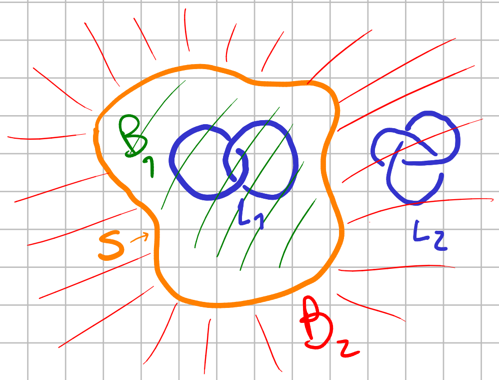
6. Linking phenomena

Split links

Def. A link $L \subset S^3$ is *split* if there are embedded 3-balls $B_1, B_2 \subset S^3$ such that

- $B_1 \cap B_2 = \partial B_1 = \partial B_2 =: S$ and $B_1 \cup B_2 = S^3$,
- $L \cap S = \emptyset$
- Each link $L_i := L \cap B_i$ is non-empty



The 2-sphere S is called a *splitting sphere* of L.

Prop. If $L = L_1 \cup L_2$ as above, then

$$\pi_1(X_L) \cong \pi_1(X_{L_1}) * \pi_1(X_{L_2}).$$

free product

Pf. We show: $\pi_1(S^3 \setminus L) \cong \pi_1(S^3 \setminus L_1) * \pi_1(S^3 \setminus L_2)$.

Since $S^3 \setminus L_1 = \left((S^3 \setminus L_1) \setminus \mathring{B}_2 \right) \underset{S}{\cup} B_2$, simply connected

Seifert-van Kampen tells us that the inclusion-induced map

$$\pi_1\left((S^3 \setminus L_1) \setminus \mathring{B}_2 \right) \longrightarrow \pi_1(S^3 \setminus L_1)$$

is an isomorphism. Similarly, $\pi_1\left((S^3 \setminus L_2) \setminus \mathring{B}_1 \right) \cong \pi_1(S^3 \setminus L_2)$.

Finally, since

$$S^3 \setminus L = \left((S^3 \setminus L_1) \setminus \mathring{B}_2 \right) \underset{S}{\cup} \left((S^3 \setminus L_2) \setminus \mathring{B}_1 \right),$$

we obtain

$$\pi_1(S^3 \setminus L) \cong \pi_1\left((S^3 \setminus L_1) \setminus \mathring{B}_2 \right) * \pi_1\left((S^3 \setminus L_2) \setminus \mathring{B}_1 \right)$$
$$\cong \pi_1(S^3 \setminus L_1) * \pi_1(S^3 \setminus L_2).$$

$\square$

Cor. If L is an unlink of m unknots, then

$$\pi_1(X_L) \cong F_m$$

Pf. $\pi_1(X_L) \cong \ast_m \pi_1(X_U) \cong \ast_m \pi_1(\underbrace{S^1 \times D^2}_{\simeq S^1}) \cong \ast_m \mathbb{Z} = F_m$ $\square$

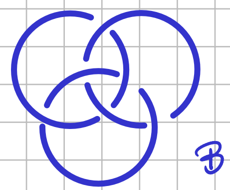
inductively split L

Cor. The Hopf link cannot be unlinked.

Pf. If L is an unlink of two unknots, then

$$\pi_1(X_L) \cong F_2 \not\cong \mathbb{Z}^2 \cong \pi_1(S^1 \times S^1 \times [0,1]) \cong \pi_1(X_H)$$

Exercise. In the Borromean link B, each two components form an unlink L of two unknots. Show that the third component represents a nontrivial element of $\pi_1(X_L)$, and conclude that B is not an unlink of three unknots.



The linking number

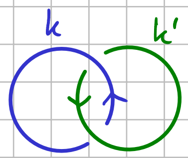
Let L be an oriented link with two components K, K'.
$H_1(S^3 \setminus K) \cong \mathbb{Z}$ has a preferred generator, represented by any meridian μ_K.

Def. The linking number $lk(K, K')$ is the integer such that, in $H_1(S^3 \setminus K)$, we have

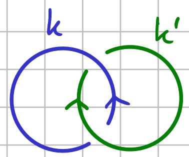
$$[K'] = lk(K, K') \cdot [\mu_K].$$

"$lk(K, K')$ counts how many times K' winds around K."

Ex.

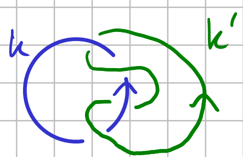


$lk(K, K') = lk(K', K) = 1$



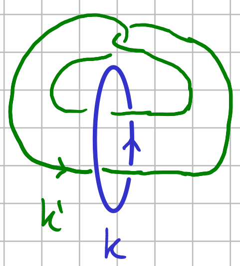
$lk(K, K') = lk(K', K) = -1$

So these links are not isotopic!



