08 - Faithfulness of the Tits presentation

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more on the
faithfulness of the Tits presentation.
See [Thomas, chapter 3.5], [Davis, Appendix D].

Prop 4.8

Sei (W,S) Coxeter System mit $|S|=n$ und sei $\rho: W \to GL_n(\mathbb{R})$ die Tits Darstellung, wie in Thm 4.2 konstruiert.

Dann gilt: ρ ist treu.

Vorbereitungen für den Beweis:

Wir betrachten die duale Darstellung $\rho^*: W \to GL(V^*)$ gegeben durch

$$\Big(\underbrace{\rho^*(\overset{\in W}{w})}_{\in GL(V^*)}(\underbrace{\varphi}_{\in V^* \text{ duales VR}})\Big)(\underbrace{v}_{\in V}) = \varphi(\rho(w^{-1})(v))$$

Es reicht zu zeigen, dass ρ^* treu ist.

Das im Beweis von Thm 4.2 konstruierte B ist eine Bilinearform.
Also können wir für alle $\varphi_i \in V^*$ definieren:

$$\varphi_i(v) := B(e_i, v).$$

Define a hyperplane H_i^* of V^* by putting:

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$$H_i^* := \{\varphi \in V^* \mid \varphi(e_i) = 0\}$$

Let $\sigma_i^* := \rho^*(s_i)$.
Then this element of $GL(V^*)$ is given by the formula:

$$\sigma_i^*(\varphi) = \varphi - 2\varphi(e_i)\cdot\varphi_i$$

for $\varphi \in V^*$.

One can check:
- $\sigma_i^*(\varphi_i) = -\varphi_i \quad \forall i$
- $(\sigma_i^*)^2 = \mathbb{1}$
- σ_i^* fixes H_i^* pointwise

Def 4.9

The chamber C associated to the Tits-representation is the following subset of V^*:

$$C = \{\varphi \in V^* \mid \varphi(e_i) \geq 0 \ \forall i \in I\}.$$

This is a simpl. cone in V^* cut out by the hyperplanes H_i^*

Example 4.10

(i) If $W = D_{2m}$, $m = m_{ij} < \infty$.
The dual space V^* may be identified with $\mathbb{E}^2$. The chamber C is a closed Euclidean sector with vertex angle $\frac{\pi}{m}$.

(ii) $W = D_\infty$, then the chamber C is the closed sector bounded by H_i^* and H_i^*.

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Case m=3 in example (i) of 4.10

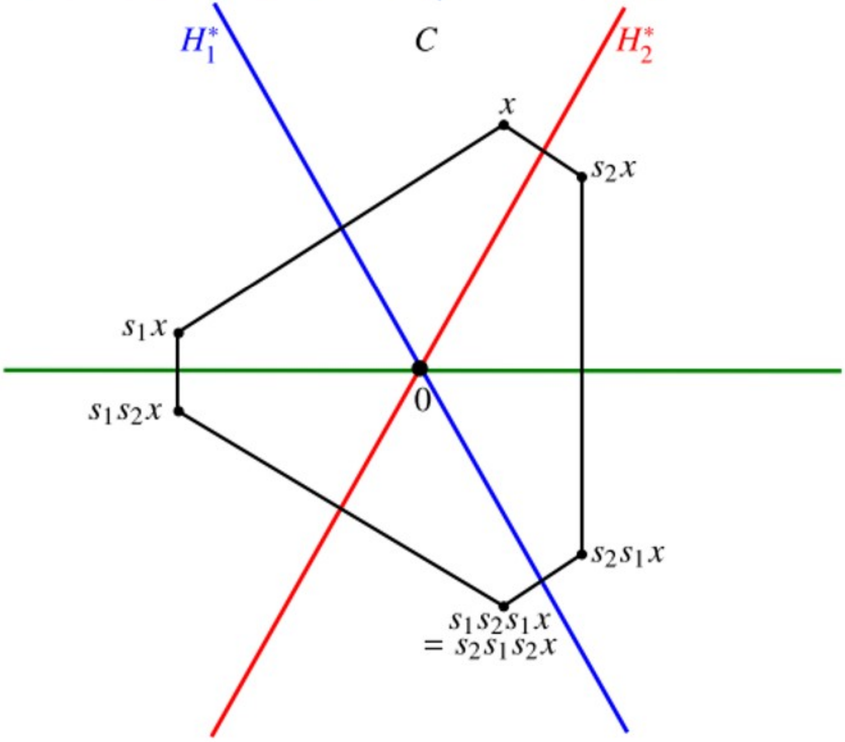

Figure 3.3. A Coxeter polytope for $W = D_6 = \langle s_1, s_2 \rangle$.

W=D∞ in case (ii) of 4.10

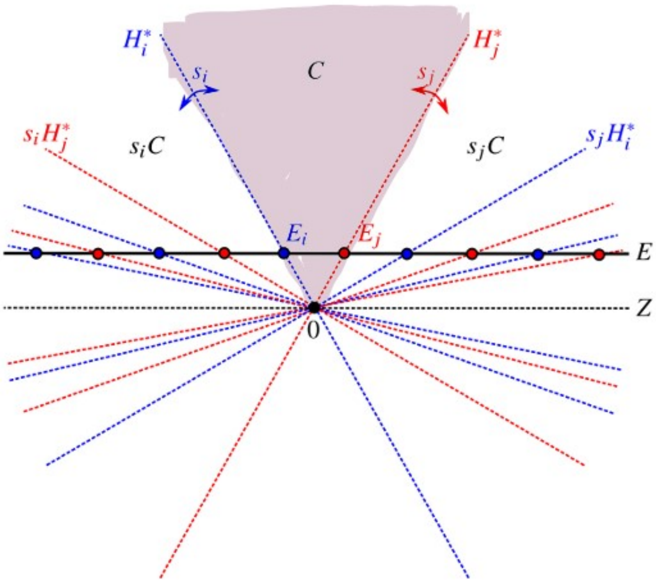

Figure 3.2. The dual space V_{ij}^* in the case $m_{ij} = \infty$, with its linear subspaces Z, H_i^* and H_j^*, and its affine subspace $E = Z + 1$. The group $W_{ij} = \langle s_i, s_j \rangle \cong D_\infty$ acts on the Euclidean space E as a geometric reflection group, generated by reflections in the codimension-1 subspaces E_i and E_j of E. The chamber C and some of its images are labelled. Compare with Figure 1.2.

