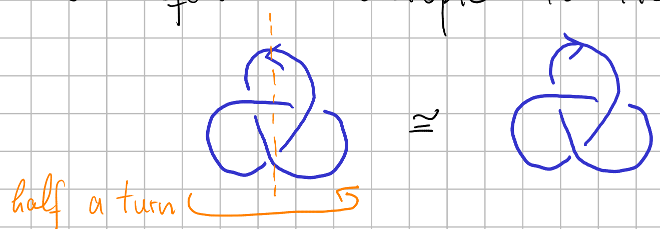
3. Seifert surfaces

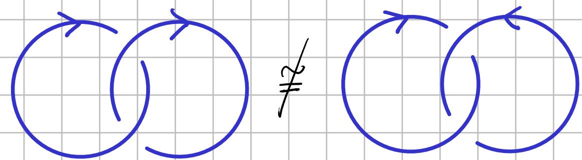
Def. • An <u>oriented link</u> is a link $L \subset S^3$ together with an orientation of L.

• Two oriented links are <u>isotopic</u> if there is an isotopy between them that preserves orientations.

Ex. • The trefoil is isotopic to its <u>reverse</u>:



• The following Hopf links are not isotopic as oriented links:



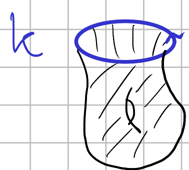
(We will later see how to justify this. Can you prove it using algebraic topology?)

Def. A <u>Seifert surface</u> for an oriented link $L \subset S^3$ is an oriented compact connected smoothly embedded surface $S \subset S^3$ such that $\partial S = L$ (as oriented manifolds).

∂S: boundary of S

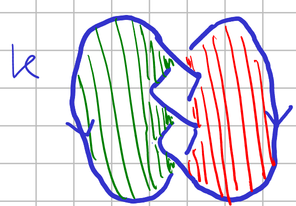
Ex. • Two Seifert surfaces for k the unknot:

A disk



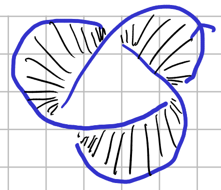
A genus-1 surface with one boundary component

• For k a trefoil:



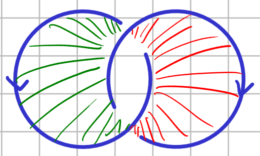
This Seifert surface is formed by connecting two disks with three half-twisted bands.

k



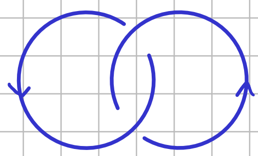
This surface is a Möbius band, which is not orientable.
So it is **not a Seifert surface**.

- The Hopf link L_1 has an annulus as a Seifert surface:



L_1

<u>Exercise</u>: Find a Seifert surface for L_2:

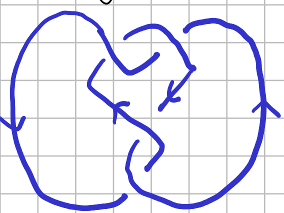


L_2

<u>Thm.</u> Every oriented link L has a Seifert surface.

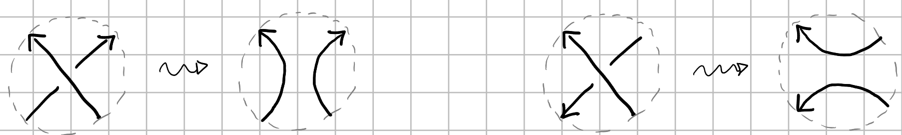
<u>Pf.</u> We will use <u>Seifert's algorithm</u> to construct it:

1. Start with a diagram D for L.

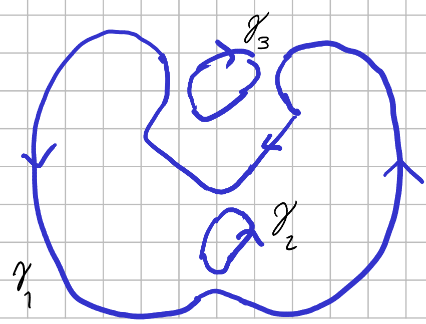


$L =$ figure-8 knot

2. <u>Resolve</u> each crossing of D, that is, replace it with two disjoint oriented arcs, matching the orientation of all incoming/outgoing arcs:

