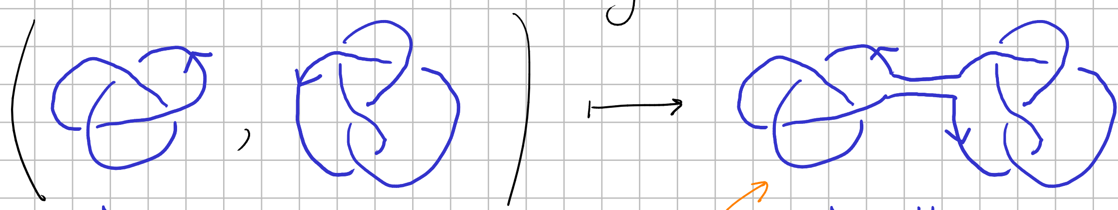
4. The connected sum of knots

We wish to define an operation on isotopy classes of oriented knots, which intuitively looks as follows:



K_1 K_2 $K_1 \# K_2$

Orientation compatible with that of K_1 and K_2

This operation will be called the <u>connected sum</u>.

Such a "definition by pictures" does not make it clear that $K_1 \# K_2$ is well-defined.

A rigorous construction uses a new ingredient.

<u>Def</u>. If N is a smooth submanifold of M, with M and N oriented, we say (M,N) is an <u>oriented manifold pair</u> of dimension (m,n), where $m = \dim(M)$ and $n = \dim(N)$.

<u>The Disk Theorem for pairs</u>.

Let (M,N) be an oriented (m,n)-manifold pair, with N connected, and $\partial M = \partial N = \emptyset$.

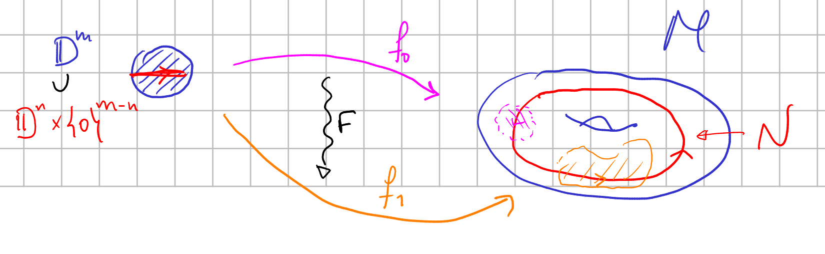
($\mathbb{D}^m$ closed unit ball in $\mathbb{R}^m$)

Let $f_0, f_1 : (\mathbb{D}^m, \mathbb{D}^n \times \{0\}^{m-n}) \hookrightarrow (M, N)$ be two orientation-preserving smooth embeddings of pairs.

Then there is smooth isotopy

$$F : (\mathbb{D}^m \times [0,1], \mathbb{D}^n \times \{0\}^{m-n} \times [0,1]) \to (M, N)$$

from f_0 to f_1.



By a version of the isotopy extension theorem for pairs, there is a smooth isotopy

$$G : (M, N) \times [0,1] \longrightarrow (M, N)$$

with $G_0 = id_M$ and $G \circ (f_0 \times id)$ a smooth isotopy from f_0 to f_1.

<u>Cor</u>. Fix $n \leq m$, and for each $i \in \{1,2\}$, let

- (M_i, N_i) be an oriented (m,n)-manifold pair with N_i connected and $\partial M_i = \partial N_i = \emptyset$.
- $f_i : (\mathbb{D}^m, \mathbb{D}^n \times \{0\}^{m-n}) \longrightarrow (M_i, N_i)$ a smooth embedding, with f_1 orientation-preserving and f_2 orientation-reversing, such that $f_i^{-1}(N_i) = \mathbb{D}^n \times \{0\}^{m-n}$.
- $M_1 \# M_2 := \left(M_1 \setminus f_1(0)\right) \sqcup \left(M_2 \setminus f_2(0)\right) / \sim$

