# 05 The deletion and exchange condition - part 2

Freitag, 5. April 2024 12:13

The goal is still to prove theorem 3.16.
We collect some further Lemma.

Lemma 3.20

Assumptions as in 3.19.
For each $r \in R = WSW^{-1}$ the set $X \setminus H_r$ has at most 2 components.

Proof:
Since $R = WSW^{-1}$ we have $r = wsw^{-1}$ for some $w \in W$, $s \in S$.
Now $H_r = H_{wsw^{-1}} = w \cdot H_s$ by def. of $H_r$.
As $W$ acts by isometries on $X = Cay(W, S)$ it suffices to check that $X \setminus H_s$ has at most 2 components for all $s \in S$.
To see this we prove that $\forall s \in S$ and $\forall v \in V(X)$ either the vertex $v$ or exclusive $sv$ is in the same component exclusive as the identity vertex $\mathbb{1}$.

If this was not the case we would find vertices $v_1 \neq v_2$ s.th. $v_1 = s \cdot v_2$ and $v_1, v_2$ are in the same component of $X \setminus H_s$

vertices $v_1 \neq v_2$ s.th. $v_1 = s \cdot v_2$ and $v_1, v_2$ are in the same component of $X \setminus H_s$.
But then $sv_1$ and $sv_2$ would be in two other distinct components of $X \setminus H_s$.
Then every path from $sv_1$ to $sv_2$ would cross $H_s$ and hence every path from $v_1$ to $v_2$ would cross $sH_s = H_s$ — which is a contradiction.

So let $v$ be a vertex and $\underline{w} = (s_1, \dots, s_k)$ be a reduced expression corresponding to a minimal path from $\mathbb{1}$ to $v$.
Let $r_1, \dots, r_k$ be the corresp. sequence of elements in $R$.
If $s \neq r_i$ for all $i$, then $\mathbb{1}$ and $v$ are in the same component of $X \setminus H_s$.
So assume that $s = r_i$ for some $i$.
Then Lemma 3.19 implies that $s \neq r_j$ $\forall j \neq i$ (as the word $\underline{w}$ is reduced).
Now $(s, s_1, \dots, s_k)$ corresponds to a path from $\mathbb{1}$ to $s \cdot v$ with sequence of elements in $R$ given by

$$s, r_1', r_2' \dots r_k' \quad \text{where } r_j' = s r_j s.$$

$s, r_1', r_2' \dots r_k'$ where $r_j' = s r_j s$.

Notice: as $r_i = s$ we have $r_i' = s \cdot s \cdot s = s$.

And: $r_j' = s r_j s \neq s \ \forall j \neq i$ as $r_j \neq s$.

Lemma 3.19 implies that we can delete exactly two instances of $s$ inside the sequence $(s, r_1', \dots, r_k')$ to obtain a word corresp. to a path from $\mathbb{1}$ to $sv$ which does not cross $H_s$.

Hence $\mathbb{1}$ and $sv$ are in the same component of $X \setminus H_s$. $\square$

For a word $\underline{w}$ in $S$ and $\mu$ the associated path in $X = Cay(W, S)$, $r \in R$, let
$$n(r, \underline{w}) := \# \ \mu \text{ crosses } H_r.$$

## Lemma 3.21

Assume $(W, S)$ is a Coxeter system.
Then $\forall w \in W$ and $\forall r \in R$ any word for $w$ crosses the wall $H_r$ the same number of times mod 2.

of times mod 2.

i.e. if $\underline{w}$ and $\underline{w}'$ are two words for $w \in W$, then

$$(-1)^{n(r,\underline{w})} = (-1)^{n(r,\underline{w}')}.$$

<u>Sketch of a proof</u> (see [D] for details).

Goal: we need to establish a well-defined homomorphism:

$$\varphi : W \longrightarrow \mathrm{Sym}(R \times \{\pm 1\})$$

given by:

for $\varepsilon \in \{\pm 1\}$ and $\underline{w}$ a word for $w$ one has

$$\varphi(w)(r,\varepsilon) = (wrw^{-1}, (-1)^{n(r,\underline{w})} \cdot \varepsilon).$$

We start by defining such a map on $S$:

for $s \in S$ put $\varphi(s) \in \mathrm{Sym}(R \times \{\pm 1\})$ to be

$$\varphi(s)(r,\varepsilon) = (srs, (-1)^{\delta_{rs}} \cdot \varepsilon).$$

here $\delta_{rs}$ is the Kronecker delta.

We have: $\varphi(s) \circ \varphi(s) = \mathrm{id}_{R \times \{\pm 1\}}$

and $\varphi$ is a bijection.

