# Sheet 6

Due date: May 31

**Problem 1** (Special subgroups)**.** Recall that given a Coxeter system $(W, S)$ and a subset $T \subseteq S$, the **special subgroup** $W_T$ is the subgroup of $W$ generated by $T$. Show that $(W_T, T)$ is a Coxeter system with the same values $m_{t,t'}$ as in $(W, S)$.

**Problem 2** (The barycentric subdivision)**.** Given a simplicial complex $X$, its **barycentric subdivision** $X'$ is the simplicial complex whose vertices are the nonempty simplices of $X$, and whose $k$-dimensional simplices are the sets $\{\sigma_0, \sigma_1, \ldots, \sigma_k\}$ with $\sigma_0 \subset \sigma_1 \subset \ldots \subset \sigma_k$.

(a) Describe the boundary of a tetrahedron combinatorially as a simplicial complex $X$ on four vertices, by explicitly listing all simplices.

(b) Draw (the realization of) $X'$. How many simplices does $X'$ have in each dimension?

**Hint:** Start by drawing the tetrahedron $X$, then place each vertex of $X'$ at the center of the corresponding simplex of $X$.

**Problem 3** (Counting cosets)**.** Recall that the Coxeter group of type $\mathrm{A}_n$ is isomorphic to $\mathrm{Sym}(n+1)$.

(a) For the Coxeter system of type $\mathrm{A}_3$, how many special cosets of each rank are there?

(b) If you have solved Problem 2, these numbers should look familiar. Is this a coincidence? Get to the bottom of this mystery!

**Hint:** Your answer should involve the words "Coxeter complex".

(c) You should have already noticed that the Coxeter group of type $\mathrm{A}_3$ has two special subgroups of type $\mathrm{A}_2$ and one of type $\mathrm{A}_1 \times \mathrm{A}_1$. Can you locate their Coxeter complexes in the pictures you have just drawn?

**Problem 4** (Joins of Coxeter complexes)**.** The **join** $X * Y$ of two simplicial complexes $X, Y$ is the simplicial complex with vertex set $\mathrm{V}(X * Y) = \mathrm{V}(X) \sqcup \mathrm{V}(Y)$, and whose simplices are the sets of the form $\sigma \sqcup \tau$, with $\sigma$ a simplex of $X$ and $\tau$ a simplex of $Y$. Observe that the join operation is associative, commutative, and has the empty simplicial complex as the identity element.

(a) Recall that $\Delta^n$ is the simplicial complex on $n+1$ vertices, and where every subset of $\mathrm{V}(\Delta^n)$ is a simplex. Given $n, m \in \mathbb{N}$, what is $\Delta^n * \Delta^m$?

(b) Denote by $\partial\Delta^n$ the simplicial complex obtained from $\Delta^n$ by discarding its only $n$-dimensional simplex. Draw (the realization of) the iterated join $(\partial\Delta^1)^{*n}$, for $n \in \{1, 2, 3\}$.

(c) Given two Coxeter systems $(W, S), (W', S')$, show that there is a canonical isomorphism of simplicial complexes

$$\Sigma(W \times W', S \sqcup S') \cong \Sigma(W, S) * \Sigma(W', S').$$

(d) Draw the Coxeter complex of type $\mathrm{A}_1 \times \mathrm{A}_1 \times \mathrm{A}_1$.