5. Link exteriors

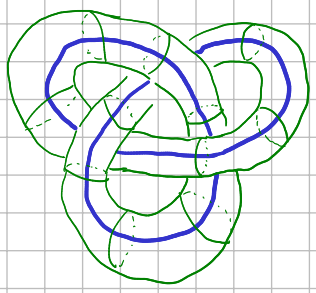
Recall: Given a link $L \subseteq S^3$, the oriented diffeomorphism type of $S^3 \setminus L$ is an invariant of L.

It is usually more convenient to work with compact spaces, so we will often use a slightly different space.

Def. Let M be a smooth m-manifold, and N a submanifold of dimension n. Assume also $\partial N = \emptyset$.

- A <u>tubular map</u> for N is a smooth embedding
$$\tau : N \times \mathbb{D}^{m-n} \longrightarrow M$$
such that $\tau(-, 0) : N \longrightarrow M$ is the inclusion of N.
- The image of such a map is called a <u>tubular neighborhood</u> of N.

Ex.



A tubular neighborhood for a trefoil in S^3.

The core curve of a Möbius band does not have a tubular neighborhood.

(because its normal bundle is not trivial.)

Fact. (Existence of tubular neighborhoods)
Every oriented link L has an orientation-preserving tubular map $\tau : L \times \mathbb{D}^2 \hookrightarrow S^3$.

(Links have trivial normal bundles.)

(We will soon see a uniqueness statement.)

Def. Let $L \subset S^3$ be a link and $\tau : L \times D^2 \to S^3$ a tubular map for L, let N be its image.
The oriented compact 3-manifold $X_L := S^3 \setminus \mathring{N}$ is called a link exterior of L.
Its boundary $\partial X_L = \partial N \cong L \times S^1$ is a disjoint union of tori.

Exercise.
- Show that the inclusion $X_L \hookrightarrow S^3 \setminus L$ is a homotopy equivalence.
- Show that X_L is connected.

Meridians

Assume now L is oriented, with m components.
The Mayer-Vietoris sequence associated to the decomposition

$$S^3 = X_L \underset{\partial X_L}{\cup} N$$

yields the exact sequence

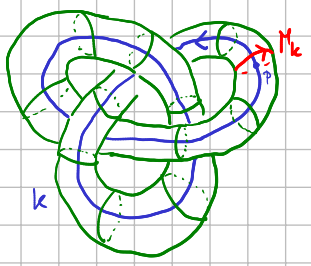
$$\underbrace{H_2(S^3)}_{0} \to H_1(\partial X_L) \xrightarrow{\cong} H_1(N) \oplus H_1(X_L) \to \underbrace{H_1(S^3)}_{0}.$$

$$H_1(\partial X_L) \cong \mathbb{Z}^{2m} \qquad H_1(N) \cong H_1(L) \cong \mathbb{Z}^m$$

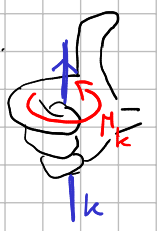
So $H_1(X_L) \cong \mathbb{Z}^m$. We will give an explicit $\mathbb{Z}$-basis.

For each component $k \subseteq L$, choose $p \in k$.
Consider the curve

$$\mu_k := \tau(\{p\} \times S^1) \subset \partial X_L$$



- The orientation of $\{p\} \times S^1$ defines an orientation of μ_k.
  Read it using the "right hand rule".



- The class $[\mu_k] \in H_1(\partial N)$ is independent of the chosen p.

Prop. The set $\{[\mu_k] \in H_1(X_L) \mid k \text{ component of } L\}$ is a $\mathbb{Z}$-basis of $H_1(X_L)$.

Pf.
- From the known homology of $\partial X_L \cong \bigsqcup_m (\mathbb{S}^1 \times \mathbb{S}^1)$, the classes $[\mu_k] \in H_1(\partial X_L)$ span a direct summand A, so $H_1(\partial X_L) = A \oplus B$ with $A, B \cong \mathbb{Z}^m$.
- Each μ_k is null-homologous in N, so the isomorphism $H_1(\partial X_L) = A \oplus B \longrightarrow H_1(N) \oplus H_1(X_L)$ has the form

$$\begin{array}{c} \\ H_1(N) \\ H_1(X_L) \end{array} \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 0 & * \\ * & * \end{bmatrix} \end{array}.$$

In particular, the restriction $A \longrightarrow H_1(X_L)$ is an isomorphism. □

Def. An oriented curve in $\mathbb{S}^3 \setminus L$ isotopic to μ_k will be called a meridian associated to k.