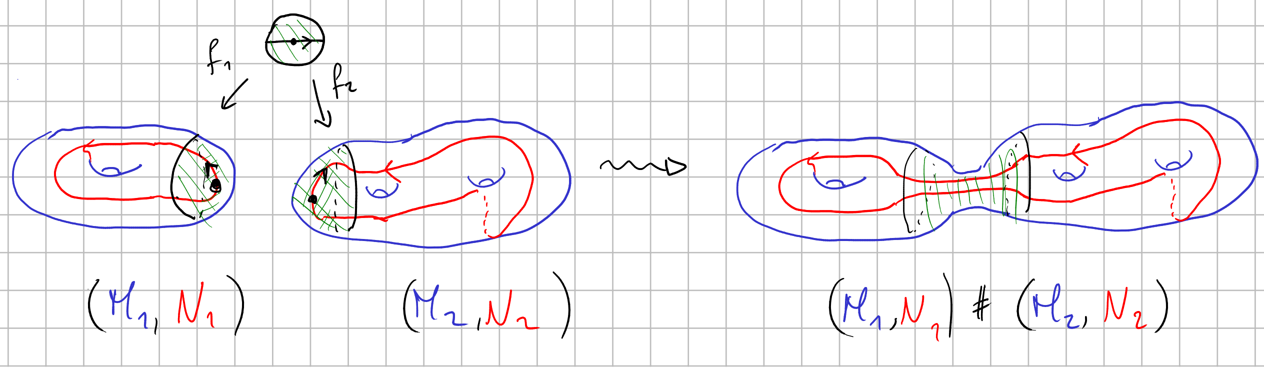
where we identify, for every $t \in]0,1[$ and $x \in \mathbb{S}^{m-1}$

$$f_1(tx) \sim f_2((1-t)x)$$

- $N_1 \# N_2 \subseteq M_1 \# M_2$ the image of $(N_1 \setminus f_1(0)) \sqcup (N_2 \setminus f_2(0))$.

Then $(M_1 \# M_2, N_1 \# N_2)$ is an oriented (m,n)-manifold pair whose oriented diffeomorphism type is independent of the maps f_1, f_2.

We call it the <u>connected sum</u> of (M_1, N_1) and (M_2, N_2)



Moreover: • $\#$ is well-defined (up to or. pres. diffeo) on orientation-preserving diffeo. classes of pairs:

$$(M_i', N_i') \cong (M_i, N_i) \Rightarrow (M_1', N_1') \# (M_2', N_2') \cong (M_1, N_1) \# (M_2, N_2)$$

• $\#$ is commutative, associative, and has $(S^m, S^n \times \{0\}^{m-n})$ as identity element.

BACK TO KNOTS!

Given (isotopy classes of) oriented knots $K, K' \subset S^3$, we have

$$(S^3, K) \# (S^3, K') \underset{\substack{\text{or. pres.}\\ \text{diff.}}}{\cong} (S^3, \underbrace{K \# K'}_{\cong S^1})$$

The pair $(S^3, K \# K')$ is only well-defined up to orientation-preserving diffeomorphism. But any two orientation-preserving diffeomorphisms of S^3 are smoothly isotopic (recall Cerf's Theorem).

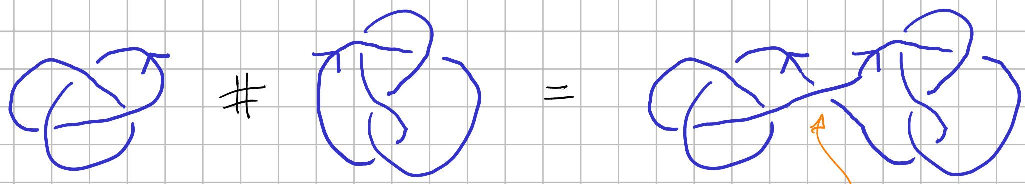
So the oriented knot $K \# K' \subset S^3$ is well-defined up to isotopy.

Upshot:

The connected sum $\#$ is a well-defined operation on the set of isotopy classes of oriented knots.

Moreover, $\#$ is associative, commutative, and has the unknot as identity element.

In practice, it is very easy to draw the connected sum of two oriented knots given as diagrams.
Just pay attention to the orientations!

Ex. 

We needed to add this crossing so orientations match!
How do we choose the over/under strand?

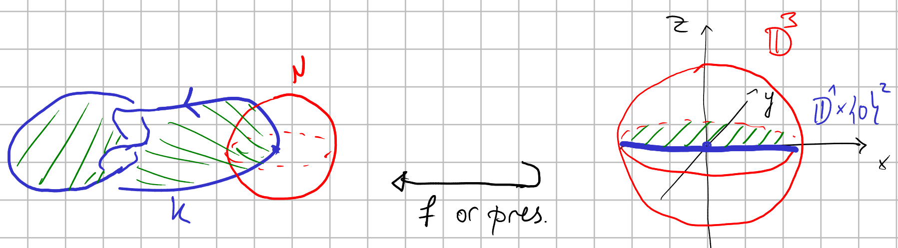
Theorem. The knot genus is additive with respect to #: for all oriented knots K, K', we have

$$g(K \# K') = g(K) + g(K')$$

Pf. $(\leq)$ Let S, S' be minimal genus Seifert surfaces for K and K'.

