06 The Deletion and Exchange condition - part 3

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We finish the proof of Theorem 3.16.

<u>Proof of (3) $\Rightarrow$ (4)</u>: deletion condition $\Rightarrow$ exchange condition.

Assume W is generated by a set S of distinct involutions.
Assume the deletion condition holds.

Suppose $(s_1, \dots, s_k)$ is a reduced word for w.
Let $s \in S$. Then $\ell(sw) \leq \ell(s) + \ell(w) = k+1$.

- $\ell(sw) = k+1$: there is nothing to prove.
- $\ell(sw) < k+1$: In this case the word $(s, s_1, s_2, \dots, s_k)$ for sw is non-reduced as it has $k+1$ letters.

  Since $(s_1, \dots, s_k)$ is reduced one of the two deleted letters must be s.

  Hence there exists i s.th.

  $$sw = ss_1 \cdots s_k = \hat{s}\, s_1 \cdots \hat{s}_i \cdots s_k$$
  $$= s_1 \cdots \hat{s}_i \cdots s_k .$$

  Multiplication on the left by s yields:

  $w = ss_1 \cdots \hat{s}_i \cdots s_k$ as desired. $\square$ (3) $\Rightarrow$ (4)

We finish the proof of 3.16 a little later.
To prepare us we need to learn about

3.24 The word problem

Let G be a finitely generated group with finite generating set S.
Does there exist an algorithm to determine whether two words in S represent the same group element?

Not all groups have solvable WP!
However, Coxeter groups do. We will not prove it, but explain the statement of the relevant theorem.

Def 3.25 braid moves

W generated by a set S of distinct involutions.
Given $s,t \in S$, $s \neq t$, $m_{st} := \mathrm{ord}(st)$.
If $m_{st} < \infty$, then a braid move on s and t swaps a subword $\underbrace{stst\ldots}$ in some word in S for the subword $\underbrace{tsts\ldots}_{m_{st}\text{-many letters}}$.

m_{st} – many letters.

Rmk: By construction $\underbrace{stst\ldots}_{m_{st}-} = \underbrace{tsts\ldots}_{m_{st}\text{ many}}$ in W.

Exercise: look at the examples given in Ex 8.11 and explain what a braid move does to a path determined by a given word in S.

Thm 3.26 Tits' solution to the word problem

Suppose W is generated by a set S of distinct involutions. Suppose the exchange condition holds. Then

(1) A word $(s_1, \ldots, s_k)$ in S is reduced if and only if it cannot be shortened by a sequence of
  a) braid moves and
  b) deletions of subwords (s, s), $s \in S$.

(2) two reduced expressions represent the same group element $w \in W$ if and only if they are related by a finite sequence of braid moves.

Using this theorem we prove (V) $\Rightarrow$ (I)

Using this theorem we prove (4) $\Rightarrow$ (1) in theorem 3.16

Assume W is generated by a set S of distinct involutions and assume also that the exchange condition holds.

Prove: (W,S) is a Coxeter system.

Let $m_{ij}' := \mathrm{ord}(s_i s_j)$ in W.

Let further (W', S') be the Coxeter system with $S' = \{ s_i', i \in I \}$ in bijection with S via $s_i \mapsto s_i'$ and with Coxeter matrix $M' = (m_{ij}')_{ij}$.

Then there exists a surjective homomorphism $\phi : W' \longrightarrow W : s_i' \longmapsto s_i$.

To prove (W,S) is a Coxeter system we show that ϕ is injective.

Suppose $1 \neq w' \in \ker(\phi)$. Let w be represented by a reduced expression $(s_{i_1}' \dots s_{i_k}')$ in S'.

By construction $\phi(w')$ is represented by the word $(s_{i_1}, \dots, s_{i_k})$ in S.

Since $\phi(w') = 1$ in W this word is <u>not</u> reduced. By theorem 3.26(1) the word since $\phi(w') = \mathbb{1}$ in W this word is <u>not</u> reduced. By theorem 3.26(1) the word can be reduced by moves of type a) and b). But then also $(s'_{i_1}, \dots, s'_{i_k})$ could be reduced.

Hence $(s_{i_1} \dots s_{i_k})$ is reduced if and only if $(s'_{i_1} \dots s'_{i_k})$ is and thus $w' = \mathbb{1}$ in W' already. This is saying that ϕ is injective.

$\square$

This completes the proof of theorem 3.16!

<u>Rmk: Dehn's classical problems:</u>

the word-, conjugacy- and isomorphism problems.

$\rightsquigarrow$ if time permits we will learn more about one of them