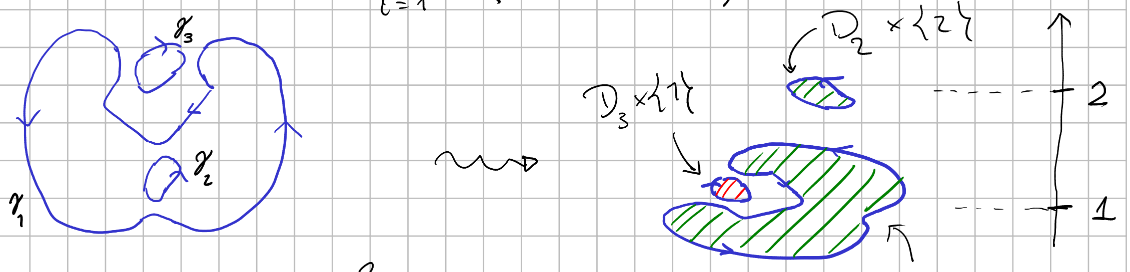


We obtain a union of disjoint oriented circles $\gamma_1, \dots, \gamma_k$ in the plane, called "Seifert circuits".



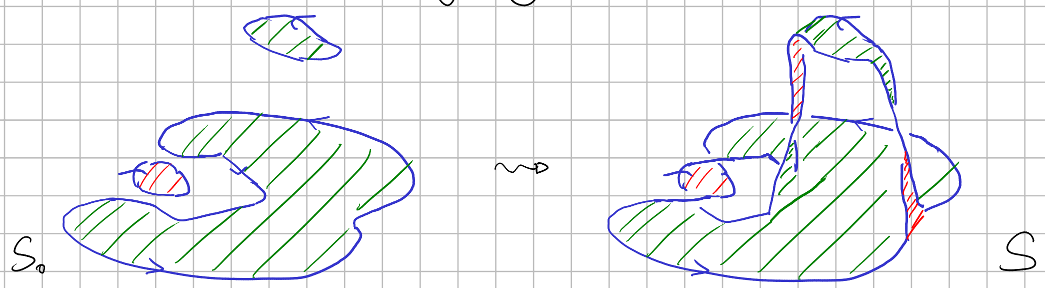
Fact. Each Seifert circuit γ_i bounds a disk $D_i \subset \mathbb{R}^2$.
We orient D_i compatibly with γ_i. (D_i is a "Seifert disk")

3. Let n_i be the "nestedness" of D_i, that is,
$$n_i = \# \{ j \in \{1, \dots, k\} \mid D_i \subseteq D_j \},$$
and define
$$S_0 = \bigcup_{i=1}^{k} \left(D_i \times \{n_i\} \right) \subset \mathbb{R}^3$$

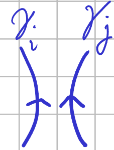


$n_1 = n_3 = 1$ $\quad n_2 = 2$

4. Restore the crossings by adding half-twisted bands.

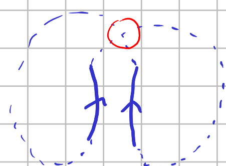


The resulting S is oriented: Consider a (resolved) crossing

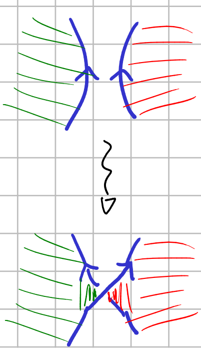


Obs. γ_i, γ_j are distinct Seifert circuits.

Otherwise, the outgoing end of each arc would connect to the incoming end of the other. This requires $\gamma_i = \gamma_j$ to self-intersect.