Extend to words $\underline{w} = (s_1, \dots, s_k)$ by putting

$$\varphi(\underline{w}) = \varphi(s_1) \circ \varphi(s_2) \circ \dots \circ \varphi(s_k)$$

$$\varphi(\underline{w}) = \varphi(s_k)\circ\varphi(s_{k-1})\circ \dots \circ\varphi(s_1).$$

By induction on $k$ check:

$$\varphi(\underline{w})(r,\varepsilon) = \left(s_k \cdots s_1 r s_1 \cdots s_k,\ (-1)^{n(r,\underline{w})}\cdot\varepsilon\right).$$

Finally prove that $\underline{w}\longmapsto\varphi(\underline{w})$ induces a homomorphism $\varphi: W\to \mathrm{Sym}(R\times\{\pm 1\})$.
In order to see this one checks that if $\underline{w}$ is the word for a relation in the Coxeter presentation of $W$, then $\varphi(\underline{w})$ is trivial:

- $\underline{w}=(s,s)$: then $\varphi(\underline{w}) = \varphi(s)\circ\varphi(s) = id$
- $\underline{w} = \underbrace{(s_i,s_j,\dots)}_{2m_{ij}\text{ letters}}$ $\underline{\text{z.z.}}$ $n(r,\underline{w})$ is even

to see this let $W_{ij} = \langle s_i, s_j\rangle < W$.
Since $s_i, s_j$ are distinct involutions and their product has finite order $m$ where $m$ divides $m_{ij}$, the group $W_{ij}$ is a finite dihedral group.
If $r\notin W_{ij}$, then $n(r,\underline{w})=0$.
Otherwise $n(r,\underline{w}) = 2m_{ij}/m$ which is even. $\square$

Corollary 3.22 (1) $\Rightarrow$ (2) in Thm 3.16.

That is:

$(W,S)$ Coxeter system $\Rightarrow$ $(X,R)$ reflection system.

Proof:

By 3.13 5) Lemma $(X,R)$ is a pre-reflection system. It remains to prove that $\forall r \in R$ the graph $X \setminus H_r$ has two components.

Lemma 3.20 $\Rightarrow$ at most 2 components.

It hence remains to prove that $H_r$ separates $X$.

Using arguments as in the proof of 3.20 it suffices to prove $H_s$ separates $\forall s \in S$.

Lemma 3.21 $\Rightarrow$ any path from $\mathbb{1}$ to $s$ crosses $H_s$ an odd number of times.

$\rightsquigarrow$ Hence at least once. $\square$

We now aim to prove (2) $\Rightarrow$ (3) in Thm 3.16:

Assume $W$ is generated by distinct involutions in $S$.

Put $X := Cay(W,S)$ and $R := WSW^{-1}$.

Put $X := \mathrm{Cay}(W,S)$ and $R := WSW^{-1}$.

**Lemma 3.23** (see [D] Le 3.2.14)

Assume $W, S, X, R$ are as above and the pair $(X,R)$ is a reflection system.
Suppose $\underline{w} = (s_1, \dots, s_k)$ is a word with associated reflections $r_1, \dots, r_k$.
Then $\underline{w}$ is reduced if and only if the reflections $r_i$ are pairwise distinct.

**Proof:**

Suppose first $r_i = r_j$ for $i \neq j$. Then Le 3.19 implies that $\underline{w}$ is not reduced.

Now let $w = s_1 \cdots s_k$ and put

vertices in $\mathrm{Cay}(W,S)$

$$R(1,w) = \left\{ r \in R \;\middle|\; \begin{array}{l} 1 \text{ and } w \text{ are in} \\ \text{distinct components} \\ \text{of } X \setminus H_r \end{array} \right\}.$$

Then $\forall r \in R(1,w)$ any path from $1$ to $w$ crosses $H_r$ at least once.

But then $r = r_i$ for some $i$.

Hence $\ell(w) \geq R(1,w)$.

Now since $(X,R)$ is a reflection system each $X \setminus H_r$ with $r \in R$ has exactly two each $X \setminus H_r$ with $r \in R$ has exactly two components.

This implies $r_i \in R(\mathbb{1}, w)$ for all refl. $r_i$ associated with the word.

This implies, since all $r_i$ are pairwise distinct, that $|R(\mathbb{1}, w)| \geq k$.

On the other hand $l(w) \leq k$.

$\Rightarrow$ $l(w) = k$ and the word $(s_1, \ldots, s_k)$ is reduced. $\square$

The deletion condition (3) now follows from Lemma 3.19 and Lemma 3.23.

$\square$ (2) $\Rightarrow$ (3)