Coxeter groups and buildings, Summer 2024
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Sheet 4

Due date: May 17

Problem 1 (Breaking dihedral groups)**.** Recall the descrpition of the dihedral groups D_n as semidirect products.

(a) Give a formula for semidirect products of the form

$$N_1 \times (N_2 \rtimes H) \cong (N_1 \times N_2) \rtimes H,$$

and use it to show that if $n \in \mathbb{N}$ is odd, then $\mathrm{D}_{2n} \cong \mathrm{D}_n \times \mathbb{Z}/2$.

(b) Show that if $n \in \mathbb{N}$ is even, then $\mathrm{D}_{2n} \ncong \mathrm{D}_n \times \mathbb{Z}/2$.
Hint: Look at $2n$-torsion.

(c) Do we have $\mathrm{D}_\infty \cong \mathrm{D}_\infty \times \mathbb{Z}/2$?

Problem 2 (At the edge of the galaxy)**.** Read Definition 3.1 of

Y. Santos Rego, P. Schwer, *The galaxy of Coxeter groups*, Journal of Algebra, DOI: `10.1016/j.jalgebra.2023.12.006` (2023).

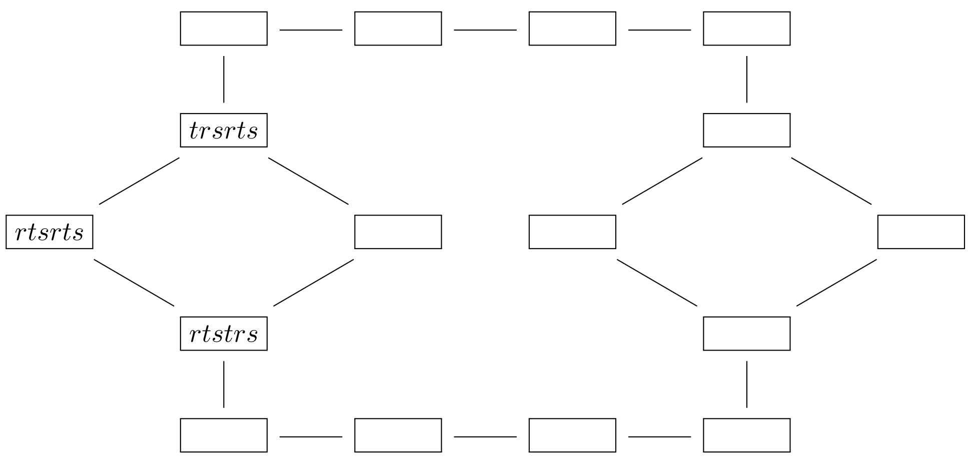
(a) What is an edge in the Coxeter galaxy?

(b) Give an example of two Coxeter systems of different rank whose Coxeter groups are isomorphic.
Hint: Somewhere in this exercise sheet there is an infinite family of examples.

Problem 3 (Real-time strategy)**.** Consider the Coxeter group of type A_3

$$G = \langle r, s, t \mid r^2 = s^2 = t^2 = (rs)^3 = (st)^3 = (rt)^2 = 1 \rangle.$$

(a) Use Tits's solution to the word problem to show that the word $rtsrts$ is reduced.
Hint: Tits's algorithm reduces us to a finite search, but still large enough that we need to be methodical! A graph like the following might help us keep track of where we have been.



(b) Decide which among the following words represent the same element of G:

$$rstrst, \quad rtsrts, \quad srtsrt, \quad strstr, \quad trstrs, \quad tsrtsr.$$

Problem 4 (The Tits bilinear form)**.** Given a Coxeter matrix $M = (m_{ij})$ of rank n, we defined a symmetric bilinear form B on $V := \mathbb{R}^n$ by

$$B(e_i, e_j) := -\cos\left(\frac{\pi}{m_{ij}}\right).$$

(a) Show that for the Coxeter system of type $\tilde{\mathrm{A}}_2$

$$\langle s_1, s_2, s_3 \mid \forall i, j \colon s_i^2 = 1, (s_i s_j)^3 = 1 \rangle,$$

the bilinear form B is degenerate (that is, for some $v \in V$, the linear map $B(v, -)$ is zero).

(b) Show that if (V, B) is linearly isometric to $\mathbb{R}^n$ with the usual inner product, then for every three indices i, j, k, we have

$$\frac{1}{m_{ij}} + \frac{1}{m_{jk}} + \frac{1}{m_{ki}} > 1.$$

Hint: For $u_i, u_j, u_k \in \mathbb{R}^n$ the vectors corresponding to $e_i, e_j, e_k \in V$, consider the piecewise-geodesic path in the unit sphere $\mathbb{S}^{n-1}$ going from u_i, to $-u_j$, to u_k, to $-u_i$.