$lk(K, K') = 2$

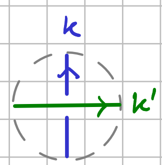
- If μ_K is a meridian of K, then $lk(K, \mu_K) = 1$.
- If $L = K \sqcup K'$ is split, then $lk(K, K') = 0$.



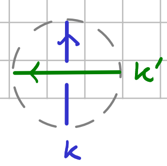
$lk(K, K') = 0$
(though the Whitehead link is nontrivial)

Given a diagram D of $L = K \sqcup K'$, the crossings of K' over K come in two types. Let

$n_+ :=$ number of positive crossings

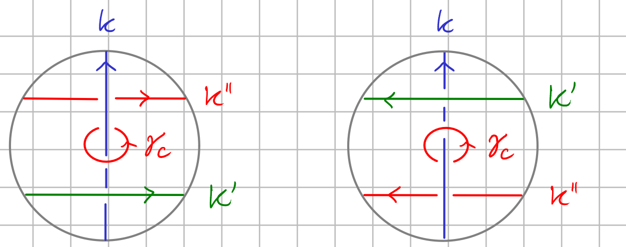


$n_- :=$ number of positive crossings

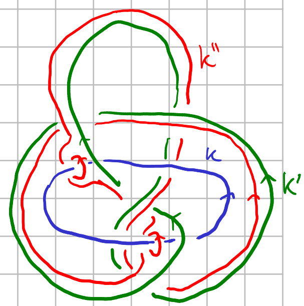


Prop. $lk(K, K') = n_+ - n_-$.

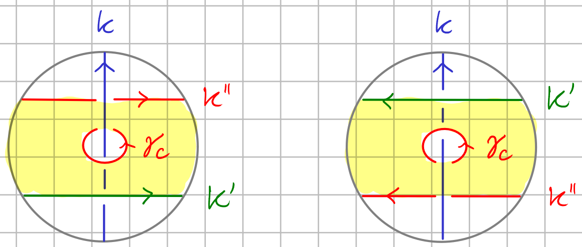
Pf. Add to the diagram a knot K'' parallel to K', (say following K' to its left) except that at every crossing of K' over K, the knot K'' crosses **under** K. For each such crossing c, add also a small circle γ_c around K as shown:

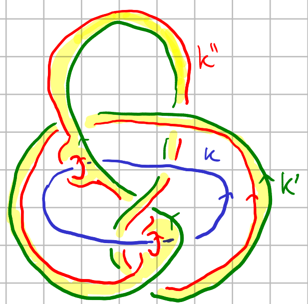


Ex.



Obs: 1. In $H_1(S^3 \setminus K)$, we have $[K'] = [K''] + \sum_c [\gamma_c]$:





The indicated surface exhibits K' as homologous to $K'' \cup \bigcup_c \gamma_c$.

2. K'' lies entirely "behind" K', so $K \cup K''$ is a split link. In particular, $lk(K, K'') = 0$.

3. For each crossing c, we have:

$$lk(K, \gamma_c) = \begin{cases} 1 & \text{if } c \text{ is positive} \\ -1 & \text{if } c \text{ is negative.} \end{cases}$$

So: $lk(K, K') = lk(K, K'') + \sum_c lk(K, \gamma_c)$ (1)

$= \sum_c lk(K, \gamma_c)$ (2)

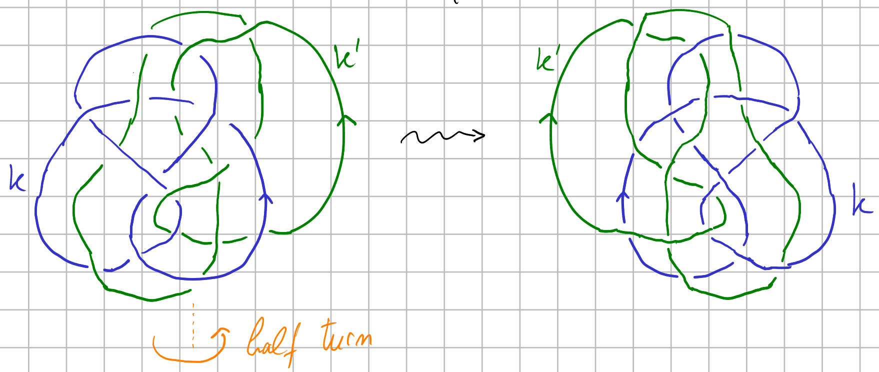
$= n_+ - n_-$ (3) $\square$

Exercise. The formula $lk(K, K') = n_+ - n_-$ is sometimes used as the definition of the linking number, in which case one must then show it is diagram-independent. Use Reidemeister's Theorem to perform this verification.

