4 The Tits representation

4.1 Definition

A representation of a group G on a $\mathbb{K}$-VS V is a group homomorphism $\rho: G \longrightarrow GL(V)$.
The repr. ρ is faithful if it is injective.

Faithful representations realize G as a subgroup of a linear group.

The goal is to prove the following result:

Thm 4.2

Let I be a finite set, $S = \{s_i\}_{i \in I}$, $\mathcal{M}$ Coxeter mx.
Let $W = \langle S \mid (s_i s_j)^{m_{ij}},\ \forall i,j \in I \rangle$.
Then there is a (faithful) representation

$$\rho: W \longrightarrow GL_n(\mathbb{R})$$

with $n = |S| = |I|$ s.th.

(i) $\forall i \in I$ $\sigma_i := \rho(s_i)$ is a linear involution with fixed set a hyperplane in $\mathbb{R}^n$.
$\rho(s_i)$ need not be an orthogonal reflection

(ii) $\forall i \neq j$ the product $\sigma_i \sigma_j$ has order m_{ij}.

We explicitly construct this representation:

Definition 4.3

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Let (W,S) be as in the statement of Thm 4.2. and suppose $I=\{1,\dots,n\}$, $V\cong\mathbb{R}^n$, $GL(V)=GL(n,\mathbb{R})$.

We define a symmetric bilinear form B on V:

$$B(e_i,e_j):=\begin{cases}-\cos\left(\frac{\pi}{m_{ij}}\right) & \text{if } m_{ij} \text{ finite}\\ -1 & \text{if } m_{ij}=\infty\end{cases}$$

Note: $B(e_i,e_i)=1$, $B(e_i,e_j)\leq 0$ if $i\neq j$.

Define hyperplanes H_i and involutions σ_i by putting

$$H_i=\{v\in V \mid B(e_i,v)=0\}$$

$$\sigma_i: V\to V: \quad \sigma_i(v)=v-2B(e_i,v)e_i$$

Then σ_i is a linear map with

- $\sigma_i(e_i)=-e_i$
- $\mathrm{Fix}(\sigma_i)=H_i$
- σ_i preserves B

Prop. 4.4 Notation as in 4.3

(1) the product $\sigma_i\sigma_j$ has order m_{ij}

(2) the map $s_i\longmapsto\sigma_i$ extends to a homomorphism $W\to GL(V)$.

Proof: (2) follows from (1) since we already observed that $\sigma_i^2=id$.

To see (1) argue as follows:

let $V_{ij}:=\mathrm{span}(e_i,e_j)$. Then σ_i and σ_j preserve

To see (ii) argue as follows

Let $V_{ij} := \mathrm{span}(e_i, e_j)$. Then σ_i and σ_j preserve the space V_{ij} by definition.

Consider the restriction of $\sigma_i\sigma_j$ to V_{ij}.

<u>Case 1 $m_{ij} < \infty$</u> Put $v := \lambda_i e_i + \lambda_j e_j \in V_{ij}$, $v \neq 0$

Then
$$B(v,v) = \lambda_i^2 - 2\lambda_i\lambda_j \cos\left(\frac{\pi}{m_{ij}}\right) + \lambda_j^2$$
$$= \left(\lambda_i - \lambda_j \cos\left(\frac{\pi}{m_{ij}}\right)\right)^2 + \lambda_j^2 \sin\left(\frac{\pi}{m_{ij}}\right)^2 > 0.$$

Hence B is positive definite on V_{ij}. (possibly not on all of V!)

This allows us to identify V_{ij} with $\mathbb{R}^2$ and $B|_{V_{ij}}$ with the standard inner product.

Then the restrictions $\sigma_i|_{V_{ij}}$ and $\sigma_j|_{V_{ij}}$ are orthog. reflections in the lines $H_i \cap V_{ij}$ and $H_j \cap V_{ij}$ through the origin.

$H_i \cap V_{ij}$ is perpendicular to e_i

$H_j \cap V_{ij}$ is — " — e_j.

We obtain the angle between these lines by computing:
$$B(e_i, e_j) = -\cos\left(\frac{\pi}{m_{ij}}\right) = \cos\left(\pi - \frac{\pi}{m_{ij}}\right)$$

Hence the angle between e_i, e_j is $\left(\pi - \frac{\pi}{m_{ij}}\right)$ and thus the angle between the lines amounts to $\frac{\pi}{m_{ij}} = \pi - \left(\pi - \frac{\pi}{m_{ij}}\right)$.

But then the product of two orthogonal reflections along lines meeting at an angle $\frac{\pi}{m_{ij}}$ is a rotation around the intersection point of order m_{ij}.