Fact. Each point of K has a closed neighborhood N such that

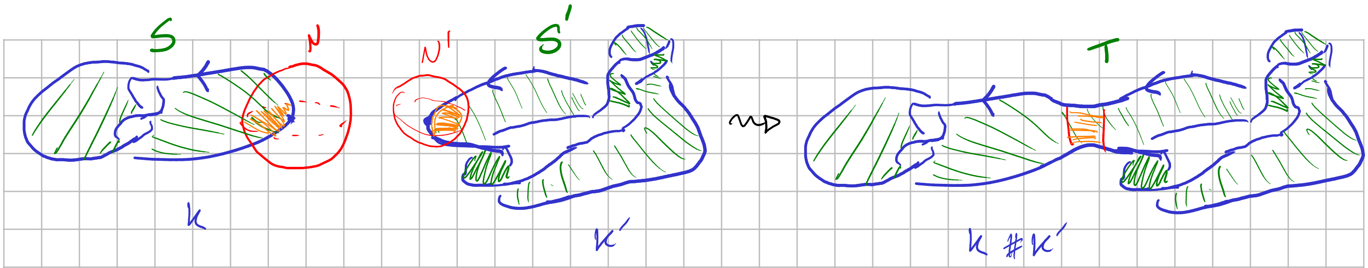
$$(N, N\cap S, N\cap K) \overset{\text{or. pres diff.}}{\cong} (\mathbb{D}^3, \mathbb{D}^3 \cap \mathbb{D}^1 \times \mathbb{R}_{\geq 0} \times \{0\}, \mathbb{D}^1 \times \{0\}^2)$$



Similarly, K', S' have N', f' orientation-reversing embedding.

Use f, f' to form the connected sum $K \# K'$.

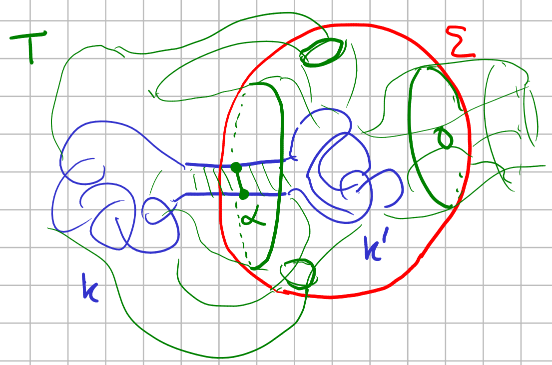
Then the image T of $S \setminus f(0) \cup S' \setminus f'(0)$ in $S^3 \# S^3$ is a Seifert surface for $K \# K'$.



We have $g(T) = g(S) + g(S')$.

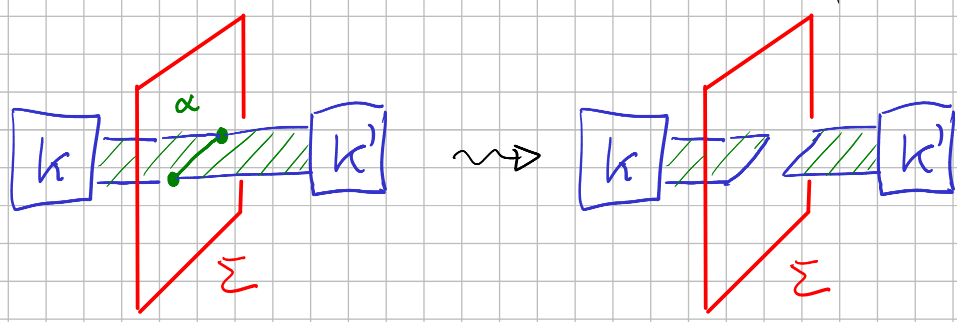
($\geq$) Let T be a minimal genus Seifert surface for $k \# k'$, let $\Sigma \subset S^3$ be a 2-sphere exhibiting $k \# k'$ as a connected sum.

After a small perturbation of Σ, may assume Σ and T are transverse (This uses the Transversality Theorem).