Cor. The linking number is symmetric:

$$lk(K, K') = lk(K', K)$$

Pf. Given a diagram of $K \cup K'$, consider the diagram obtained by rotating everything by a half turn around an axis in the projection plane:



This diagram change interchanges crossings of K' over K with crossings of K over K'. The signs of the crossings are preserved. □

Exercise. • How does $lk(K, K')$ change when K and K' are replaced by their mirror-images?

• What if we reverse the orientation of one of the knots?

If $L, L' \subset S^3$ are disjoint oriented links, define

$$lk(L, L') := \sum_{\substack{K \text{ comp. of } L \\ K' \text{ comp. of } L'}} lk(K, K')$$

If L is fixed, $lk(L, L')$ depends only on $[L'] \in H_1(X_L)$, and it varies $\mathbb{Z}$-linearly with $[L']$

<u>Upshot</u>: Given two disjoint subspaces $X, Y \subseteq S^3$, the linking number defines a symmetric bilinear form

$$\begin{aligned} H_1(X) \times H_1(Y) &\longrightarrow \mathbb{Z} \\ ([L], [L']) &\longmapsto lk(L, L'). \end{aligned}$$

Longitudes

Let L be an oriented link and K one of its components.

Recall that we have a canonical isomorphism

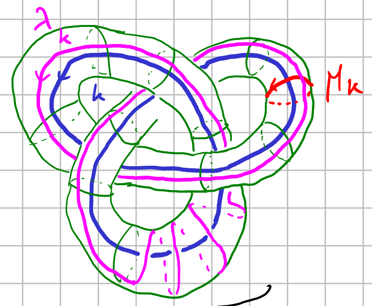
$$H_1(\partial X_K) \cong \underbrace{H_1(X_K)}_{\langle [\mu_K] \rangle_{\mathbb{Z}}} \oplus H_1(K)$$

Let $\lambda_K \subset \partial X_K$ be an oriented curve whose class $[\lambda_K] \in H_1(\partial X_K)$ maps to $(0, [K])$

<u>Def</u> Any curve in X_L isotopic to λ_K will be called a <u>longitude</u> of K.

<u>In other words</u>: One produces a longitude for K by choosing a curve $\lambda_K \subset \partial X_K$ that is "parallel to K" <u>and</u> satisfies

$$lk(K, \lambda_K) = 0$$



Twists added to ensure $lk(K, \lambda_K) = 0$

So we obtain a $\mathbb{Z}$-basis for ∂X_L:

$$\{ [\mu_K], [\lambda_K] \mid K \text{ component of } L \}$$

<u>Warning</u>: λ_K is null-homologous in X_K but not necessarily in X_L! In fact, a longitude of one component may be a meridian of another. Can you think of an example?

<u>Exercise</u> A curve $\lambda \subset \partial X_K \subset S^3$ is a longitude iff it is the transverse intersection of ∂X_K with a Seifert surface S for K

[Hint: $(\Leftarrow)$ If $\lambda = \partial X_K \cap S$, what can one deduce, homologically, from each component of $S \setminus\!\setminus \lambda$? ("cut along")

$(\Rightarrow)$ Use the previous part to produce a longitude λ' given by a Seifert surface S', and the fact that any two homologous curves on a torus are isotopic, to produce a Seifert surface S with $\lambda = S \cap \partial X_K$.]

Prop. Let S be a Seifert surface for an oriented knot K. Given an oriented curve $\gamma \subset S^3 \setminus K$ transverse to S,

$$lk(K,\gamma) = \underbrace{S \cdot \gamma}_{\text{signed intersection count}}$$

Pf. Build X_K using a "small enough" tubular neighborhood of K, so $\gamma \subset X_K$ and ∂X_K intersects S transversely along a longitude λ_K.
(Fact: Such a tubular neighborhood exists.)

Let $S_0 := S \cap X_K$.

$n := lk(K,\gamma)$, so $[\gamma] = n \cdot \underbrace{\iota_*}_{\iota:\, \partial X_K \hookrightarrow X_K}[\overbrace{M_K}^{\text{meridian in } \partial X_K}] \in H_1(X_K)$.

Then:

$$\begin{aligned}
S \cdot \gamma &= S_0 \cdot \gamma \\
&= \langle \underbrace{PD(\overbrace{[S_0]}^{\in H_2(X_K,\partial X_K)})}_{\in H^1(X_K)}, \overbrace{[\gamma]}^{\in H_1(X_K)} \rangle \\
&= n \cdot \langle PD([S_0]), \iota_*[M_K] \rangle \\
&= n \cdot \langle \underbrace{PD(\underbrace{[\partial S_0]}_{\in H_1(\partial X_K)})}_{\in H^1(\partial X_K)}, \underbrace{[M_K]}_{\in H_1(\partial X_K)} \rangle \\
&= n \langle PD([\lambda_K]), [M_K] \rangle \\
&= n \underbrace{(\lambda_K \cdot M_K)}_{1} \\
&= lk(K, \lambda_K)
\end{aligned}$$

$\square$