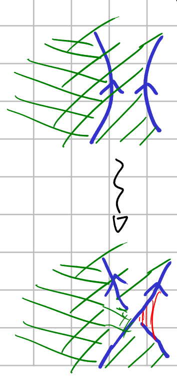
Case 1: $D_i \cap D_j = \emptyset$



D_i, D_j oppositely oriented wrt $\mathbb{R}^2$

Attached band respects orientations

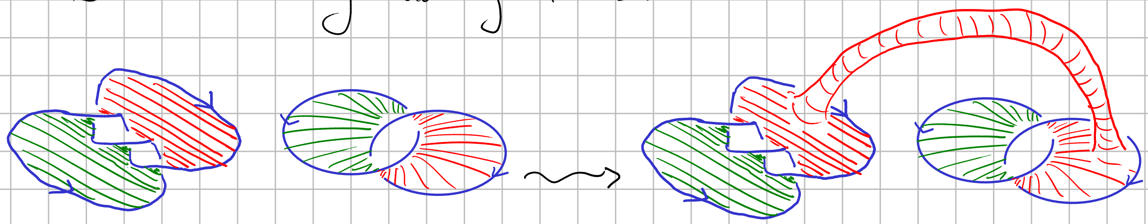
Case 2: $D_i \subset D_j$



D_i, D_j equioriented w.r.t. $\mathbb{R}^2$

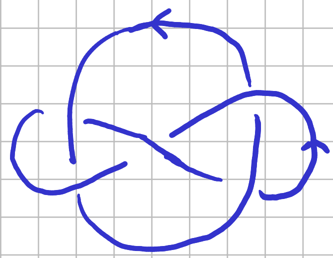
Attached band respects orientations

5. If L has multiple components, then S might not yet be connected. In that case, it is possible to make S connected by adding "tubes".



(Here we are using the fact that $S^3 \setminus S$ is connected.) $\square$

<u>Exercise</u>. Sketch a Seifert surface for the Whitehead link:



The knot genus

Recall: The classification of compact orientable surfaces:

For each $g \in \mathbb{N}$, let S_g be the smooth surface given by

$g=0$ $\qquad$ $g=1$ $\qquad$ $g=2$ $\qquad$ $g=3$ $\qquad \dots$

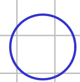
$\mathbb{S}^2$ $\qquad$ $\mathbb{S}^1 \times \mathbb{S}^1$

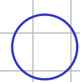
Let S be a compact orientable connected surface with $k \in \mathbb{N}$ boundary components. Then there is a unique $g \geq 0$ such that S is homeomorphic/diffeomorphic to S_g minus the interior of (any) k disjoint closed disks.

We say g is the genus of S.

Def. The genus $g(K)$ of a knot K is the minimal genus of a Seifert surface for K.

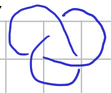
(this is preserved by ambient isotopy, so it is a knot invariant.)

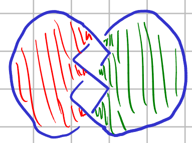
Ex. • $g($  $) = 0$

• Fact: If $g(K) = 0$, then $K \cong$ .

(This is not entirely obvious! One has to show that every smoothly embeded disk in $\mathbb{R}^3$ is smoothly isotopic to the standard disk in the xy-plane.

A rigorous argument uses the implicit function theorem.)

- $g($  $) = 1$

($\leq$) The surface  has genus 1. (Why?)

($\geq$) We know (from 3-colorability) that the trefoil is not an unknot.

Prop. Suppose a knot k has a diagram with c crossings and s Seifert circuits. Then:

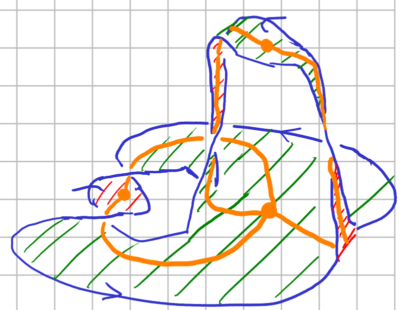
$$g(k) \leq \frac{1}{2}(c-s+1)$$

Pf sketch. Use the following facts:

(1) The Euler characteristic of an orientable surface S of genus g with k boundary components is

$$\chi(S) = 2-2g-k$$

(2) if S is constructed using Seifert's algorithm on a knot diagram, then is homotopy-equivalent to its spine Γ.



Graph with:
- vertices: Seifert disks
- one edge D_i •——• D_j for each half-twisted band between them.

Exercise.
- Work out the details of this proof.
- Compute the genus of all twist knots.