$T \cap \Sigma$ is a disjoint union of circles and one arc α connecting two points of $k \# k'$.

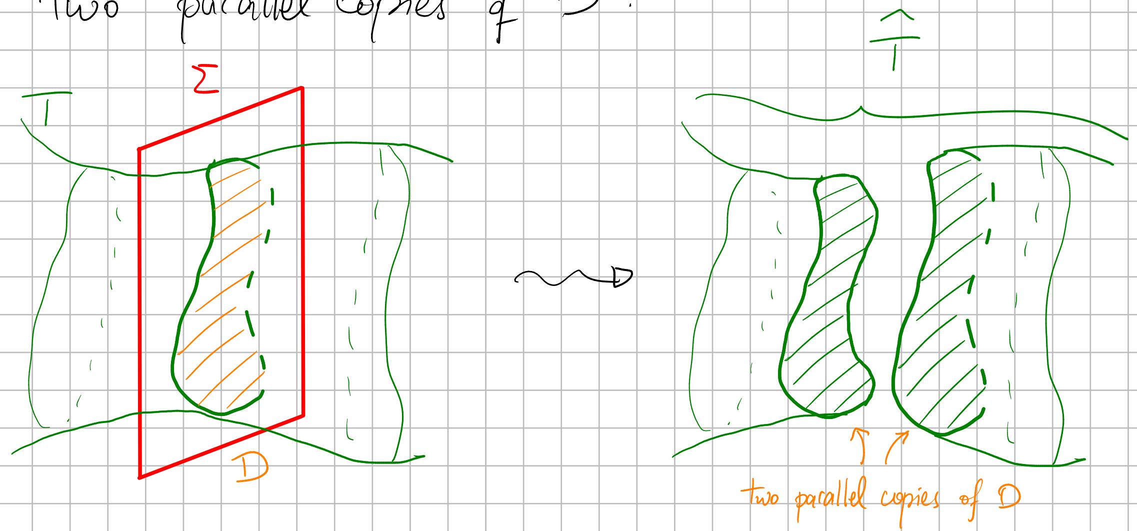
If there are no circles in $T \cap \Sigma$, cap T along α:



Since the resulting surface is disjoint from Σ, it is the union of two disjoint surfaces: a Seifert surface S for k and a Seifert surface S' for k'.

Thus $g(k) + g(k') \leq g(S) + g(S') = g(T) = g(k \# k')$.

Now suppose there are circles in $T \cap \Sigma$.
Each of them bounds two discs in Σ: one that contains α and one that doesn't.
Let D be an innermost disc among those that do not contain α. In other words, the interior of D is disjoint from T.
Consider the surface $\hat{T}$ obtained from capping T with two parallel copies of D:



<u>Claim</u>: $\hat{T}$ has two components.
Otherwise, $\hat{T}$ would be connected, and thus a Seifert surface for $K \# K'$. But this operation of capping with disks decreases genus by 1 (why?). This contradicts minimality of T.

So let T' be the component of $\hat{T}$ containing $K \# K'$. T' is a Seifert surface and $T' \cap \Sigma$ has fewer circles than $T \cap \Sigma$.
Repeat until we reach the case where there are no circles. $\square$

<u>Cor</u>. No nontrivial knot has an inverse for $\#$.

Def A nontrivial oriented knot K is called prime if for every oriented knots K_1, K_2 with $K \cong K_1 \# K_2$, we have $K_1 \cong \bigcirc$ or $K_2 \cong \bigcirc$.

Cor. Every oriented knot of genus 1 is prime

Cor. Every oriented knot K has a decomposition as a connected sum of prime knots.

Pf. The unknot is the empty connected sum.
If K is prime, we are done.
If K is nontrivial and not prime, write

$$K = K_1 \# K_2,$$

with K_1, K_2 nontrivial. In particular,

$$g(K_1) < g(K) \quad \text{and} \quad g(K_2) < g(K).$$

By induction on genus, K_1 and K_2 can be expressed as a connected sum of primes. $\square$

In fact, more is true:

The prime decomposition theorem

Each oriented knot K has a unique decomposition into primes.

