5 A little bit of geometry of Coxeter groups

A geometric realization of finite and affine Coxeter groups

From the Tits representation we obtain in some cases a canonical representation of Coxeter groups as "honest" reflection groups on S^n, respectively some $\mathbb{R}^n$.

We now state the corresponding result without proofs. For details see • Humphreys Sec 6.4, 6.5
or • Davis, Chapter 6.

(or Thomas, Sec 3.5, for a summary)

Def 5.1
A Coxeter system (W,S) is <u>reducible</u> if $S = S' \sqcup S''$ with S', S'' nonempty, proper subsets of S, such that $m_{ij} = 2$ $\forall s_i \in S'$ and $s_j \in S''$.
We say (W,S) is <u>irreducible</u> if it is not reducible.

Bem 1) If (W,S) is reducible and S', S'' as above, then $W \cong \langle S' \rangle \times \langle S'' \rangle$.
i.e. reducability yields an "obvious"/visible direct product decomposition of the Coxeter

direct product decomposition of the Coxeter group. The factors correspond to the connected components of the Coxeter diagram of W.

2) If (W,S) is reducible the group W may still split as a direct product.
However, the factors may not correspond to subsets of S.
E.g. $D_{12} \hat{=}$  and $D_{12} \cong D_6 \times \mathbb{Z}_2$.

Let in the following B denote the bilinear form introduced with the Tits representation.

Thm 5.2

(W,S) irreducible, $n=|S|$, then:

(1) B is positive definite $\Leftrightarrow$ W is finite.

In this case W is a geometric reflection grp generated by reflections along sides of a spherical simplex in $\mathbb{S}^{n-1}$.
If s_i, s_j are reflections along codim-1 faces F_i and F_j, then these faces meet at an angle of $\frac{\pi}{m_{ij}}$.

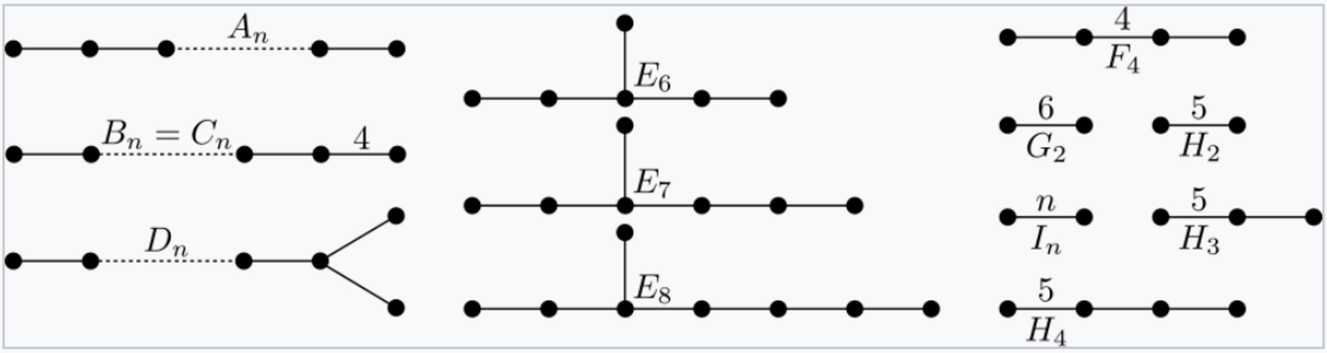
(2) B is positive semi-definite of corank 1 $\Leftrightarrow$ W is a Euclidean reflection grp.

In this case W is generated by refl.

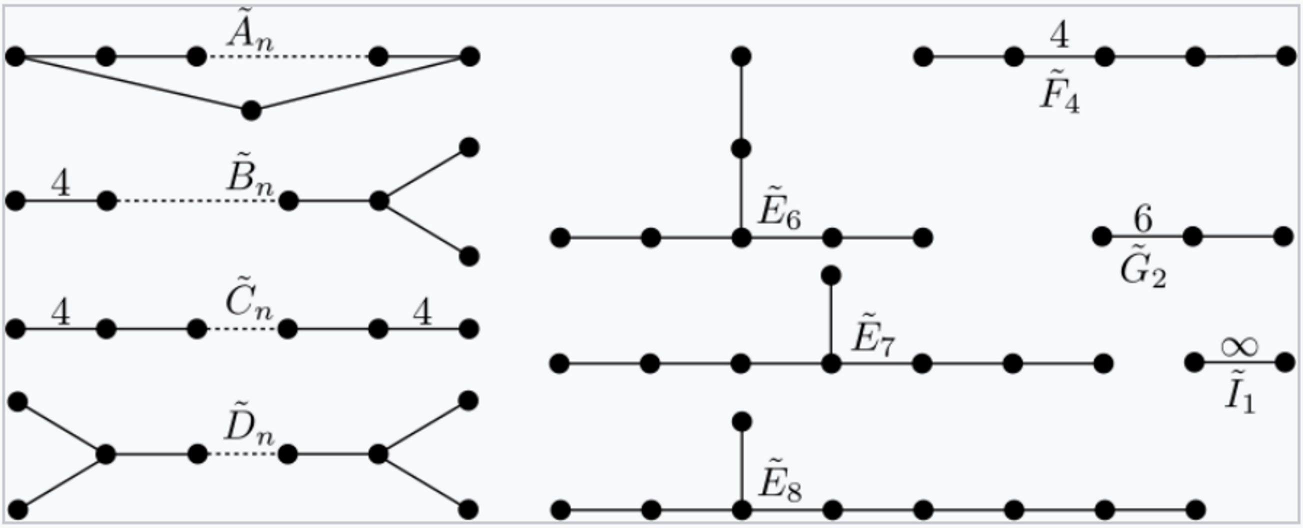
In this case W is generated by refl. along the codim-1 faces of a simplex in $\mathbb{E}^{n-1}$. Again reflections s_i, s_j along faces F_i, F_j meet at angle $\frac{\pi}{m_{ij}}$ in case $n \geq 3$.
If $n=2$, then the simplex is an interval and m_{ij} must be ∞. ($W \cong D_\infty$ then).

Idea of proof: generalize the construction we have seen in the D_∞ case.
In case (2) the affine space $\mathbb{E}^{n-1}$ is always a slice across the Tits-cone – as we have seen for the D_∞-case.

5.3. Classification of finite and affine Coxeter groups



Coxeter graphs of the finite Coxeter groups



Coxeter diagrams for the affine Coxeter groups

Since we always want a geometric object to act on (not only in the finite/affine case) we now define Coxeter complexes. This needs some preparation.

<u>Def 5.4</u> (W,S) Coxeter system. For all $T \subset S$ define the <u>special subgroup W_T</u> of W as the group $W_T := \langle T \rangle \leq W$.
For $T = \emptyset$ we put W_T to be the trivial group.

Special subgroups are also called <u>(standard) parabolic subgroups</u>.

We also use the following <u>notation</u>:
Let I be an indexing set for S. Then put $W_J := \langle s_j \mid j \in J \rangle$ for all subsets $J \subseteq I$.

Thm 5.5 (without proof) uses the Tits representation

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uses the Tits representation

ÜA Humphr. Sec 5.5

(W,S) Coxeter system. Then

(1) (W_T, T) is a Coxeter system $\forall T \subseteq S$.

(2) $\forall T \subseteq S$ and $w \in W_T$ we have

$l_T(w) = l_S(w)$.

And any reduced expression for w uses only letters in T.

Hence $\mathrm{Cay}(W_T, T)$ embeds isometrically as a convex subgraph of $\mathrm{Cay}(W,S)$.

(3) If $T, T' \subseteq S$, then $W_T \cap W_{T'} = W_{T \cap T'}$ and $\langle W_T, W_{T'} \rangle = W_{T \cup T'}$.

(4) The map $T \longmapsto W_T$ is a bijection on

$\{\text{subsets of } S\} \longrightarrow \{\text{special subgroups of } W\}$

Many properties can be seen on the Coxeter diagram and read directly from S.

We will now construct a simplicial complex from the set of special subgroups.

Def 5.6 special cosets

(W,S) a Coxeter system. A special coset is a set of the form $w\langle T\rangle$ with $T\subseteq S$, $w\in W$. The set $\{w\langle T\rangle \mid w\in W, T\subseteq S\} =: \Sigma = \Sigma(W,S)$ forms a partially ordered set with the ordering being reverse inclusion, i.e.