Figure 3.2. The dual space V_{ij}^* in the case $m_{ij} = \infty$, with its linear subspaces Z, H_i^* and H_j^*, and its affine subspace $E = Z + 1$. The group $W_{ij} = \langle s_i, s_j \rangle \cong D_\infty$ acts on the Euclidean space E as a geometric reflection group, generated by reflections in the codimension-1 subspaces E_i and E_j of E. The chamber C and some of its images are labelled. Compare with Figure 1.2.

Main ingredient for the proof of Prop 3.5 is the following:

Thm 4.11 (Tits) See e.g. [D] Appendix D

Let $w \in W$. If $wC^\circ \cap C^\circ \neq \emptyset$ then $w = \mathbb{1}$.

Idea of proof

Use the following sets:

$$A_s := \{ w \in W \mid wC^\circ \subset H_s \} \quad \forall s \in S$$

where $H_s := \{ \xi \in V^* \mid e_i(\xi) > 0 \}$ ← open half-space in V^*

$\uparrow$ linear form on V^*

then prove

4.12 Lemma D.1.5 in Davis

Given $s, t \in S$ and $w \in W_{\{s,t\}}$, then $w(A_s \cap A_t)$ is either contained in A_s or in sA_s. (P)

In the second case $l(sv) = l(v) - 1$.

in the second case $x(sv) = x(v)$ ...

(Done by explicit investigation of ρ)
~ 1 page

Then use the following lemma by Tits:

4.13 Tits' Lemma [D, Lemma 4.8.3]

Suppose that for all distinct $s, t \in S$
(P) is satisfied, then $(W, S, \{B_s\}_{s\in S})$ satisfies (P),
moreover:
given $w \in W$, $s, t \in S$, put $B := \bigcap_{s\in S} B_s$
if $wB \subset B_s$ and $wtB \not\subset B_s$
then $sw = wt$.

„D"

⤳ Key point: translate faithfulness of ρ
to a geometric question
about cones

Consequences of the faithful representation:

Def 4.14 The Tits-cone of W is the subset

$$\bigcup_{w\in W} wC \quad \text{of } V^*,$$ with C as in Def. 4.9.

Example 4.15

(i) $W = D_{2m}$, m finite. Then $V^* \cong E^2 = \bigcup_{w \in W} wC$.

i.e. Tits cone is the whole space.

(ii) $W = D_\infty$, $V^* = V_{ij}^*$ then the Tits cone is the set $\{\varphi \in V_{ij}^* \mid \varphi(e_i + e_j) > 0\} \cup \{0\}$.

In the figure the Tits cone is the open half-space bounded by Z containing the chamber C.

Cor 4.16

A Coxeter group may be identified with a discrete subgroup of $GL_n(\mathbb{R})$.

Proof: Since ρ is faithful it suffices to prove that $\rho(W)$ is a discrete subgroup of $GL_n(\mathbb{R})$.

We will in fact prove that $\rho^*(W)$ is a discrete subgrp of $GL(V^*)$ and we do so in proving that the identity element has an open neighbourhood in $GL(V^*)$ containing no other elements of $\rho^*(W)$.

interior of chamber C

Consider the action on $C^\circ \subset C \subset V^*$.

Let $\varphi \in C^\circ$. Put

$$U := \{g \in GL(V^*) \mid g(\varphi) \in C^\circ\}.$$

Then U is open and contains the identity.

Then U is open in $GL(V^*)$ and contains the identity.

By Theorem 4.11 we have $U \cap \rho^*(W) = \{id\}$.
Hence $\rho^*(W)$ is a discrete subgroup of $GL(V^*)$ and hence $\rho(W)$ is also discrete in $GL(V)$. $\square$

Next: What kind of consequences does linearity of the representation have?

Def. 4.17 A group G is linear if there exists a faithful representation $\rho : G \to GL_n\mathbb{R}$ for some $n \in \mathbb{N}$.

Cor. 4.18 Coxeter groups and their subgroups are linear.

↑ follows directly from Tits' representation and its properties.

Linearity has strong consequences.
(see: Bogdan Nica: Linear groups (2013) arXiv: 1306.2385)

We will state two after some preparatory definitions.

e.g. finite, torsion free,

definitions.

e.g. finite, torsion free, abelian, solvable, ...

Def. 4.19

Let P be a group theoretic property. We say a group G has <u>virtually P</u> if G has a finite index subgroup which satisfies P.

Ex. Every finite extions G of H has virtually all properties the group G itself has.

Thm 4.20 <u>Selberg's Lemma</u>

Finitely generated linear groups are vitually torsion free.

Def. 4.21

A group G is <u>residually finite</u> if for each non-trivial $g \in G$ there exists a finite group H_g and a homomorphism $\varphi_g : G \to H_g$ s.th. $\varphi_g(g) \neq \mathbb{1}$.

(many other, equivalent, definitions exist).

Thm 4.22 <u>Malcev's Thm</u>

Finitely generated linear groups are residually finite.

Cor. 4.23 Coxeter groups are residually finite and virtually torsion free.