11 - Local properties of Coxeter Complexes

Montag, 8. April 2024 13:46

We will now study local properties of Coxeter complexes, e.g. "around" vertices.

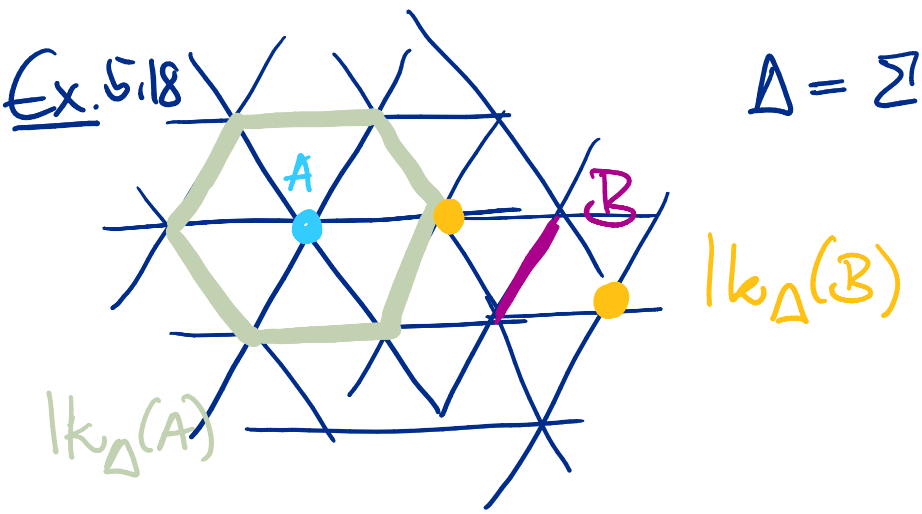
Def 5.17 (Links)

The link of a simplex A in a simplicial complex Δ is the subcomplex $lk_\Delta(A)$ of Δ consisting of all simplices B in Δ s.th. $A \cap B = \emptyset$ and $A \cup B$ is a simplex in Δ.

A and B are joinable i.e. have a common least upper bound

Rmk The maximal simplices in $lk_\Delta A$ in a chamber cplx Δ are in bijection with the chambers of Δ having A as a face.

Ex. 5.18



$\Delta = \Sigma(W,S)$ where (W,S) is a Coxeter system of type $\tilde{A}_2$

Lemma 5.19

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Let Δ be a chamber complex and A a simplex in Δ. Then $lk_\Delta(A)$ is isomorphic to the subposet $\Delta_{\geq A} \subseteq \Delta$ via the map

$$B \longmapsto B \cup A \qquad \text{for } B \in lk_\Delta(A).$$

Proof:

This follows directly from the definition of the poset structure on Δ and $lk_\Delta(A)$. $\square$

Prop. 5.20

Let $\Sigma = \Sigma(W,S)$ be some Coxeter complex.
Let $S' = S - \lambda(A)$ for a fixed simplex A in Σ.
Let $W' = \langle S' \rangle$.
Then $lk_\Sigma(A)$ is isomorphic (as labeled chamber complex) to $\Sigma(W', S')$.

Proof

First convince yourself that indeed (W', S') is a Coxeter system (e.g. using the Def via presentations and Coxeter matrices).

Using the W-action we may assume that A is a face of the fundamental chamber $C = W_\emptyset = \{1\}$.

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By Lemma 5.19 $\mathrm{lk}_\Sigma(A) \cong \Sigma_{\geq A}$, this is that the link is isomorphic to the poset containing all special cosets that are contained in W' (ordered by reverse inclusion).

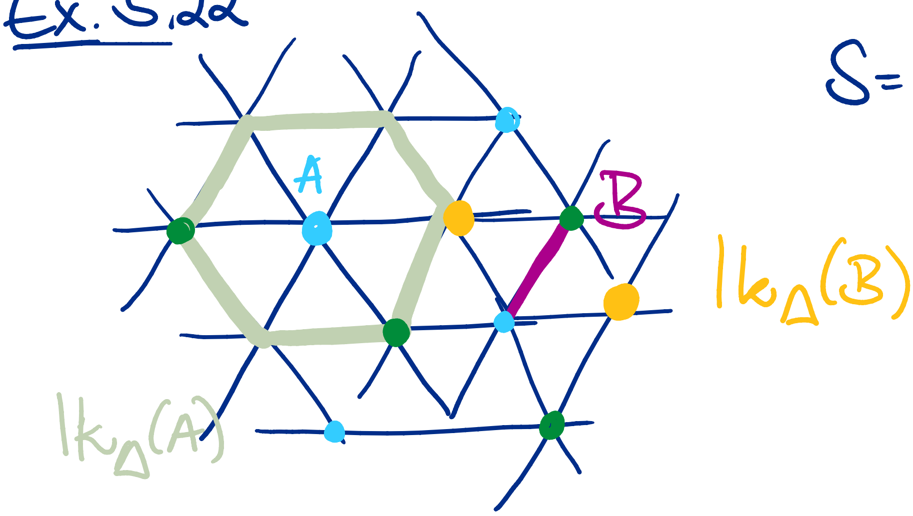
But these special cosets are exactly the ones in $\Sigma(W', S')$. Hence $\mathrm{lk}_\Sigma(A) \cong \Sigma(W', S')$.

