UNIVERSITÄT HEIDELBERG ZUKUNFT SEIT 1386

Sheet 2

Due date: May 3

Problem 1 (Word metrics)**.** Fix a group G with generating set S. Assume S is **symmetric** (that is, $S = S^{-1}$), though not necessarily comprised of involutions.

(a) Prove that the function

$$\begin{aligned} d_S\colon G \times G &\to \mathbb{R}_{\geq 0} \\ (g,h) &\mapsto l_S(g^{-1}h) \end{aligned}$$

is a metric on G.

(b) Let H be a second group, with symmetric generating set T. Show that if T is finite, then every group homomorphism $f\colon H \to G$ is Lipschitz with respect to the metrics d_T and d_S.

Reminder: A function $f\colon X \to Y$ between metric spaces $(X, d_X), (Y, d_Y)$ is called **Lipschitz** if there is a constant $L \in \mathbb{R}_{\geq 0}$ such that

$$\forall x, x' \in X \quad d_Y(f(x), f(x')) \leq L\, d_X(x, x').$$

(c) Conclude that if S and S' are finite symmetric generating sets for G, then the metrics $d_S, d_{S'}$ on G are **bilipschitz equivalent**, that is, there are constants $\alpha, \beta > 0$ such that

$$\forall g, h \in G \quad \alpha\, d_S(g,h) \leq d_{S'}(g,h) \leq \beta\, d_S(g,h).$$

Remark: Any properties of the metric space (G, d_S) that are preserved by bilipschitz equivalence are therefore intrinsic to G (that is, independent of S). This fact justifies the study of finitely generated groups as metric spaces, and lies at the heart of geometric group theory!

Problem 2 (Orders and presentations)**.** Consider the group

$$G := \langle a, b \mid a^{12} = 1, ba^2b^{-1} = a \rangle.$$

(a) What is the order of b?

(b) Show that $a^3 = 1$, so a has order 1 or 3.

Remark: From the presentation, did you expect it to be 12? We will see that for the relations $(s_is_j)^{m_{ij}} = 1$ in the presentation given by a Coxeter system, the exponent m_{ij} is indeed the order of s_is_j.

(c) Prove that a has order 3.

Hint: Find an isomorphism $G \to \mathbb{Z}/3 \rtimes \mathbb{Z}$. To prove injectivity, you might want to show that every element of G can be expressed in the form $a^m b^n$, with $m, n \in \mathbb{Z}$.

Problem 3 (Cayley graphs for dihedral groups)**.** Recall from Sheet 1 the ismorphisms

$$\mathrm{D}_n \cong \mathbb{Z}/n \rtimes \mathbb{Z}/2 \quad (n \in \mathbb{N}), \qquad \mathrm{D}_\infty \cong \mathbb{Z} \rtimes \mathbb{Z}/2$$

given by $s_1 \mapsto r, s_2 \mapsto rt$, where s_1, s_2 are the Coxeter generators, r is the generator of $\mathbb{Z}/2$, and t is a generator of the cyclic group $\mathbb{Z}/n$ or $\mathbb{Z}$.

(a) Draw the Cayley graph of D_5 with respect to the generating set $\{s_1, s_2\}$, and with respect to $\{t, r\}$.

(b) Draw the Cayley graph of D_∞ with respect to $\{s_1, s_2\}$ and to $\{t, r\}$.

Hint: The description as a semidirect product allows you to list all group elements without repetition. Later, we will see that this is also possible from a Coxeter-type presentation.

Problem 4 (Pre-reflection systems)**.** Give an example of a pre-reflection system for a Coxeter group that is not a reflection system.

Hint: If you don't have any ideas, you can try to make sense of the following picture: