



Sheet 5

Due date: May 24

Problem 1. (Tits' solution to the word problem) The goal of this exercise is to prove Tits' solution to the word problem. Let (G, S) be a group together with a generating set of involutions satisfying the exchange condition.

- (a) Use the exchange condition to show that for every $s, t \in S$ and reduced words

$$w_1 = sw'_1, \quad w_2 = tw'_2$$

representing the same element $g \in G$, the product st has finite order m , and there is a reduced word for g of the form

$$w = \underbrace{st \dots}_{m \text{ letters}} w'.$$

- (b) Deduce by induction on word length that any two reduced S -words for the same element are related by a sequence of braid moves.
- (c) Show that every S -word w can be transformed into a reduced word by a sequence of braid moves and cancellation of syllables ss for $s \in S$.

Hint: Apply the exchange condition to the maximal final sub-word of w that is reduced.

Problem 2 (Residual finiteness on the nose). Show by the definition that the infinite dihedral group D_∞ is residually finite.

Problem 3 (Divisible groups). A group G is called **divisible** if for every $g \in G$ and every $n \in \mathbb{N}_{\geq 0}$ there is $h \in G$ such that $h^n = g$.

- (a) Give an example of a divisible group.
- (b) Show that no divisible group is residually finite.

Problem 4 (A finitely generated, non-residually finite group). Let G be the group of bijections of \mathbb{Z} generated by the shift $\sigma: n \mapsto n + 1$ and by the transposition $\tau = (01)$.

- (a) Show that G contains an embedded copy of every symmetric group $\text{Sym}(n)$.

Hint: Use Problem 2 (a) from Sheet 3.

- (b) Let $\varphi: G \rightarrow H$ be a homomorphism with H finite. Show that for large enough $n \in \mathbb{N}$, we have

$$\text{Alt}(n) \leq \ker(\varphi).$$

- (c) Conclude that G is not residually finite.