$\square$

Remark 5.21 Links are visible

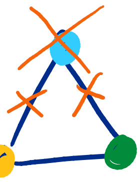
This proposition has a nice interpretation on the Coxeter diagram:
Let A be a simplex in $\Sigma = \Sigma(W, S)$ and denote by $\Gamma_\Sigma = (V, E)$ the Coxeter diagram of (W, S). Note that V is in bijection with S.
Then the type $\lambda(A)$ is a subset of S.
For all $s \in \lambda(A)$ delete the vertex s in Γ (and also all outgoing (open) edges).
The resulting diagram is the diagram corresponding to (W', S') in Prop. 5.20, that is the type of $\mathrm{lk}_\Sigma(A)$ as a Coxeter cplx.

Ex. 5.22



$S = \{ \bullet \;\; \bullet \;\; \bullet \}$

with diagram



A has type $\bullet$ $\rightsquigarrow$ delete $\bullet$ from Γ and obtain the diagram $\bullet\!-\!\!\!-\!\bullet$ as the type of $\mathrm{lk}_\Sigma(A)$ which indeed is a Coxeter cplx of type A_2 (a hexagon).

B has type $\{\bullet, \bullet\}$ which we delete from Γ to obtain type $\bullet$ for the link. This link is a pair of vertices which is exactly the Coxeter cplx of the Coxeter group $\langle s \mid s^2 \rangle$ with diagram $\bullet$.

$\rightsquigarrow$ Say something about the interpretation on the Coxeter matrix level.

Rmk 5.23

As a general principle: The link of a

As a general principle: The link of a codim 2 simplex of type $S-\{s,t\}$ is a $2m$-gon, with $m = m(s,t)$.

Cor 5.24

The Coxeter matrix M of (W,S) can be recovered from $\Sigma = \Sigma(W,S)$ as follows: For any $s,t \in S$ with $s \neq t$ the order $m(s,t)$ is the unique number $2 \leq m \leq \infty$, s.th. the link of a simplex of type $S-\{s,t\}$ is a $2m$-gon.

In particular is W (up to isomorphism) determined by the complex Σ.

5.25 Topological excursion:

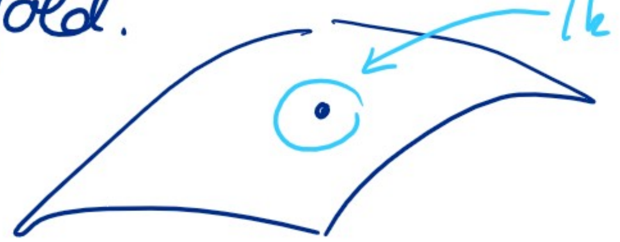
A question arises naturally (in case one is familiar with combinatorial topology): In which cases is Σ a manifold?

Triangulated manifolds are examples of chamber complexes.
And we have seen that in case W is

And we have seen that in case W is finite the geometric realization of Σ is homeom. to a (triangulated) sphere.

What happens if W is infinite?

There is one obvious restriction: Manifolds are locally compact and links of vertices in Σ must hence be spheres if Σ should have a chance to be any manifold.



Σ locally compact
$\Rightarrow$ locally finite as a complex
$\Rightarrow$ lks are Cox. cplx of <u>finite</u> grps

We hence conclude:

<u>A necessary condition</u>

for $\Sigma = \Sigma(W,S)$ to be a manifold is, that links of simplices in Σ are spheres and the corresp. Coxeter groups are finite.

Indeed one can show:

Cor The following are equivalent:

(a) Σ is a manifold

(b) Σ is locally finite, i.e. links are finite

(c) every proper special subgroup of W is finite.

Note that these conditions can be read off of the diagram by Rem. 5.21.

ÜA If (c) holds and W is infinite, then W must be irreducible.

Note that the condition (c) above is quite restrictive. There are only 3 cases for it to be satisfied:

- Σ is a sphere (and W is finite)
- Σ is some Euclidean space (and W affine and irreducible)
- Σ is some hyperbolic space $\mathbb{H}^n$ in which case W is generated by the reflections along the faces of a closed simplex in $\mathbb{H}^n$ all of whos vertices are in the interior of $\mathbb{H}^n$.

$\rightsquigarrow$ see Bourbaki 4-6 for details.