# 04 The deletion and exchange condition - part 1

Donnerstag, 4. April 2024 16:21

Continuation of Chapter 3:

We will now introduce reflection systems.

**Def 3.12**

A <u>pre-reflection system</u> for a group $G$ is a pair $(X, R)$ where $X$ is a connected, simple graph on which $G$ acts by graph automorphisms and $R$ is a subset of $G$ s.th. the following are satisfied

(1) each $r \in R$ is an involution

(2) $R$ is closed under conjugation i.e. $\forall g \in G$ and $\forall r \in R$: $grg^{-1} \in R$.

(3) $R$ generates $G$

(4) $\forall$ edges $e$ in $X$ there exists $r_e \in R$ s.th. $r_e$ swaps the two endpoints of the edge $e$.

(5) $\forall r \in R$ there exists at least one edge in $X$ which is flipped by $r$.

**3.13 Example**

Put $X = \mathrm{Cay}(W, S)$

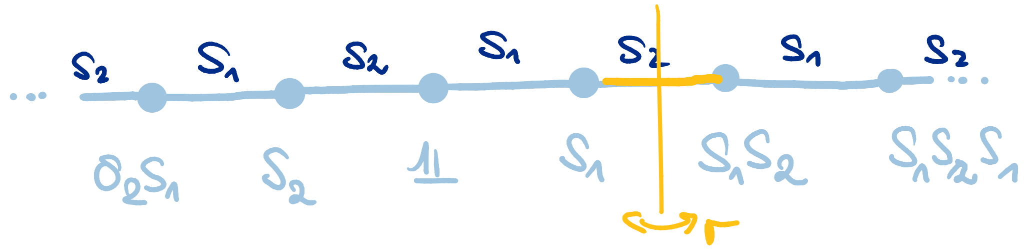
Put $X = \mathrm{Cay}(W,S)$

and put $R := WSW^{-1} = \{gsg^{-1} \mid g \in W, s \in S\}$.

Then $(X,R)$ is a pre-reflection system in all cases 1)–3) of Example 3.8.

1) $D_\infty = \langle s_1, s_2 \mid s_i^2 \rangle$ $\quad S := \{s_1, s_2\}$

$\mathrm{Cay}(D_\infty, S)$
$\shortparallel$
$W$




The "reflection" swapping the ends $s_1$ and $s_1s_2$ of the yellow edge is $r := s_1s_2s_1 \in R$

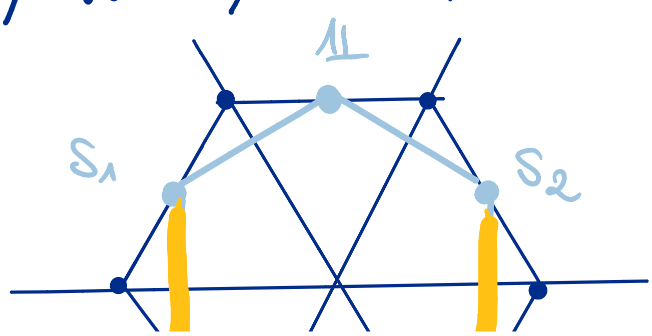
compute $r \cdot s_1 = s_1s_2\cancel{s_1} \cdot \cancel{s_1} = s_1s_2$ and

$r \cdot s_1s_2 = s_1\cancel{s_2}\cancel{s_1} \cdot \cancel{s_1}\cancel{s_2} = s_1$

More generally, the edge $\{g, gs\}$ is swapped by the element $gsg^{-1} \in R$.

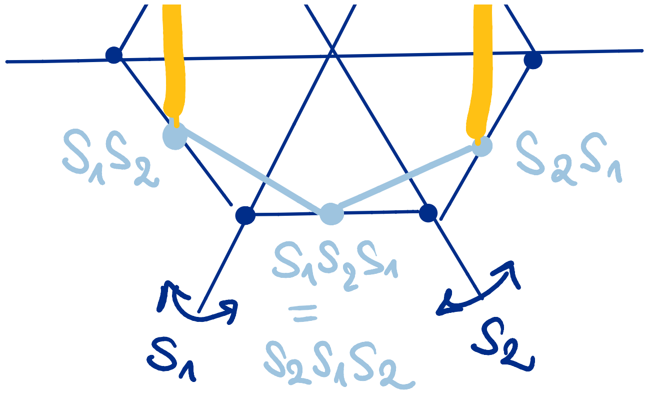
This works for examples 2) and 3) also.

