Sheet 7

Due date: June 7

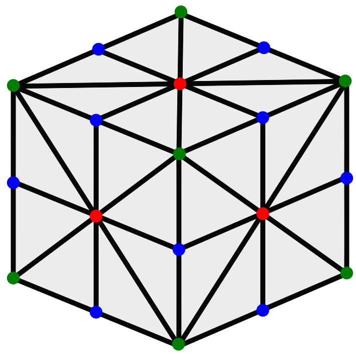
Problem 1 (Cayley and Coxeter)**.** Recall that given a Coxeter system (W,S), the group W acts on the Coxeter complex $\Sigma := \Sigma(W,S)$, with each $g \in W$ mapping $w\langle S'\rangle \mapsto gw\langle S'\rangle$.

(a) Determine the stabilizer of each simplex and conclude that the action $W \curvearrowright \Sigma$ is faithful.

(b) The Cayley graph $\Gamma := \mathrm{Cay}(W,S)$ is a 1-dimensional simplicial complex with a faithful W-action. Regarding Γ and Σ as posets, exhibit a W-equivariant inclusion $\Gamma \hookrightarrow \Sigma^{\mathrm{op}}$. What is the relation between the edge labels in Γ by S and the labeling λ of Σ we constructed in class?

(c) Describe the length function $l_S \colon W \to \mathbb{N}$ in terms of galleries in Σ.

Problem 2 (Coxeter diagram from Coxeter complex)**.** The following triangulation of the boundary of a cube, together with the vertex labeling indicated by the colors, is a Coxeter complex. Draw the corresponding Coxeter diagram.



Problem 3 (Joinable simplices)**.** Recall that two simplices σ_1, σ_2 of a simplicial complex X are **joinable** if their union is also a simplex.

(a) Show that in this case, $\mathrm{lk}_X(\sigma_1 \cup \sigma_2) \subseteq \mathrm{lk}_X(\sigma_1) \cap \mathrm{lk}_X(\sigma_2)$, and find a counterexample to the converse inclusion.

(b) Show that if σ_1, σ_2 are joinable and $\sigma_1 \cap \sigma_2 = \emptyset$ (so σ_2 is a simplex of $\mathrm{lk}_X(\sigma_1)$), then

$$\mathrm{lk}_{\mathrm{lk}_X(\sigma_1)}(\sigma_2) = \mathrm{lk}_X(\sigma_1 \cup \sigma_2).$$

A simplicial complex is called **flag** if it has the following property: for every finite set of vertices σ such that every two-element subset of σ is a simplex, the set σ itself is a simplex.

(c) Show that the barycentric subdivision of a simplicial complex (cf. Problem 2 from Sheet 6) is always flag.

(d) Show that if X is flag and σ_1, σ_2 are joinable in X, then $\mathrm{lk}_X(\sigma_1 \cup \sigma_2) = \mathrm{lk}_X(\sigma_1) \cap \mathrm{lk}_X(\sigma_2)$.

It turns out that Coxeter complexes are always flag, although we have not developed the tools to prove this.

Problem 4 (Manifolds and irreducibility)**.** Suppose (W,S) is a Coxeter system with W infinite, and whose Coxeter complex $\Sigma(W,S)$ is a manifold. Show that (W,S) is irreducible.