Sheet 0

Classroom exercises

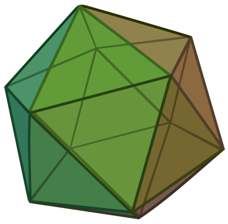
Problem 1 (Group actions)**.** Stare at a face of a cube. There are 4 rotations of the cube (including the identity) that preserve this face, and the cube has 6 faces. Note that $4 \times 6 = 24$. Stare at an edge of the cube. There are 2 rotations of the cube that preserve this edge, and the cube has 12 edges. Note that $2 \times 12 = 24$. Stare at a vertex of the cube. There are 3 rotations of the cube that preserve this vertex, and the cube has 8 vertices. Note that $3 \times 8 = 24$.

(a) Write down the definition of an **action** of a group G on a set X.

(b) Given an action $G \curvearrowright X$, the **stabilizer** of a point $x \in X$ is the subgroup $\operatorname{Stab}_G(x) := \{g \in G \mid gx = x\}$, and the **orbit** of x is $Gx := \{gx \in X \mid g \in G\}$. Show that if G is finite, then for each $x \in X$, we have $|G| = |Gx| \times |\operatorname{Stab}_G(x)|$.

(c) Why does the cube seem to like the number 24?

(d) What is the order of the group of orientation-preserving symmetries of the icosahedron?



Problem 2 (Counting k-frames)**.** Fix a prime power q, and let $k \leq n$ be nonnegative integers. A **k-frame** in an n-dimensional vector space is an ordered tuple of k linearly independent vectors.

(a) How many k-frames are there in $(\mathbb{F}_q)^n$?

(b) Compute the order of the **general linear group** $\mathrm{GL}_n(\mathbb{F}_q)$, which consists of the invertible linear maps $f\colon (\mathbb{F}_q)^n \to (\mathbb{F}_q)^n$, and of the **special linear group**

$$\mathrm{SL}_n(\mathbb{F}_q) := \{f \in \mathrm{GL}_n(\mathbb{F}_q) \mid \det(f) = 1\}.$$

(c) Show that the number of k-dimensional subspaces of $(\mathbb{F}_q)^n$ is

$$\frac{(q^n - 1)(q^{n-1} - 1)\dots(q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1)\dots(q - 1)}.$$

Problem 3 (The projective plane over $\mathbb{F}_q$)**.** Given a prime power p, denote by $\mathcal{P}$ and $\mathcal{L}$, respectively, the set of points and lines in the projective plane over $\mathbb{F}_q$. Prove that:

(a) $|\mathcal{P}| = |\mathcal{L}| = q^2 + q + 1$.

(b) Each $p \in \mathcal{P}$ is incident with precisely $q + 1$ lines.

(c) Each $l \in \mathcal{L}$ is incident with precisely $q+1$ points.

Problem 4 (Anatomy of the Heawood graph)**.** The projective plane over $\mathbb{F}_2$ is called the **Fano plane**, and its incidence graph Δ is the **Heawood graph**. Together with the labeling of its vertices as "point" or "line", it is an example of a **building**.

(a) Draw Δ, labeling each vertex with the corresponding subspace of $(\mathbb{F}_2)^3$. Try to do it without looking at your lecture notes!

(b) Each edge of Δ is called a **chamber**. How many chambers does Δ have?

(c) Each hexagonal subgraph of Δ, which corresponds to three 1-dimensional subspaces spanning $(\mathbb{F}_2)^3$, is called an **apartment**. How many apartments does Δ have?