2) $W = \mathrm{Sym}(3)$, $S = \{s_1, s_2\}$ as shown




$W = \langle S \mid s_i^2, s_1s_2^3 \rangle = D_6$

$(s_1s_2)^3 = 1$




$(s_1s_2)^3 = \mathbb{1}$

$\Leftrightarrow s_1s_2s_1s_2s_1s_2 = \mathbb{1}$

$\Leftrightarrow s_1s_2s_1 = s_2s_1s_2$

Both yellow edges are swapped by

$r := s_1s_2s_1$

This is due to the fact that $s_1s_2s_1 = s_2s_1s_2$

3) $W = \langle s_0, s_1, s_2 \mid s_i^2, (s_is_j)^3\ i \neq j \rangle$

The Cayley graph is the dual graph of the tiling corresponding to $W$:

$Cay(W, S)$, $S = \{s_0, s_1, s_2\}$

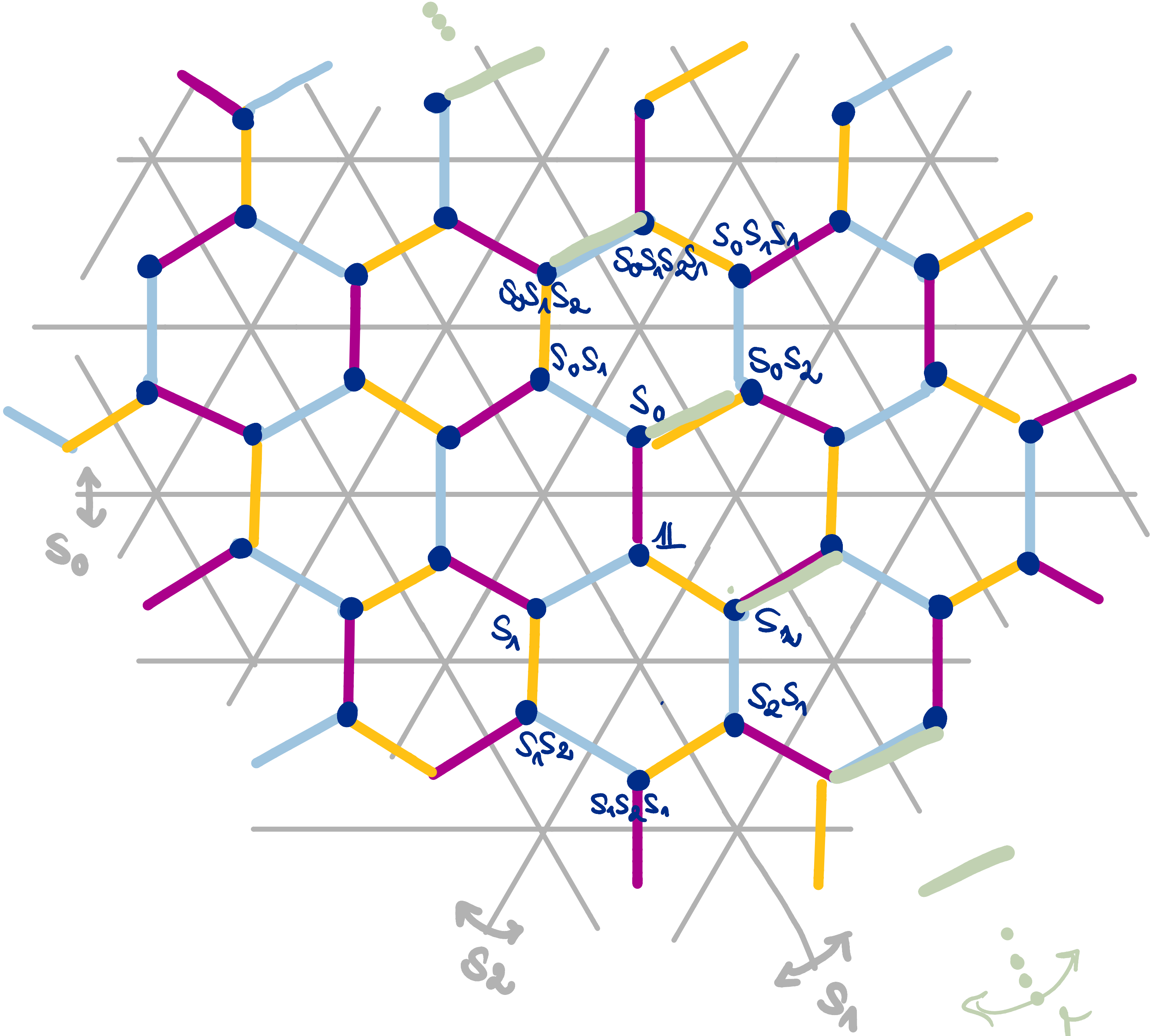
Put again $R = WSW^{-1}$.

now every $r \in R$ swaps infinitely many edges in this example.

