2. Colorability of knots [D. Silver, A colorful approach to knot theory]

By an <u>arc</u> of a link diagram D, we mean a segment of D connecting two consecutive undercrossings (or a circle component).

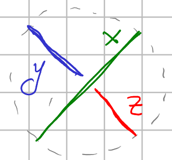
<u>Def.</u> Given $n \in \mathbb{N}$ and a knot diagram D, an <u>n-coloring</u> of D is a function

$$c : \{\text{arcs of } D\} \longrightarrow \mathbb{Z}/n$$

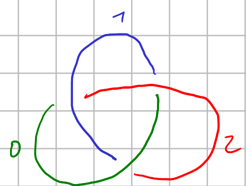
think of $\mathbb{Z}/n$ as a set of "colors"

such that at each crossing, the color x of the over-strand and the colors y, z of the under-strands satisfy

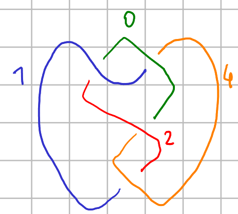
$$2x = y + z$$



<u>Ex.</u>



A 3-coloring



A 5-coloring

Another 5-coloring

We will be interested in the case where $n =: p$ is prime.

<u>Obs.</u>
- The set of p-colorings of D forms a $\mathbb{Z}/p$-vector space $V_p(D)$.
- $V_p(D)$ contains a 1-dimensional subspace $T_p(D)$ consisting of the constant p-colorings.

<u>Def.</u> The <u>Fox p-coloring space</u> of D is

$$\mathrm{Col}_p(D) := V_p(D) / T_p(D)$$

Exercise. Show that for every knot diagram D,

$$Col_2(D) = 0$$

Obs. Fix an arc α_0 of D.

Then each element of $Col_p(D)$ has a unique representative $c \in V_p(D)$ such that $c(\alpha_0) = 0$.

Thus:

$$Col_p(D) \cong \{ c \in V_p(D) \mid c(\alpha_0) = 0 \}$$

Thm. If D and D' are diagrams that represent isotopic knots, then

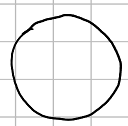
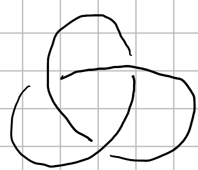
$$Col_p(D) \cong Col_p(D')$$

In other words, $\dim(Col_p(-))$ is a knot invariant! (proof coming soon)

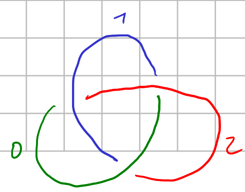
Def. A knot K is <u>p-colorable</u> if for some (hence every) diagram D representing K we have

$$\dim(Col_p(D)) \geq 1$$

Ex. Let $p = 3$,

$D =$  , $D' =$ 

- D admits only constant 3-colorings, so
$$Col_3(D) = 0$$
- D' admits a non-constant 3-coloring, so
$$\dim(Col_3(D')) \geq 1$$



(Exercise: What is this dimension?)

<u>So</u>: The (left-handed) trefoil is not trivial.

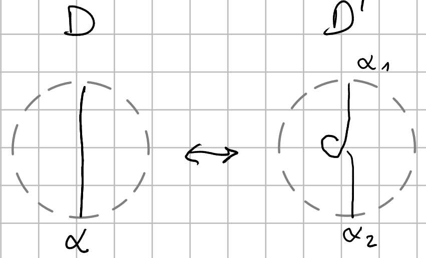
<u>Pf of Thm.</u>

We show that $\dim(V_p(D)) = \dim V_p(D')$ whenever D, D' differ by a Reidemeister move.

Then Reidemeister's Theorem implies $\dim(V_p(D)) = \dim(V_p(D'))$ whenever D, D' represent isotopic knots, and so

$$\dim(Col_p(D)) = \dim(V_p(D)) - 1 = \dim(V_p(D')) - 1 = \dim(Col_p(D')).$$

$\boxed{R1}$



For every p-coloring c' of D', we have

$$2c'(\alpha_1) = c'(\alpha_1) + c'(\alpha_2)$$
$$\Leftrightarrow \quad c'(\alpha_1) = c'(\alpha_2)$$

Given a p-coloring c of D, define a p-coloring c' of D' by

$$c'(x) = \begin{cases} c(\alpha) & \text{if } x = \alpha_1 \text{ or } x = \alpha_2 \\ c(x) & \text{otherwise} \end{cases}$$

This c' is indeed a p-coloring, that is, <u>all</u> crossing relations are satisfied (why?).

The map $\qquad V_p(D) \longrightarrow V_p(D')$

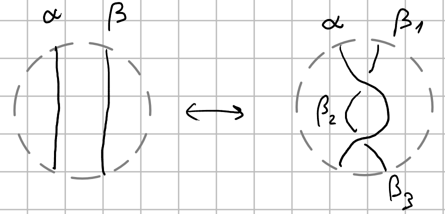
$$c \longmapsto c' ,$$

is an isomorphism, with inverse $c' \mapsto c$, where

$$c(x) = \begin{cases} c'(\alpha_1) = c'(\alpha_2) & \text{if } x = \alpha \\ c'(x) & \text{if } x \neq \alpha . \end{cases}$$

(Why is the condition $c'(\alpha_1) = c'(\alpha_2)$ crucial?)

$\boxed{R2}$ $\qquad D \qquad\qquad D'$



If $c' \in V_p(D')$, then:

$$2c'(\alpha) = c'(\beta_1) + c'(\beta_2) = c'(\beta_2) + c'(\beta_3)$$
$$\text{(in particular, } c'(\beta_1) = c'(\beta_3)\text{)}$$

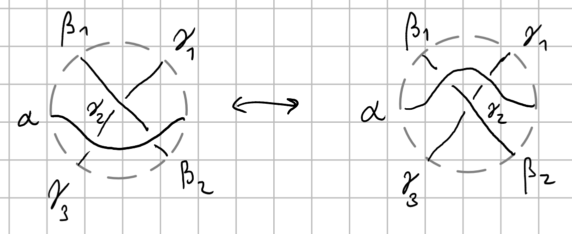
Given $c \in V_p(D)$, define $c' \in V_p(D')$ by

$$c'(x) = \begin{cases} c(\beta) & \text{if } x = \beta_1 \text{ or } x = \beta_3, \\ 2c(\alpha) - c(\beta) & \text{if } x = \beta_2, \\ c(x) & \text{otherwise.} \end{cases}$$

This assignment has an inverse $c' \mapsto c$, where

$$c(x) = \begin{cases} c'(\beta_1) = c'(\beta_3) & \text{if } x = \beta \\ c'(x) & \text{if } x \neq \beta . \end{cases}$$

$\boxed{R3}$ D D'



Exercise!

Hint. To define assignments $c \mapsto c'$ and $c' \mapsto c$ as before, one is forced to preserve the colors of $\alpha, \beta_1, \beta_2, \gamma_1$ and γ_3.
What about γ_2? $\square$

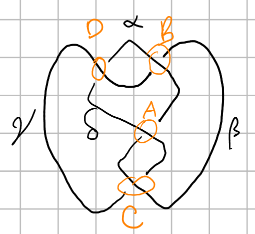
Computing $Col_p(D)$

D is not just a circle

Let D be a nontrivial diagram with n crossings for a knot K.
We encode the coloring relations in an $n \times n$ matrix M_0 over $\mathbb{Z}$, where

rows $\longleftrightarrow$ crossings columns $\longleftrightarrow$ arcs

Ex.



$$M_0 = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{cccc} \alpha & \beta & \gamma & \delta \end{array}$$

$$M_0 = \begin{pmatrix} -1 & -1 & 0 & 2 \\ 2 & -1 & -1 & 0 \\ 0 & 2 & -1 & -1 \\ -1 & 0 & 2 & -1 \end{pmatrix} \begin{matrix} A \\ B \\ C \\ D \end{matrix}$$

Then for every prime p, $V_p(D)$ is the kernel of the mod-p reduction of M_0.

