03 Cayley graphs and word length

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§3 A combinatorial characterization of Coxeter grps

The main goal of this section is to describe a Coxeter group in terms of two core combinatorial properties:
the exchange and deletion property.

We start with some standard notions from combinatorial group theory

3.1 Notation / standing assumptions

G is a group with generating set S.
Assume $S = S^{-1} := \{ s^{-1} \mid s \in S \}$ for simplicity.
in a Coxeter system this is automatically the case.
We will distinguish words in S, i.e. lists of generators $(s_1, \dots, s_n)$ and group elements $s_1 \cdot \ldots \cdot s_n \in G$, where $s_i \in S$ $\forall i$.

Def 3.2 Word length

With G, S as above the word length with resp. to S (or simply S-length) of an element $g \in G$ is given by:

$$l_S(g) := \min\{ n \in \mathbb{N} \mid \exists s_1, \dots, s_n \in S \text{ s.th. } g = s_1 \cdots s_n \}$$

If $l(g) = n$ and $g = s_1 \cdots s_n$ then we call

If $\ell(g)=n$ and $g=s_1\cdots s_n$ then we call the word $(s_1,\dots,s_n)$ a reduced word or reduced expression for g.

We then define the word metric on G with respect to S by putting, for $g,h\in G$,

$$d_S(g,h)=\ell_S(g^{-1}h).$$

3.3 Universality property

Recall this from general facts about presentations of groups.

Let (W,S) be a Coxeter system.
If G is a group and $f:S\to G$ is a map s.th. $\left(f(s_1)\cdot f(s_2)\right)^{m(s_1,s_2)}=1 \quad \forall s_1,s_2\in S$

then $\exists!$ extension of f to a group homomorphism $f:W\to G$.

3.4 Lemma

Let (W,S) be a Coxeter system and consider the map $\varepsilon: s\mapsto -1 \quad \forall s\in S$.
Then ε extends to a group homomorphism $\varepsilon: W\to\{+1,-1\}$.

seen as a multiplicative grp.

multiplicative grp.

Proof:

We need to check that $(\varepsilon(s_1)\cdot\varepsilon(s_2))^{m(s_1,s_2)} = 1$ for all $s_1,s_2\in S$. Then 3.3 implies the assertion.
By definition $\varepsilon(s_i) = -1$ and hence

$$\varepsilon(s_1)\cdot\varepsilon(s_2) = +1 \quad \forall\, s_1,s_2\in S. \qquad \square$$

Rmk: Equivalently there is an epimorphism (which we also call ε):

$$\varepsilon : W \to \mathbb{Z}/2\mathbb{Z}$$

← as an additive group

Cor 3.5

Each $s\in S$ is an involution.

3.6 Properties of l_S

(W,S) a Coxeter system, $l := l_S$ the S-length.
Then for all $w\in W$ we have:

(i) $\varepsilon(w) = (-1)^{l(w)}$

(ii) $l(uw) = l(u) + l(w) \mod 2$

(iii) $l(sw) = l(w) \pm 1 \quad \forall\, s\in S$

(iv) $l(w^{-1}) = l(w)$

(v) $|\,l(u) - l(w)\,| \le l(uw) \le l(u) + l(w)$

(v) $|\ell(u)-\ell(w)| \leq \ell(uw) \leq \ell(u)+\ell(w)$

(vi) $\ell(uw^{-1})$ is indeed a metric on W.

Proof: items (i) – (iii) are direct consequences of Lemma 3.4.

The rest is left as an exercise!

⇝ see Homework sheet. $\square$

ÜA: Check that the left-action of G on itself is isometric wrt this metric.

This length function is indeed a metric of a geometric object, which we now introduce:

Def 3.7 The Cayley graph $Cay(G,S)$ of a group G with respect to a (finite) generating set S has as vertices the set S and as edges the set

$$\{\{g,gs\} \mid g\in G, s\in S, s^2=1\}$$

involutions ≙ undirected edges

$$\cup \{(g,gs) \mid g\in G, s\in S, s^2\neq 1\}$$

directed edges for non-involutions

We think of an edge in $Cay(G,S)$ as being colored by s.

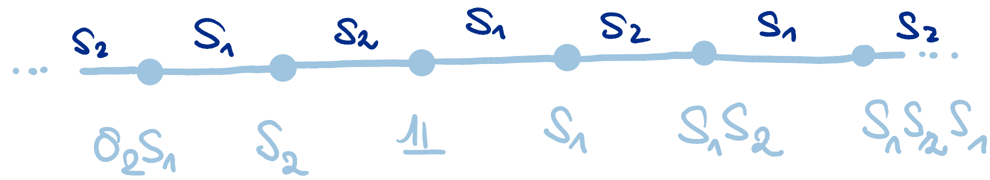
being colored by s.

<u>Rmk</u> if s is an involution, i.e. $s^2 = \mathbb{1}$, then gsg^{-1} swaps g and gs and is the unique isomorphism of $\mathrm{Cay}(G,S)$ which flips this edge.

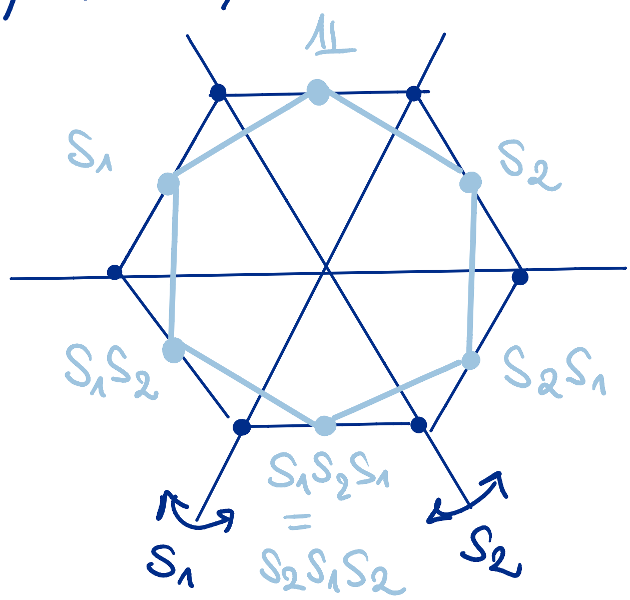
3.8 Examples

1) $D_\infty = \langle s_1, s_2 \mid s_i^2 \rangle \qquad S := \{s_1, s_2\}$

$\mathrm{Cay}(D_\infty, S)$



2) $W = \mathrm{Sym}(3)$, $\quad S = \{s_1, s_2\}$ as shown



$W = \langle S \mid s_i^2, s_1 s_2^3 \rangle = D_6$

$(s_1 s_2)^3 = \mathbb{1}$

$\Leftrightarrow \; s_1 s_2 s_1 s_2 s_1 s_2 = \mathbb{1}$

$\Leftrightarrow \; s_1 s_2 s_1 = s_2 s_1 s_2$

3) $W = \langle s_0, s_1, s_2 \mid s_i^2, (s_i s_j)^3 \; i \neq j \rangle$

The Cayley graph is the dual graph of the tiling corresponding to W:

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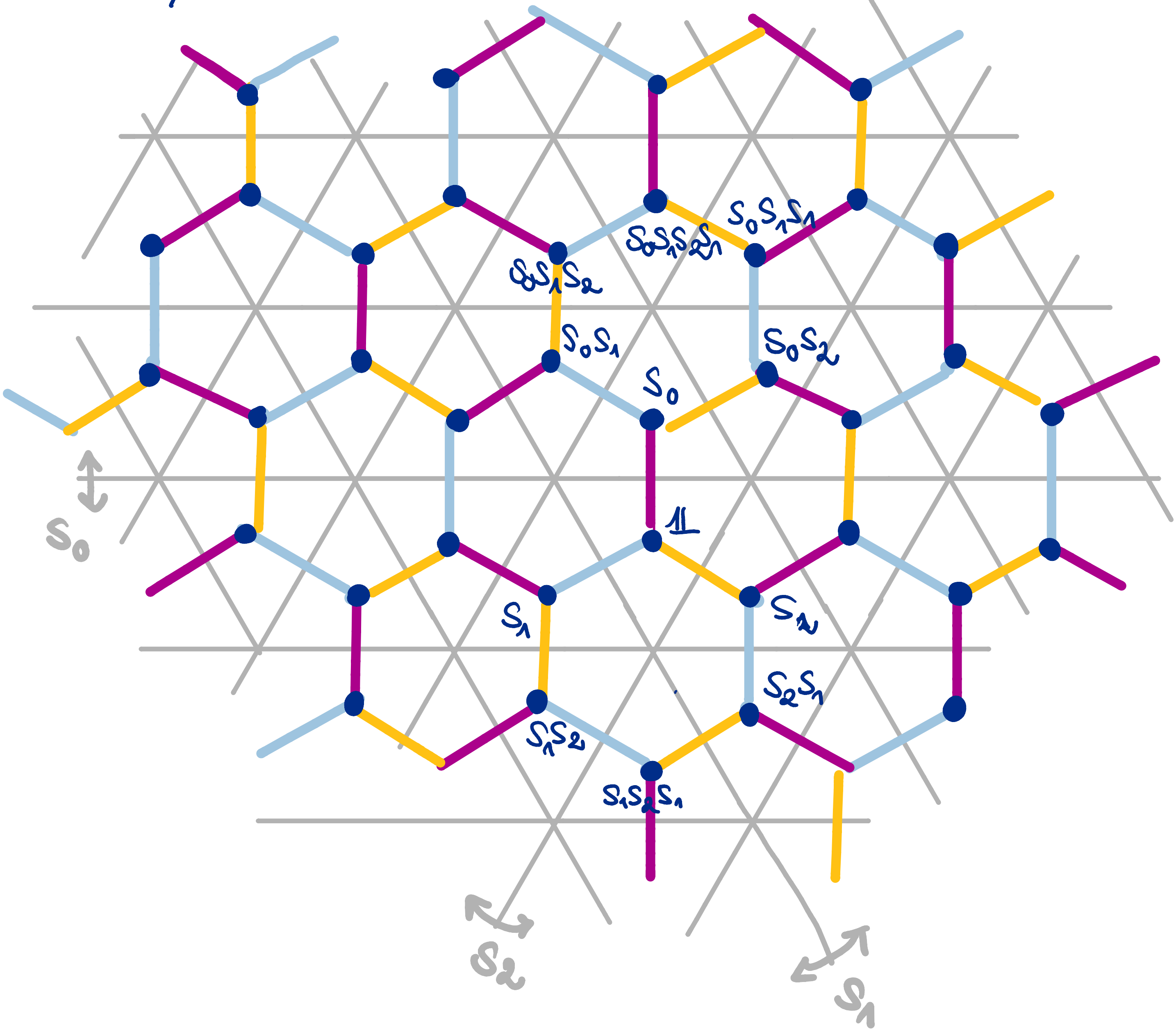
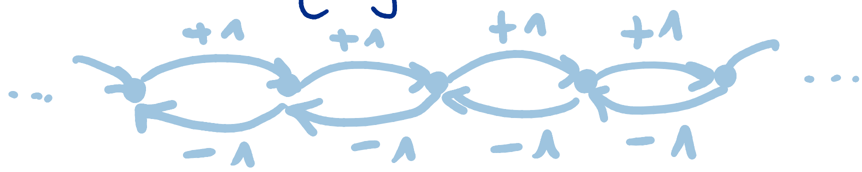
$\mathrm{Cay}(W,S)$, $S=\{s_0,s_1,s_2\}$



4) A non-Coxeter example:

take $G=\mathbb{Z}$ and $S=\{\pm 1\}$ ← no involutions

$\mathrm{Cay}(G,S) =$ 

3.9 Remark

The word metric is precisely the path metric

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The word metric is precisely the path metric on $Cay(W,S)$.

That is $d(g,h) = \ell_S(g^{-1}h)$

$= \#$ edges in a minimal path from g to h in $Cay(g,h)$.

Moreover, a reduced word $(s_{i_1}, s_{i_2}, \dots, s_{i_n})$ corresponds to a minimal path from $\mathbb{1}$ to $s_{i_1} \cdot \ldots \cdot s_{i_n} = g$ in $Cay(W,S)$.

This path passes through the vertices

$\mathbb{1}, s_{i_1}, s_{i_1}s_{i_2}, s_{i_1}s_{i_2}s_{i_3} \dots, g$

(in this order).

Lemma 3.10

In a Coxeter system (W,S) the generators in S are pairwise distinct.

This is <u>not</u> a priori clear from the definition. We will see a proof in the next two weeks.

Prop. 3.11

The Cayley graph $Cay(W,S)$ of a Coxeter system (W,S) is indeed a graph which is the Cayley graph Cay(W,S) of a Coxeter system (W,S) is indeed a graph which is connected, simple* and w/o double edges.

* recall: a graph is <u>simple</u> if it does not contain any loops. i.e. edges are sets not multisets.

<u>Proof:</u>

Cay(W,S) is connected by definition.

By Cor 3.5 the set S consists of involutions and $1 \notin S$ by definition.

Hence there are no loops and all edges are undirected and of the form $\{g, gs\}$.

Thus Cay(G,S) is a connected, simple graph.

By Lemma 3.10 there are no double edges. $\square$