UNIVERSITÄT HEIDELBERG ZUKUNFT SEIT 1386

Sheet 1

Due date: April 26

Problem 1 (Products of Coxeter groups)**.** Let W_1 and W_2 be Coxeter groups.

(a) Show that the direct product $W_1 \times W_2$ is a Coxeter group. A Coxeter group that cannot be expressed as a direct product of non-trivial Coxeter groups is called **irreducible**.

(b) Show that the free product $W_1 * W_2$ is a Coxeter group.

Problem 2 (The semidirect product)**.** Let N, H be groups.

(a) Given an action $\alpha\colon H \to \mathrm{Aut}(N)$ of H on N by group automorphisms, we define the (external) **semidirect product** $N \rtimes_\alpha H$ (written simply $N \rtimes H$ if there is no room for confusion) as the group with underlying set $N \times H$ and group operation given by

$$(n_1, h_1) \cdot (n_2, h_2) := (n_1\, \alpha(h_1)(n_2),\, h_1 h_2).$$

Show that $N \rtimes H$ is indeed a group, fitting into a short exact sequence of the form

$$\mathbb{1} \to N \to N \rtimes H \to H \to \mathbb{1}.$$

What happens if α is the trivial action?

(b) Consider a short exact sequence of groups

$$\mathbb{1} \to N \xrightarrow{f} G \xrightarrow{g} H \to \mathbb{1},$$

and show the following conditions are equivalent:

1. This short exact sequence **splits**, that is, there is a homomorphism $s\colon H \to G$ such that $s \circ g = \mathrm{id}_H$ (called a **section**).
2. There is an action $\alpha\colon H \to \mathrm{Aut}(N)$ and an isomorphism $\varphi\colon G \to N \rtimes_\alpha H$ making the following diagram commute:

$$\begin{array}{ccccc} N & \xrightarrow{\ f\ } & G & \xrightarrow{\ g\ } & H \\ \| & & \downarrow{\scriptstyle\varphi} & & \| \\ N & \longrightarrow & N \rtimes H & \longrightarrow & H \end{array}.$$

In this case we also say G is a semidirect product of N and H.

(c) Give an example of a short exact sequence of groups that does not split.

Problem 3 (Dihedral groups)**.** Recall that for each $n \in \mathbb{N}_{\geq 1} \cup \{\infty\}$, the **dihedral group** D_n is the group of isometries of the regular n-gon (the "∞-gon" being $\mathbb{Z}$). We have seen in class that these are precisely the Coxeter groups of Rank 2:

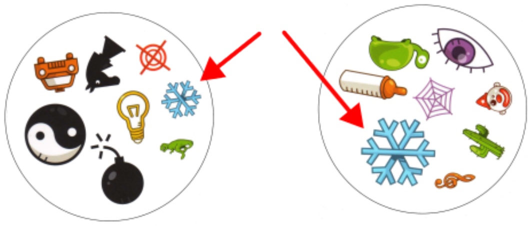
$$\mathrm{D}_n \cong \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^n = 1 \rangle,$$

(where "$(s_1 s_2)^\infty = 1$" is interpreted as no relation). Prove they can also be described as semidirect products:

$$\mathrm{D}_n \cong \mathbb{Z}/n \rtimes \mathbb{Z}/2 \quad \text{(for all } n \in \mathbb{N}_{\geq 1}), \qquad \mathrm{D}_\infty \cong \mathbb{Z} \rtimes \mathbb{Z}/2,$$

where the nontrivial element of $\mathbb{Z}/2$ acts by sending $1 \mapsto -1$.

Problem 4 (Dobble)**.** The game Dobble is played with a deck of cards where each card contains 8 different pictures, and every two cards have precisely one picture in common.



How can one design such a deck with 57 distinct cards? **Hint:** Look for inspiration in Sheet 0.