Fact. Each row of M_0 is a linear combination of the others. (Challenging exercise)

$\rightsquigarrow$ We may delete any row of M_0 and this remains true.

<u>Recall</u>: For every arc α_0:

$$\mathrm{Col}_p(D) \cong \{ c \in V_p(D) \mid c(\alpha_0) = 0 \}$$

$\rightsquigarrow$ We may delete any column of M_0 to obtain $\mathrm{Col}_p(D)$ as the kernel of the resulting matrix.

<u>Upshot</u>: Let M be an $(n-1)\times(n-1)$ matrix obtained by deleting (any) one row and one column from M_0, and M_p its mod-p reduction

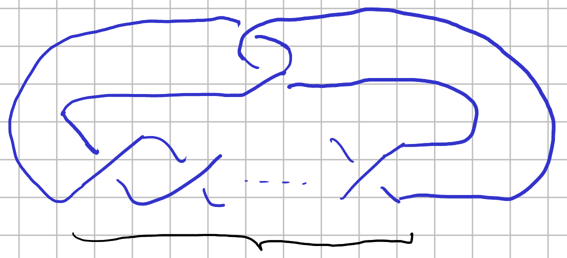
Then $\mathrm{Col}_p(D) \cong \ker(M_p)$. In particular,

$$K \text{ is } p\text{-colorable} \iff \det(M) \equiv 0 \pmod p$$

<u>Cor</u>. The set of primes p for which K is p-colorable is finite

<u>Pf</u>. Suppose this set is infinite.
Then $\det(M)$ is divisible by infinitely-many primes.
Thus $\det(M) = 0$.
In particular, $\det(M) \equiv 0 \pmod 2$ and so K is 2-colorable.
But no knot is 2-colorable! $\square$

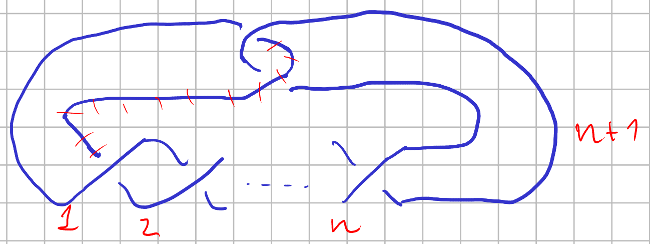
Ex. Let $n \in \mathbb{N}$. The <u>n-twist knot</u> K_n has diagram:



n crossings

(We have already encountered K_0, K_1 and K_2. Do you recognise them?)

Let's number all except one arc:



The corresponding matrix is

$$
M = \begin{array}{c} \\ 1 \\ 2 \\ \\ \\ \\ n \\ n+1 \end{array}
\begin{array}{c}
\begin{array}{cccccc|c} 1 & 2 & & & & n & n+1 \end{array} \\
\left(\begin{array}{cccccc|c}
2 & -1 & & & & & \\
-1 & 2 & -1 & & & & \\
 & -1 & 2 & -1 & & & \\
 & & -1 & 2 & -1 & & \\
 & & & & \ddots & -1 & \\
 & & & & -1 & 2 & -1 \\
\hline
-1 & & & & & & 2
\end{array}\right)
\end{array}
$$

<u>Exercise</u>. Show that $\det(M) = 2n+1$.

<u>Hint</u>: Show first that the upper left $n \times n$ matrix has determinant $n+1$.

<u>So</u>: If $n \geq 1$, then $\det(M)$ has some prime divisor p. Thus K_n is p-colorable.

So K_n is not the unknot!