E.g. take $r = s_0s_2s_0$ then obviously the edge $s_0s_2$ ——— $s_0$ is swapped by $r$. But also all other edges marked the same color.

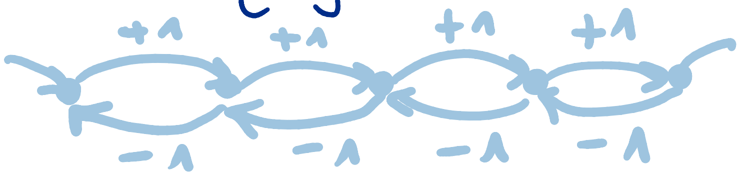
Again elements in $R$ correspond to actual hyperplane reflections.

actual hyperplane reflections.




4) A non-Coxeter example:

take $G = \mathbb{Z}$ and $S = \{\pm 1\}$ ← no involutions

$Cay(G, S) =$ ...  ...



Q: Is there a pre-reflection system

Q: Is there a pre-reflection system for this example?

5) Lemma: $(W,S)$ any Coxeter system, put $X := Cay(W,S)$ and $R := WSW^{-1}$, then $(X,R)$ is a pre-reflection system.

Proof: Recall that $X$ is connected and simple as a graph.
The element $wsw^{-1}$ flips the edge $\{w, ws\}$ and no other element in $R$ does so, too. $\square$

Def 3.14 Walls and reflection systems.
Given a pre-reflection system $(X,R)$ for a group $G$. Then the wall $H_r$ for $r \in R$ is the set of (midpoints of) edges in $X$ flipped by $r$.
A pre-reflection system is a reflection system if in addition
(6) for each $r \in R$ the complement $X \setminus H_r$ of the wall of $r$ has exactly two components.

two components.

**Rmk** Note that each $r \in R$ in a reflection system flips the two components of $X \setminus H_r$.

We will prove that all pre-reflection systems for Coxeter grps defined as in Example 3.11 are in fact reflection systems.

**Def 3.15** deletion & exchange condition

Let $W$ be generated by a set of involutions. We say $W$ satisfies the deletion condition if for all words $(s_1, \dots, s_k)$ in $S$ with $\ell(s_1 \cdot \dots \cdot s_k) < k$ there exists indices $i,j$ s.th.

$$s_1 \cdots s_k = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k .$$

these letters are deleted.

$W$ satisfies the exchange condition if the following holds:
Suppose $(s_1, \dots, s_k)$ is a reduced word for $w$, then $\forall s \in S$ we have either $\ell(sw) = k+1$ or there exists an index $i$ s.th.

$$w = s \cdot s_1 \cdot \dots \cdot \hat{s}_i \cdots s_k .$$

$w = s \cdot s_1 \cdot \ldots \cdot \hat{s}_i \cdots s_k.$

**Thm 3.16**

Let $W$ be a group generated by a set of involutions $S$.
Put $X := \mathrm{Cay}(W,S)$ and $R := WSW^{-1}$.

Then the following are equivalent:

(1) $(W,S)$ is a Coxeter system.
(2) $(X,R)$ is a reflection system
(3) $(W,S)$ satisfies the deletion condition.
(4) $(W,S)$ satisfies the exchange condition.

This is how we will prove this theorem.

**Rmk:** We will use the fact that the involutions in $S$ in a Coxeter system $(W,S)$

## 3.17 Words and paths in $X = \mathrm{Cay}(W,S)$

Suppose $W$ is generated by involutions $S$. Suppose further that all involutions in $S$ are distinct. Then we have a bijection:

$$\{\text{minimal words in } S\} \longleftrightarrow \{\text{minimal edge paths in } X\}$$

$$\left\{ \text{words in } S \right\}^{\text{minimal}} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{l} \text{edge paths in } X \\ \text{starting at } \mathbb{1} \end{array} \right\}^{\text{minimal}}$$

minimal / minimal

$(s_1, s_2, \dots s_k) \longmapsto$ path with vertices $s_1, s_1 s_2, s_1 s_2 s_3, \dots, s_1 s_2 \cdots s_k$

## 3.18 Words and reflections and crossed walls

Assumptions as in 3.17.

Every word $(s_1, s_2, \dots s_k)$ in $S$ has an associated sequence of reflections:

$$r_1 = s_1$$
$$r_2 = s_1 s_2 s_1$$
$$r_3 = s_1 s_2 s_3 s_2 s_1$$
$$\vdots$$
$$r_k = s_1 s_2 \dots s_{k-1} s_k s_{k-1} \dots s_2 s_1.$$

Reflection $r_i$ flips the edge between the vertices $s_1 \cdots s_{i-1}$ and $s_1 \cdots s_i$ in $X$.

We say the path associated with $(s_1, \dots, s_k)$ (as defined in 3.17) crosses the wall $H_r$ for some $r \in R = WSW^{-1}$ if $r = r_i$ for some $i$ and $r_i$ defined as above.

key to prove (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3).

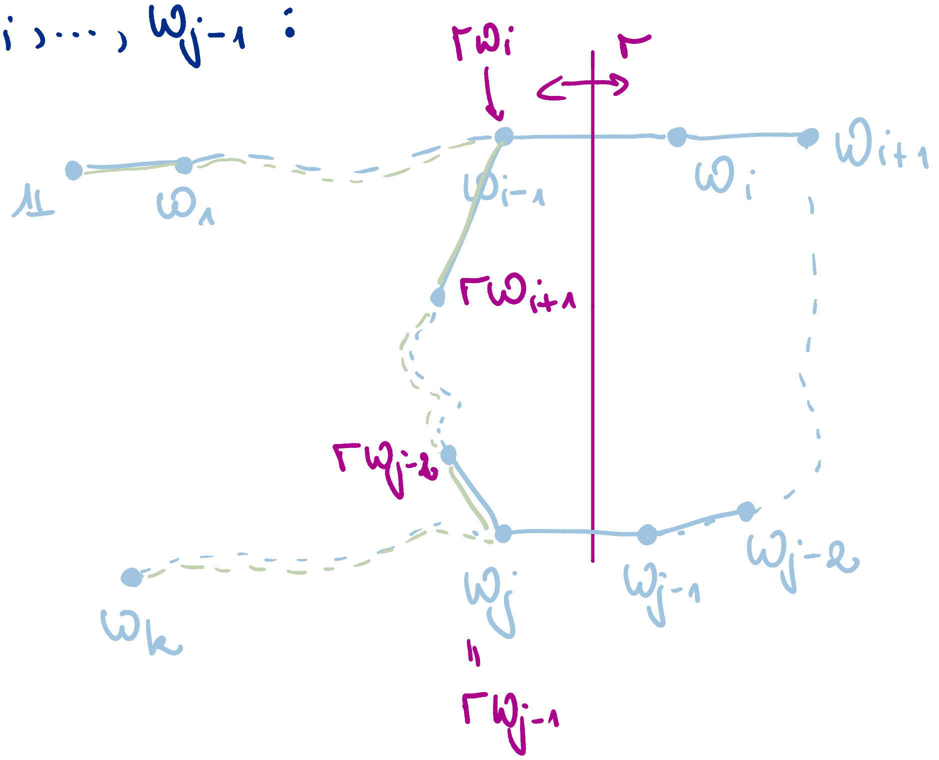
### Lemma 3.19

W generated by a set $S$ of distinct

$W$ generated by a set $S$ of distinct involutions.
$X = \mathrm{Cay}(W,S)$, $R = WSW^{-1}$.
Let $(s_1, \dots, s_k)$ be a word in $S$ and let $r_1, \dots, r_k$ be the associated reflections.
If $r_i = r_j$ for some $i<j$, then

$$s_1 \cdots s_k = s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_k.$$

<u>Proof</u>: Suppose $r = r_i = r_j$.
Put $w_p := s_1 \cdots s_p \;\; \forall 1 \le p \le k$.
Let $\gamma$ be the path determined by $(s_1, \dots, s_k)$.
Apply $r$ to the sub-path on the vertices $w_i, \dots, w_{j-1}$:




Concatenating the image with sub-paths

Concatenating the image with sub-paths of $\gamma$ connecting $\mathbb{1}$ and $w_{i-1}$ as well as $w_j$ and $w_k$, we obtain a new path from $\mathbb{1}$ to $w_k$ which is shorter.

Since the action of $W$ on $X = Cay(W, S)$ preserves types of edges this new path corresponds to the word

$$(s_1, s_2, \dots s_{i-1}, s_{i+1}, \dots, s_{j-1}, s_{j+1}, \dots, s_k).$$

Hence $w = s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_k$. $\square$