Coxeter groups and buildings, Summer 2024
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Sheet 3

Due date: May 10

Problem 1 (Graphs for Coxeter systems)**.** Read Definition 3.5.1 of a **Coxeter diagram** (sometimes also called **Coxeter-Dynkin diagram**) in

> M.W. Davis, *The Geometry and Topology of Coxeter Groups*, Princeton University Press (2008).

A free version of the book is available to download from the author's webpage at `https://people.math.osu.edu/davis.12/davisbook.pdf`.

Read also the definition of a **Coxeter graph** in Section 3.1 of

> T.A. Schroeder, Coxeter Groups and the Davis Complex, in A. Wootton, V. Peterson, C. Lee (eds), *A Primer for Undergraduate Research*, Foundations for Undergraduate Research in Mathematics. Birkhäuser, Cham. (2017).

You should have access through the University's library via
`https://doi.org/10.1007/978-3-319-66065-3_1`.

(a) Explain the difference between these two definitions and illustrate with a Coxeter system of your choice.

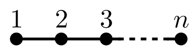
(b) Suppose Γ_1 and Γ_2 are Coxeter diagrams for the Coxeter groups W_1, W_2. What is the Coxeter group represented by the disjoint union $\Gamma_1 \sqcup \Gamma_2$? What if Γ_1 and Γ_2 are instead Coxeter graphs?

Hint: This should remind you of Problem 1 from Sheet 1.

Problem 2 (Type A_n)**.** Given $n \in \mathbb{N}$, the Coxeter system of **type** A_n has rank n and Coxeter matrix $(m_{ij})_{i,j\in\{1,\ldots,n\}}$ given by

$$m_{ij} = \begin{cases} 1 & \text{if } j = i, \\ 3 & \text{if } j = i \pm 1, \\ 2 & \text{otherwise.} \end{cases}$$

Equivalently, by the following Coxter diagram:



The goal of this exercise is to show that the symmetric group $\mathrm{Sym}(n+1)$, together with the set S of transpositions between adjacent elements $s_1 = (1\,2), s_2 = (2\,3), \ldots, s_n = (n\; n+1)$, is a Coxeter system of type A_n.

(a) Show that S generates $\mathrm{Sym}(n+1)$.

(b) Verify that each product $s_i s_j$ has order m_{ij} in $\mathrm{Sym}(n+1)$ as above.

By results from class, we are left to prove that $(\mathrm{Sym}(n+1), S)$ satisfies the exchange condition.

(c) Given a permutation $\sigma \in \mathrm{Sym}(n+1)$ and $1 \leq i < j \leq n+1$, we say the pair (i, j) is an **inversion** of σ if $\sigma(i) > \sigma(j)$, and denote by $\mathrm{inv}(\sigma)$ the number of inversions of σ. For $\sigma = (5, 4, 3, 2, 1) \in \mathrm{Sym}(5)$, what is $\mathrm{inv}(\sigma)$?

(d) Show that for every $\sigma \in \mathrm{Sym}(n+1)$ and $i \in \{1, \dots, n\}$,

$$\mathrm{inv}(\sigma s_i) = \begin{cases} \mathrm{inv}(\sigma) - 1 & \text{if } (i, i+1) \text{ is an inversion of } \sigma, \\ \mathrm{inv}(\sigma) + 1 & \text{otherwise.} \end{cases}$$

Use this to show that $l_S(\sigma) = \mathrm{inv}(\sigma)$.

Hint: Prove one inequality at a time. Each of the two cases is helpful for one direction.

(e) Prove $(\mathrm{Sym}(n+1), S)$ satisfies the exchange property.

Hint: If $(i, i+1)$ is an inversion of $\sigma = s_{i_1} \dots s_{i_m}$, with $\sigma(i) = b > a = \sigma(i+1)$, consider the first integer r such that in the permutation $\tau = s_{i_1} \dots s_{i_r}$, the pair $(\tau^{-1}(b), \tau^{-1}(a))$ is an inversion. Then compare σ with $s_{i_1} \dots \widehat{s_{i_r}} \dots s_{i_m}$.

Problem 3 (Conjugation classes of reflections)**.** Recall that the set R of **reflections** in a Coxeter system $(W, S = \{s_i\}_{i \in I})$ consists of all conjugates of the standard generators:

$$R := \langle wsw^{-1} \mid s \in S, w \in W \rangle.$$

In particular, R is a union of conjugacy classes in W, but how many? Maybe there is one conjugacy class for each $s \in S$? Or maybe they are all conjugate?

(a) Show that for every $i, j \in I$ with m_{ij} odd, the generators s_i and s_j are conjugate.

(b) Show that if m is not odd (that is, $m \in 2\mathbb{N} \cup \{\infty\}$), then in the Coxeter group of type $\mathrm{I}_2(m)$

$$W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^m \rangle,$$

the elements s_1 and s_2 are not conjugate.

Hint: Find a group G and a homomorphism $W \to G$ mapping s_1 and s_2 to elements that are *clearly* not conjugate. (Further hint: Distinct elements of an abelian group are clearly not conjugate.)

(c) Show that it is possible for two standard generators s_i, s_j in a Coxeter system with $m_{ij} \in 2\mathbb{N} \cup \{\infty\}$ to be conjugate.

(d) Let Γ_0 be the Coxeter graph of (W, S) (as in Problem 1), and let Γ be the graph obtained from Γ_0 by discarding all edges with even labels. Describe the conjugacy classes of reflections in terms of Γ.

Problem 4 (A non-Coxeter group)**.** Recall that for each $n \in \mathbb{N}_{\geq 1}$, the **Alternating group** $\mathrm{Alt}(n)$ is defined as the kernel of the sign map

$$\begin{aligned} \mathrm{Sym}(n) &\to \{\pm 1\} \\ \sigma &\mapsto \mathrm{sgn}(\sigma). \end{aligned}$$

In other words, $\mathrm{Alt}(n)$ consists of the permutations that are the product of an even number of transpositions. [1]

[1] The alternating group is also often denoted by A_n, but we avoid this nomenclature as it might cause confusion with the Coxeter group of type A_n.

(a) Show if $n \geq 5$, then $\mathrm{Alt}(n)$ is generated by involutions.

Hint: $\mathrm{Alt}(n)$ is generated by products of two transpositions. When is such a product an involution?

(b) You might have learned at some point that when $n \geq 5$, the group $\mathrm{Alt}(n)$ is simple. Use this fact to show that in this case, despite being generated by involutions, $\mathrm{Alt}(n)$ is not a Coxeter group.

Hint: In class we have seen that Coxeter groups always admit an epimorphism onto $\mathbb{Z}/2$.