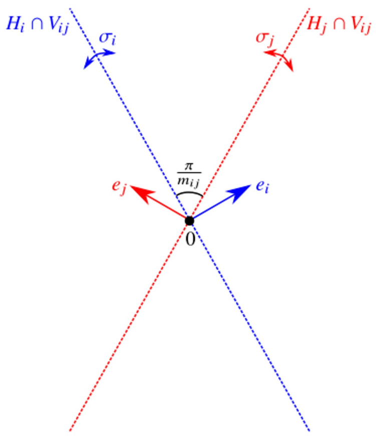

Figure 3.1. Orthogonal reflections obtained by restricting σ_i and σ_j to $V_{ij} = \mathrm{span}(e_i, e_j) \cong \mathbb{R}^2$, when m_{ij} is finite.

This implies $(\sigma_i \sigma_j)\big|_{V_{ij}}$ has order m_{ij}.

We need to see what happens on $V_{ij}^{\perp}$:

$$V_{ij}^{\perp} = \{ v' \in V \mid B(v', v) = 0 \quad \forall v \in V_{ij} \}.$$

B is positive definit on V_{ij} $\Rightarrow$ $V_{ij} \oplus V_{ij}^{\perp} = V$.

<u>Claim</u> σ_i fixes $V_{ij}^{\perp}$ pointwise.

<u>Proof</u>: Let $v \in V_{ij}^{\perp}$, then

$$\sigma_i(v) \underset{\text{def}}{=} v - 2 \underbrace{B(e_i, v)}_{= 0 \text{ as } e_i \in V_{ij}} e_i = v \quad . \quad \square$$

Same is true for σ_j.

Thus $\sigma_i \sigma_j$ has order m_{ij} on V. Hence (i)

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Thus $\sigma_i\sigma_j$ has order m_{ij} on V. Hence (i) in case 1.

Case 2: $m_{ij} = \infty$

Again let $v = \lambda_i e_i + \lambda_j e_j \in V_{ij}$. Then

$$\mathcal{B}(v,v) = \lambda_i^2 - 2\lambda_i\lambda_j \cos\left(\tfrac{\pi}{\infty}\right) + \lambda_j^2$$
$$= (\lambda_i - \lambda_j)^2 \geq 0$$

where „$=0$" $\Leftrightarrow$ $\lambda_i = \lambda_j$.

Hence $\mathcal{B}$ is positive semidefinite but not positive definite on V_{ij}.

We want to check that the product $\sigma_i\sigma_j$ has infinite order on V_{ij}:

Consider $\sigma_i\sigma_j(e_i) = \sigma_i(e_i - 2\underbrace{\mathcal{B}(e_j,e_i)}_{=1}e_j)$
$$= \sigma_i(e_i - 2e_j)$$
$$= -e_i + 2(e_j + 2e_i)$$
$$= e_i + 2(e_i + e_j).$$

By induction we obtain
$$(\sigma_i\sigma_j)^k(e_i) = e_i + 2k(e_i + e_j)$$

and hence $\sigma_i\sigma_j$ has infinite order on V_{ij} and (for the same reasons as above) also on V.

$\Rightarrow$ (1)

More on faithfulness later! $\square$

This implies the theorem.

<u>Corollary 4.5</u> Notation as in 4.3

In a Coxeter system (W,S) the generators $s \in S$ are pairwise distinct involutions of W.

<u>Proof</u> Each σ_i is an involution by construction.

Since the order of $\sigma_i \sigma_j$ is m_{ij} we obtain that σ_i and σ_j are distinct for $i \neq j$.

As ρ is a representation also s_i and s_j must be distinct. $\square$

<u>Cor. 4.6</u>

(W,S) Coxeter system with Coxeter mx $M = (m_{ij})_{ij}$.

Then the product $s_i s_j$ has order m_{ij}.

<u>Proof</u> if $m_{ij} < \infty$: then $(s_i s_j)^{m_{ij}} = \mathbb{1}$ means $s_i s_j$ has order at most m_{ij}.

Since the image $\rho(s_i s_j) = \rho(s_i)\rho(s_j) = \sigma_i \sigma_j$ has order m_{ij} the same must be true for $s_i s_j$.

in $\rho(W) \leq GL(n,\mathbb{R})$

$m_{ij} = \infty$: then $\rho(s_i s_j)$ has infinite order and so has $s_i s_j$. $\square$

<u>4.7 The geometry behind case 2 in the proof above:</u>

a long example for $W = D_\infty$.

Continue notation from above.

$m_{ij} = \infty$

$m_{ij} = \infty$

Define $A := \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

This matrix describes $B|_{V_{ij}}$ with respect to the basis e_i, e_j of $V_{ij} = \mathrm{span}(e_i, e_j) \subset V$.

We have:

$$\mathrm{null}(A) = \mathrm{span}\left(\begin{pmatrix}1\\1\end{pmatrix}\right) = \mathrm{span}(e_i + e_j)$$

1-dim $\quad L := \{ v \in V_{ij} \mid B(v,v) = 0 \}$.

$\Rightarrow$ B induces a positive definite form on the quotient: $\quad V_{ij}/\mathrm{null}(A)$.