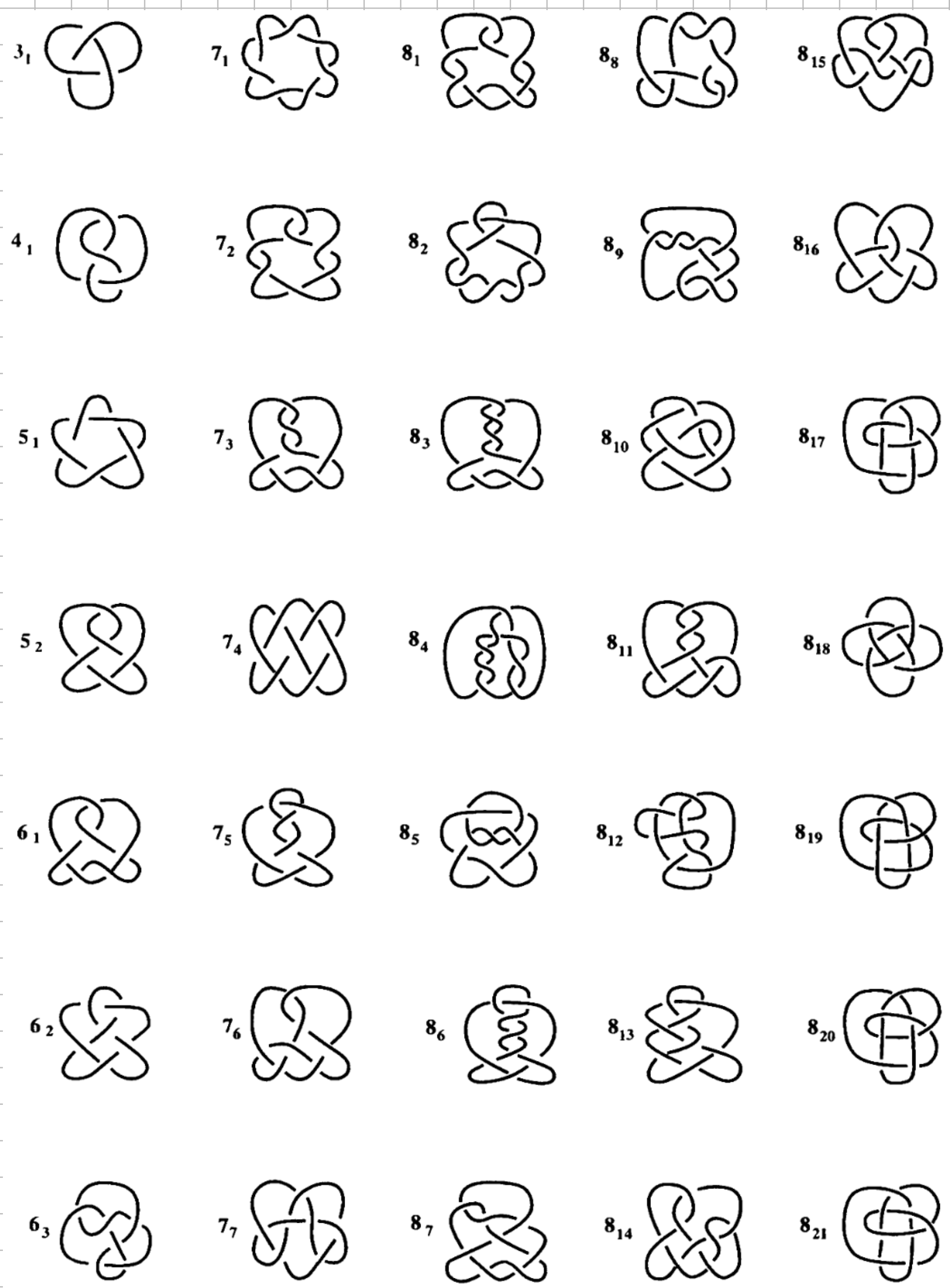
[This proof is omitted.]

The crossing number

Def. The crossing number $cr(L)$ of a link L is the minimal number of crossings in a diagram for L.

Traditionally, knot tabulations are organized by crossing number, and include only prime knots (without orientations and suppressing mirror reflections).

The prime knots K with $\mathrm{cr}(K) \leq 8$



[Lickorish, An Introduction to Knot Theory]

__Obs__. If K, K' are oriented knots, then
$$\mathrm{cr}(K \# K') \leq \mathrm{cr}(K) + \mathrm{cr}(K')$$

It is unknown whether the converse inequality holds.
But Lackenby showed (2009):
$$\mathrm{cr}(K \# K') \geq \frac{1}{152}\left(\mathrm{cr}(K) + \mathrm{cr}(K')\right).$$