# Sheet 5

Due date: May 24

**Problem 1.** (Tits' solution to the word problem) The goal of this exercise is to prove Tits' solution to the word problem. Let $(G, S)$ be a group together with a generating set of involutions satisfying the exchange condition.

(a) Use the exchange condition to show that for every $s, t \in S$ and reduced words

$$w_1 = sw_1', \qquad w_2 = tw_2'$$

representing the same element $g \in G$, the product $st$ has finite order $m$, and there is a reduced word for $g$ of the form

$$w = \underbrace{st...}_{m \text{ letters}} w'.$$

(b) Deduce by induction on word length that any two reduced $S$-words for the same element are related by a sequence of braid moves.

(c) Show that every $S$-word $w$ can be transformed into a reduced word by a sequence of braid moves and cancellation of syllables $ss$ for $s \in S$.

**Hint:** Apply the exchange condition to the maximal final sub-word of $w$ that is reduced.

**Problem 2** (Residual finiteness on the nose)**.** Show by the definition that the infinite dihedral group $\mathrm{D}_\infty$ is residually finite.

**Problem 3** (Divisible groups)**.** A group $G$ is called **divisible** if for every $g \in G$ and every $n \in \mathbb{N}_{\geq 1}$ there is $h \in G$ such that $h^n = g$.

(a) Give an example of a divisible group.

(b) Show that no nontrivial divisible group is residually finite.

**Problem 4** (A finitely generated, non-residually finite group)**.** Let $G$ be the group of bijections of $\mathbb{Z}$ generated by the shift $\sigma \colon n \mapsto n + 1$ and by the transposition $\tau = (0\,1)$.

(a) Show that $G$ contains an embedded copy of every symmetric group $\mathrm{Sym}(n)$.

**Hint:** Use Problem 2 (a) from Sheet 3.

(b) Let $\varphi \colon G \to H$ be a homomorphism with $H$ finite. Show that for large enough $n \in \mathbb{N}$, we have

$$\mathrm{Alt}(n) \leq \ker(\varphi).$$

(c) Conclude that $G$ is not residually finite.