We can identify this quotient with a Euclidean VS of dimension $\dim(V_{ij}) - 1$.

Put $W_{ij} := \langle s_i, s_j \rangle \leq W$. As $s_i s_j$ has order ∞ we have $W_{ij} \cong D_\infty$. this is a Euclidean reflection group

We will recover a natural action of W_{ij} on the real line from the Tits representation:

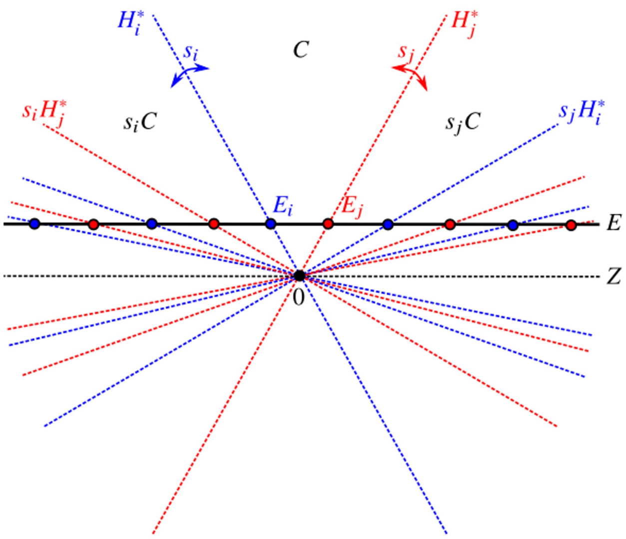

Figure 3.2. The dual space V_{ij}^* in the case $m_{ij} = \infty$, with its linear subspaces Z, H_i^* and H_j^*, and its affine subspace $E = Z + 1$. The group $W_{ij} = \langle s_i, s_j \rangle \cong D_\infty$ acts on the Euclidean space E as a geometric reflection group, generated by reflections in the codimension-1 subspaces E_i and E_j of E. The chamber C and some of its images are labelled. Compare with Figure 1.2.

Observe:

- W_{ij} acts on V_{ij} (i.e. $\rho(W_{ij})$ is a group of linear transformations of V_{ij})
- (ÜA) the representation $\rho|_{W_{ij}}$ is faithful on V_{ij}
- the group $\rho(W_{ij})$ fixes null(A) pointwise

  we say „W_{ij} fixes ....“ with slight abuse of notation

  as $s_i \cdot (e_i + e_j) = s_j \cdot (e_i + e_j) = e_i + e_j$.

Consider the dual VS

$$V_{ij}^* = \{ \text{linear functionals } \varphi : V_{ij} \to \mathbb{R} \}.$$

The group W_{ij} acts on V_{ij}^* via

$$(w \cdot \varphi)(v) := \varphi(\underbrace{w^{-1} \cdot v}_{\text{short for } (\rho(w^{-1}))(v)})$$

where $w \in W_{ij}$, $\varphi \in V_{ij}^*$, $v \in V_{ij}$.

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This action is faithful as ρ is faithful.

$\Rightarrow W_{ij} \cong D_\infty \curvearrowright V_{ij}^*$ faithfully.

Consider the codim 1 linear subspace Z of V_{ij}^*

$$Z := \{ \varphi \in V_{ij}^* \mid \varphi(e_i + e_j) = 0 \}.$$

The group $\rho(W_{ij})$ fixes $e_i + e_j$ and hence preserves Z.

We may identify Z with $V_{ij}/\mathrm{null}(A)$.

$\Rightarrow Z$ has a Euclidean structure (of $\dim(V_{ij}^*)-1$).

Define an affine subspace E of V_{ij}^* as follows:

$$E = \{ \varphi \in V_{ij}^* \mid \varphi(e_i + e_j) = 1 \}.$$

Translation of Z

Then E also has a Euclidean structure (of dimension $\dim(V_{ij}^*)-1$).

Observe that W_{ij} fixes $e_i + e_j$ and hence stabilizes E.

As a subset of V_{ij}^* the space E spans V_{ij}^*.

And W_{ij} acts faithfully on V_{ij}^*.

Hence the action on E is faithful.

Consider the codim 1 linear supspace H_i of V_{ij}^*:

$$H_i^* := \{ \varphi \in V_{ij}^* \mid \varphi(e_i) = 0 \}.$$

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$H_i^* \neq Z \Rightarrow E_i := E \cap H_i^*$ is a codim 1 hyperpl. of E i.e. a point

We have:

- $s_i e_i = -e_i$, $s_i^2 = id$

$\Rightarrow$ s_i acts as a reflection along E_i.

Same goes through for s_j and $E_j \neq E_i$.

$\Rightarrow$ W_{ij} isometrically acts on E and is generated by reflections along two endpoints E_i, E_j of an interval.

This is exactly the presentation as a Euclidean reflection group!