriented knot diagram.
Then each relator in the resulting Wirtinger presentation is a consequence of the others (and hence may be suppressed).

Proof sketch.

Regard D as living in the equatorial 2-sphere $S^2 \subset S^3$, and draw the graph Γ dual to D:

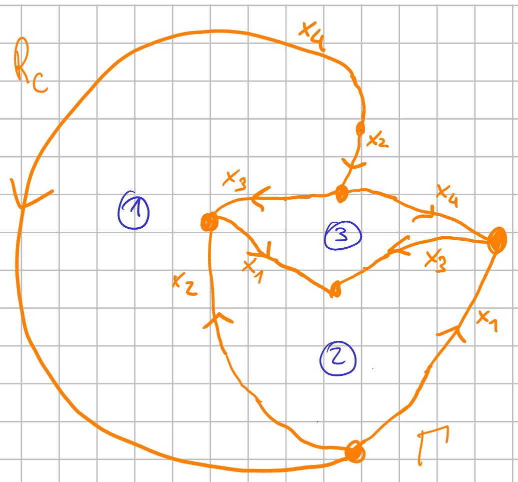




Γ has:
- vertices: components of $S^2 \setminus D$
- oriented edge $\overset{\bullet \longrightarrow \bullet}{A \quad B}$ if A is adjacent to B along a segment of some arc α having A to its right and B to its left.
  Label this edge with x_α.

Each crossing c of D corresponds to a region R_c of S^2 cut off by Γ. Its boundary ∂R_c is comprised of 4 edges, whose labels read the relation r_c.

The fact that, for each c, the region $S^2 \setminus \mathring{R}_c$ is a topological disk allows us to write r_c in terms of the other relations.



For example, let R_c be the "outside region".

$$r_c = x_4 x_1 x_4^{-1} x_2^{-1}$$

$$= \underbrace{x_4 (x_2 x_3^{-1} x_2^{-1})}_{r_1} (x_2 x_3 x_2^{-1}) x_1 x_4^{-1} x_2^{-1}$$

$$= r_1 \, x_2 x_3 x_2^{-1} x_1 \underbrace{(x_3 x_1^{-1})(x_1 x_3^{-1})}_{r_2} x_4^{-1} x_2^{-1}$$

$$= r_1 \, x_2 x_3 \, r_2 \underbrace{x_1 x_3^{-1} x_4^{-1} x_3}_{r_3} x_3^{-1} x_2^{-1}$$

$$= r_1 \, x_2 x_3 \, r_2 \, r_3 \, x_3^{-1} x_2^{-1}$$

(The disk formed by the regions 1, 2, 3 is a van Kampen diagram for the relation r_c.)

<u>Exercise</u>. Revisit the notes on Colorability and justify the unproven "Fact" that each row of the matrix M_0 is a linear combination of the others.

Colorings revisited

<u>Obs.</u> For every knot K, there is a unique group epimorphism
$$q: \pi_1(X_K) \twoheadrightarrow \mathbb{Z}/2.$$

[<u>Pf.</u> Every such homomorphism factors through the abelianization $H_1(X_K) \cong \mathbb{Z}$. And there is only one epimorphism $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2$.]

Let $n \in \mathbb{N}_{\geq 1}$, and recall the dihedral group $D_n = \mathbb{Z}/n \rtimes \mathbb{Z}/2$, which has a canonical epimorphism $D_n \twoheadrightarrow \mathbb{Z}/2$.

<u>Prop.</u> For every diagram D of a knot K, there is a bijection
$$\Phi: \{n\text{-colorings of } D\} \xrightarrow{\cong} \{\text{homomorphisms } f: \pi_1(X_K) \to D_n \text{ lifting } q\}$$

that is, fitting into
$$\begin{tikzcd} \pi_1(X_K) \arrow[r, "f"] \arrow[dr, "q"'] & D_n \arrow[d, two heads] \\ & \mathbb{Z}/2 \end{tikzcd}$$

<u>Pf.</u> Given any function $c: \{\text{arcs } \alpha \text{ of } D\} \to \mathbb{Z}/n$, define, for each arc α of D,
$$f_c(x_\alpha) := (c(\alpha), 1) \in D_n$$

generator in the Wirtinger pres.

<u>Exercise:</u> Show that this defines a homomorphism
$$f_c: \pi_1(X_K) \longrightarrow D_n$$
if and only if c is a coloring.

<u>Hint:</u> Recall that inverses in D_n are given by
$$(m,0)^{-1} = (-m,0) \quad \text{and} \quad (m,1)^{-1} = (m,1).$$

For every coloring c, the composition $\pi_1(X_K) \xrightarrow[f_c]{} D_n \twoheadrightarrow \mathbb{Z}/2$ is nontrivial, so it is q.