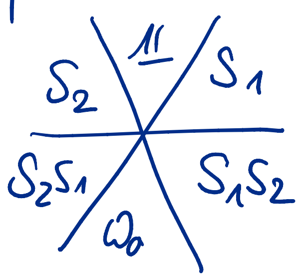
$$w_1\langle T_1\rangle \geq w_2\langle T_2\rangle \text{ if (by definition) } w_1\langle T_1\rangle \subseteq w_2\langle T_2\rangle$$

greater equal, ordering

contained as sets

Ex. 5.7

$W = Sym(3)$, $S=\{s_1,s_2\}$



special subgroups:
$W_\emptyset=\{1\}$, $W_{\{1\}}=\langle s_1\rangle$, $W_{\{2\}}=\langle s_2\rangle$
and also $W=W_S$

special cosets:

$s_1W_\emptyset$

$$\Sigma = \{W_\emptyset, \{s_1\}, \{s_2\}, \{s_1s_2\}, \{s_2s_1\}\ \{w_0\}$$
$$s_2\langle s_1\rangle,\ s_1s_2\langle s_1\rangle,\ w_0\underbrace{\langle s_1\rangle}_{=W_{\{s_1\}}},$$
$$s_1\langle s_2\rangle,\ s_2s_1\langle s_2\rangle,\ w_0\langle s_2\rangle,\ W\}$$

e.g. $s_1W_\emptyset \leq W_{\{s_1\}}$

We will show that Σ is in fact a simplicial (chamber) complex in the following sense:

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Def 5.8 Simplicial complex

A simplicial complex with vertex set V is a collection Δ of finite subsets of V, called simplices, such that every singleton $\{v\}$ is a simplex and every subset of a simplex A is also a simplex, called face of A. The cardinality $r := \# A$ is called the rank of A and $r-1$ its dimension.

We include the empty set as a simplex. It has rank 0 and dimension -1.

A subcomplex of Δ is a subset Δ' which contains, for each of its elements A, all faces of A. i.e. Δ' is a simplicial complex on its own with vertices some subset V' of V.

Note: Δ is a poset ordered by the face relation. As a poset Δ satisfies:

(i) any two simplices $A, B \in \Delta$ have a greatest lower bound $A \cap B$.

(ii) $\forall A \in \Delta$ the poset $\Delta_{\leq A}$ of faces of A is isomorphic to the poset of A is isomorphic to the poset of the subsets of $\{1, 2, \dots, r\}$ for some r.

Conversely, any poset satisfying (i) and (ii) is the face poset of a simplicial complex, where the rank 1 elements are taken to be the vertices of the simplicial complex.

Def 5.9 geometric realization

The geometric realization $|\Delta|$ of a simplicial complex Δ is a topological space partitioned into (open) simplices $|A|$, one for each non-empty A in Δ.

It is constructed as follows:

Let V be a vector space over $\mathbb{R}$ with V as a basis. Let $|A|$ be the interior of the convex hull spanned by the vertices of A, i.e. $|A|$ consists of

$$|A| = \left\{ \sum_{v \in A} \lambda_v v \;/\; \lambda_v > 0 \text{ and } \sum_{v \in A} \lambda_v = 1 \right\}$$

and we put:

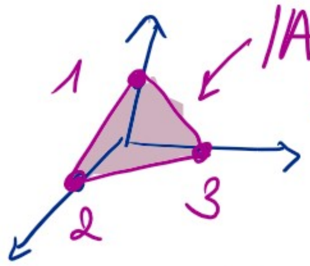
$$|\Delta| = \bigcup_{A \in \Delta} |A|$$

Ex: $V = \{1, 2, 3, 4\}$

$\Delta = \{\{1,2,3\}, \{2,3,4\}, \{1,2\}, \{2,3\}, \{1,3\},$
$\{2,4\}, \{3,4\}, \{i\}, i = 1,2,3,4\}.$

Then V is 4-dimensional

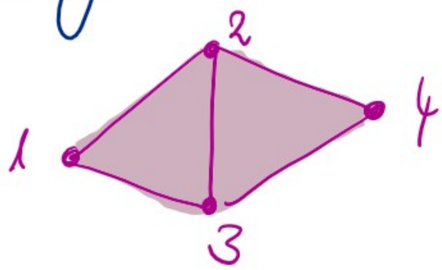
$A = \{1,2,3\}$ yields



$|A|$ is the interior of this triangle

$B = \{1,2\}$ corresponds to the open edge 

And $|\Delta|$ is given as:



inside 4-space where each vertex corresp. to a basis vector.