Exercise. Show that if $L \neq \emptyset$, then $H_2(X_L) \simeq \mathbb{Z}^{m-1}$.

(Showing $H_2(X_L)$ is torsion free might need Lefschetz duality and the Universal Coefficients Theorem!)

Fact (Uniqueness of tubular neighborhoods). Let τ, σ be orientation-preserving tubular maps for an oriented link L. Then there are a smooth isotopy $F: \mathbb{S}^3 \times [0,1] \longrightarrow \mathbb{S}^3$, and a smooth map $g: L \longrightarrow SO(2)$ s.th.:

- $F_0: \mathbb{S}^3 \longrightarrow \mathbb{S}^3$ is the identity.
- F is a homotopy rel L (that is, $F(-,t) = id_L$ for all $t \in [0,1]$)
- $F_1: \mathbb{S}^3 \longrightarrow \mathbb{S}^3$ restricts to the diffeomorphism

$$\begin{aligned} \sigma(L \times \mathbb{D}^2) &\longrightarrow \tau(L \times \mathbb{D}^2) \\ \sigma(x, v) &\longmapsto \tau(x, g(x)(v)) \end{aligned}$$

In particular, F_1 maps the tubular neighborhood given by σ to the one given by τ.

Cor. (1) For every two exteriors $X_L, X_{L'}$ of an oriented link L, there is an orientation-preserving diffeomorphism $X_L \to X_{L'}$, which sends every meridian to a meridian of the same component.
(2) If L, L' are isotopic, then there is an orientation-preserving diffeomorphism $X_L \to X_{L'}$ sending meridians to meridians

Pf. (1) Follows from uniqueness of tubular neighborhoods.
(2) Use the isotopy extension theorem to deform a tubular map of L into one of L'.

Prop. For U the unknot, X_U is a solid torus:

$$X_U \cong S^1 \times D^2.$$

Pf. Expressing $S^3 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$,
we have $U = \{(z,w) \in S^3 \mid w = 0\}$.

A tubular neighborhood of U is

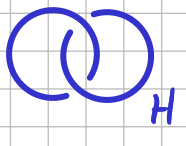
$$N_U := \{(z,w) \in S^3 \mid |w|^2 \leq \tfrac{1}{2}\},$$

(Exercise: write down a tubular map $U \times D^2 \to N_U$),
so N_U is a solid torus.

But then $X_U = \{(z,w) \in S^3 \mid |w|^2 \geq \tfrac{1}{2}\}$
$= \{(z,w) \in S^3 \mid |z|^2 \leq \tfrac{1}{2}\}$
is also obviously a solid torus. □

Exercise. Show that for H the Hopf link, X_H is a "thickened torus", that is,

$$X_H \cong S^1 \times S^1 \times [0,1]$$



Hint: You may regard $H \subset S^3 \subset \mathbb{C}^2$ as the subspace
$H = \{z = 0\} \cup \{w = 0\}$.

In the case of a knot K, the exterior X_K contains all information about K:

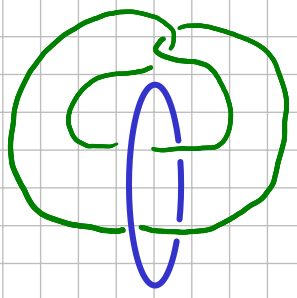
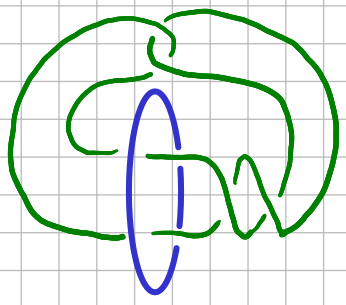
Thm (Gordon, Luecke). If K, K' are knots with $X_K \cong X_{K'}$ (or. pres. diffeo.), then $K \cong K'$.

(This is a deep theorem, which we will not prove.)

It is however not true that links are determined by their exteriors:

Exercise. Show that the following two links L, L' are not isotopic, but their exteriors satisfy $X_L \cong X_{L'}$.

L:



L':