Hence, we may define
$$\Phi: \{n\text{-colorings of } D\} \longrightarrow \{\underbrace{f: \pi_1(X_K) \to D_n \text{ lifting } q}_{*}\}$$
$$c \longmapsto f_c.$$

Φ is clearly injective. To see Φ is surjective:

every f as in $\circledast$ fits into the diagram

maps every x_α to the same class $[\mu_k]$

$$\begin{tikzcd} \pi_1(X_k) \arrow[r, "f"] \arrow[d] \arrow[dr, "q"] & D_n \arrow[d] \\ H_1(X_k) \arrow[r, two heads] & \mathbb{Z}/2 \end{tikzcd}$$

some function $\{\text{arcs}\} \to \mathbb{Z}/n$

So f is of the form $x_\alpha \mapsto (c(\alpha), 1)$.

By the "only if" part of the exercise, c is a coloring. $\square$

Link exteriors as Eilenberg-MacLane spaces

Prop. Let L be a nonempty link. Then:

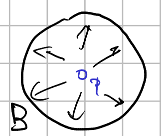
for all $k \geq 2$, $\pi_k(X_L) = 0$

$$X_L \text{ is aspherical} \iff L \text{ is not split.}$$

In particular, knot complements are aspherical.

Pf. $(\Rightarrow)$ **Obs.** If L is split, then X_L retracts to the splitting sphere S.

Pf. A ball minus any interior point $B \setminus \{p\}$ retracts to ∂B.



tubular neighborhood of L_i

So each $B_i \setminus N_i$ also retracts to S.

Two such retractions assemble to a retraction $X_L \xrightarrow{r} S$.

Choosing any diffeomorphism $\varphi : \mathbb{S}^2 \to S$, we see the composition

$$\mathbb{S}^2 \xrightarrow{\varphi} S \xhookrightarrow{\iota} X_L \xrightarrow{r} S \xrightarrow{\varphi^{-1}} \mathbb{S}^2$$

is the identity. So the identity automorphism of $\pi_2(\mathbb{S}^2)$ factors as

$$\underbrace{\pi_2(\mathbb{S}^2)}_{\cong\, \mathbb{Z}} \xrightarrow{\iota \circ \varphi} \pi_2(X_L) \xrightarrow{\varphi^{-1} \circ r} \underbrace{\pi_2(\mathbb{S}^2)}_{\cong\, \mathbb{Z}}$$

Thus $\pi_2(X_L) \neq 0$, so X_L is not aspherical.

$(\Leftarrow)$ We need two important results.

① <u>The Sphere Theorem</u> (Papakyriakopoulos). If X is a smooth orientable 3-manifold with $\pi_2(X) \neq 0$, then there is a <u>smooth embedding</u> $\mathbb{S}^2 \hookrightarrow X$ representing some nontrivial element of $\pi_2(X)$.

② <u>Generalized Schönflies Theorem</u> If $n \neq 4$, then for every smooth embedding $\mathbb{S}^{n-1} \xhookrightarrow{f} \mathbb{S}^n$, there is a self-diffeomorphism $\varphi : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ such that the composition

$$\mathbb{S}^{n-1} \xhookrightarrow{f} \mathbb{S}^n \xrightarrow{\varphi} \mathbb{S}^n$$

is the standard inclusion of the equatorial sphere.

<u>Proof of ($\Leftarrow$)</u>

Suppose L is not split.

We show first that $\pi_2(\mathbb{S}^3 \setminus L) = 0$.

By the sphere theorem, it suffices to show every smooth embedding $f : \mathbb{S}^2 \hookrightarrow \mathbb{S}^3 \setminus L$ is null-homotopic.

By Schönflies, $f(\mathbb{S}^2) \subset \mathbb{S}^3$ bounds a 3-ball on each side.

Since L is not split, it is contained in one of these balls.

The other ball is thus contained in $\mathbb{S}^3 \setminus L$,

so $f : \mathbb{S}^3 \hookrightarrow \mathbb{S}^3 \setminus L$ is null homotopic.

Now, the universal cover $Y := \widetilde{\mathbb{S}^3 \setminus L}$ satisfies:

$\pi_1(Y) = 1$, so $H_1(Y) = 0$

Since covering maps induce isomorphisms on π_2

$\pi_2(Y) = 0$, so $H_2(Y) = 0$ (by Hurewicz)

Since Y is a non-compact 3-mfd,

$H_k(Y) = 0$ for all $k \geq 3$

Thus by Hurewicz, also $\pi_k(Y) = 0$ for all $k \geq 3$.

So also $\pi_k(\mathbb{S}^3 \setminus L) = 0$ for $k \geq 3$. $\square$

<u>Some context</u>: • For G any group, a connected CW-complex X with $\pi_1(X) = G$ and $\pi_k(X) = 0$ for $k \geq 2$ is called an <u>Eilenberg-MacLane space</u> of type $K(G,1)$.

- For every group G, one can construct such a space X, and it is unique up to homotopy equivalence.

- Besides having numerous applications in algebraic topology, spaces $K(G,1)$ provide a <u>canonical</u> homotopy type associated to G.
  So every homotopy-type invariant of X (e.g. its homology modules) is automatically a group invariant of G.
  This observation is the starting point of the vast subject of group (co)homology.

<u>Upshot</u>: If L is not split, then homotopy-type invariants of X_L are in fact group invariants of $\pi_1(X_L)$.