UNIVERSITÄT HEIDELBERG ZUKUNFT SEIT 1386

Sheet 5

Due date: Nov 22

Problem 1. (Count the uncountables) Which of the following classes of groups contain uncountably many isomorphism classes?

(a) Finite groups

(b) Countable groups

(c) Finitely generated groups

(d) Finitely generated abelian groups

(e) Finitely presented groups

(f) Countably presented groups (that is, having a presentation with (at most) countably many generators and countably many relators)

(g) Free groups

Problem 2 (A 2-generated group with uncountably many quotients)**.** In this exercise we will construct a 2-generated group G whose center contains an isomorphic copy of $\bigoplus_{\mathbb{N}} \mathbb{Z}$. Define:

- the free abelian groups A and B as having $\mathbb{Z}$-bases $(a_n)_{n\in\mathbb{N}_{>0}}$ and $(b_n)_{n\in\mathbb{Z}}$, respectively.
- the $\mathbb{Z}$-bilinear map $f\colon B \times B \to A$ given on basis elements by

$$f(b_m, b_n) := \begin{cases} a_{n-m} & \text{if } n > m, \\ 0 & \text{if } n \leq m, \end{cases}$$

- the binary operation on the set $H := A \times B$, given by

$$(a, b) \cdot (a', b') := (a + a' + f(b, b'), b + b'),$$

- the map

$$T\colon H \to H$$
$$\left(a, \sum_{n\in\mathbb{Z}} \beta_n b_n\right) \mapsto \left(a, \sum_{n\in\mathbb{Z}} \beta_n b_{n+1}\right).$$

Show the following statements:

(a) The operation $\cdot$ makes H into a group.

(b) The set $A \times \{0\}$ is a central subgroup of H.

(c) H is generated by the set $\{(0, b_n) \mid n \in \mathbb{Z}\}$.
Hint: For each $n \in \mathbb{N}_{>0}$, compute the commutator $[(0, b_0), (0, b_n)]$.

(d) T is a group automorphism of H.

We now let $G := H \rtimes \mathbb{Z}$, with action $\mathbb{Z} \to \mathrm{Aut}(H)$ given by $1 \mapsto T$. Prove that G satisfies the desired properties:

(e) G is generated by $\sigma := ((0, b_0), 0)$ and $\tau := ((0, 0), 1)$.

(f) G contains a central subgroup isomorphic to $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}$, and thus uncountably many normal subgroups.

By Proposition 4.7 from class, it follows that G has uncountably many pairwise nonisomorphic quotients! We can use G to show that the same is true of the group G_0 from class, presented by two generators s, t and relations

$$\{[[s, t^n st^{-n}], s] \mid n \in \mathbb{Z}\} \cup \{[[s, t^n st^{-n}], t] \mid n \in \mathbb{Z}\}.$$

(g) Show that the assignment $s \to \sigma$, $t \to \tau$ induces an epimorphism $G_0 \twoheadrightarrow G$.

(h) Show that the subgroup of G_0 generated by $\{[s, t^n st^{-n}] \mid n \in \mathbb{Z}\}$ is central and contains a subgroup isomorphic to $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}$.

Problem 3 (Word metrics)**.** Fix a group G with generating set S, and assume S is **symmetric** (that is, $S = S^{-1}$).

(a) Prove that the function

$$\begin{aligned} d_S \colon G \times G &\to \mathbb{R}_{\geq 0} \\ (g, h) &\mapsto l_S(g^{-1}h), \end{aligned}$$

where l_S is the word length relative to S, is a metric on G.

(b) Let H be a second group, with symmetric generating set T. Show that if T is finite, then every group homomorphism $f \colon H \to G$ is Lipschitz with respect to the metrics d_T and d_S.

Reminder: A function $f \colon X \to Y$ between metric spaces $(X, d_X), (Y, d_Y)$ is called **Lipschitz** if there is a constant $L \in \mathbb{R}_{\geq 0}$ such that

$$\forall x, x' \in X \quad d_Y(f(x), f(x')) \leq L\, d_X(x, x').$$

(c) Conclude that if S and S' are finite symmetric generating sets for G, then the metrics $d_S, d_{S'}$ on G are **bilipschitz equivalent**, that is, there are constants $\alpha, \beta > 0$ such that

$$\forall g, h \in G \quad \alpha\, d_S(g, h) \leq d_{S'}(g, h) \leq \beta\, d_S(g, h).$$

Remark: This implies that any properties of the metric space (G, d_S) that are preserved by bilipschitz equivalence are therefore intrinsic to G (that is, independent of S)!