2 Definition and first properties of Coxeter groups

We start by looking at some examples of discrete isometry groups generated by reflections.

2.1 The 3 classes of spaces we are interested in:

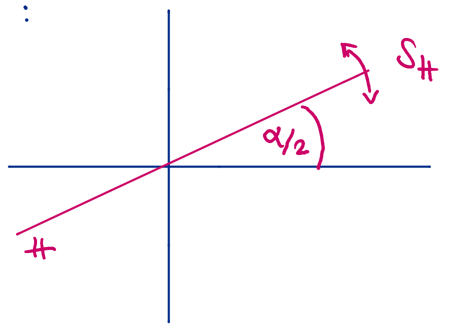
a) Euclidean space $\mathbb{E}^n$

$\mathbb{E}^n = \mathbb{R}^n$ together with the usual Euclidean metric.

We consider linear and affine hyperplanes, i.e. linear sub-Vs of dimension $n-1$ or translates of those.

For a hyperplane $H \subset \mathbb{R}^n$ we write s_H for the reflection of $\mathbb{E}^n$ along H.

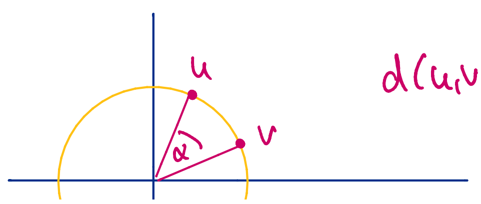
E.g. $s_H = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$ reflects along the following hyperplane in $\mathbb{E}^2$:



b) $\mathbb{S}^n$ n-dim unit sphere

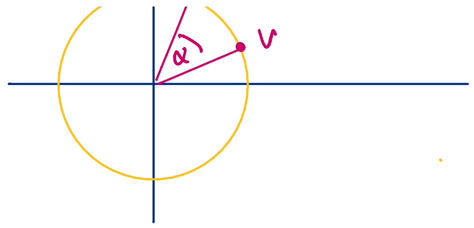
$\mathbb{S}^n = \{ v \in \mathbb{R}^{n+1} \mid \|v\| = 1 \}$

distance between $u, v \in \mathbb{S}^n$ is given by the angle at 0 enclosed by the lines through u, v.



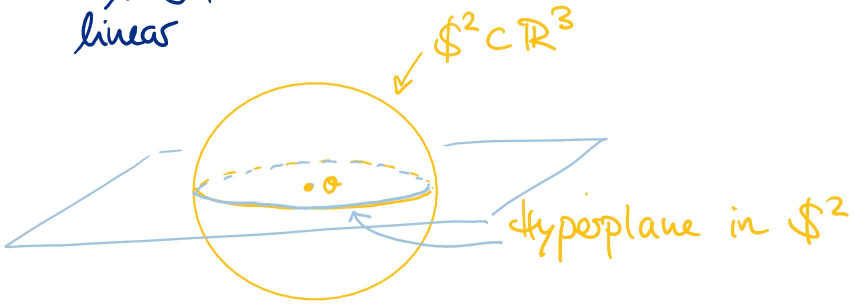
$d(u,v) = |\alpha|$

Shortest paths between points follow great circles
...



between points follow great circles on $\mathbb{S}^n$.

<u>Hyperplanes</u> in $\mathbb{S}^n$ are (by definition) the intersection of linear hyperplanes in $\mathbb{R}^{n+1}$ (through the origin) with $\mathbb{S}^n$.



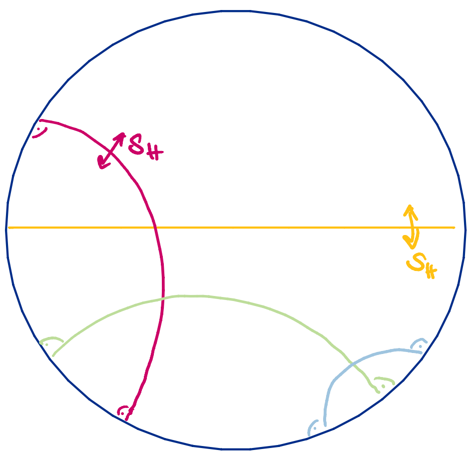
A <u>reflection</u> s_H of $\mathbb{S}^n$ is the restriction of a reflection along a hyperplane H in $\mathbb{R}^n$ to the corresponding hyperplane in $\mathbb{S}^n$.

c) $\mathbb{H}^n$ n-dim hyperbolic space

Don't worry if you've never seen this! We don't need details at the moment.
It is the unique n-dim. Riemannian mfld of constant sectional curvature -1.

One model of $\mathbb{H}^2$: <u>the Poincaré disc</u>

take an open unit disc. The metric on this disc is such, that points closer to the boundary are further apart.



⇝ Diameter of the disc becomes infinite!

Hyperplanes correspond to pieces of circles

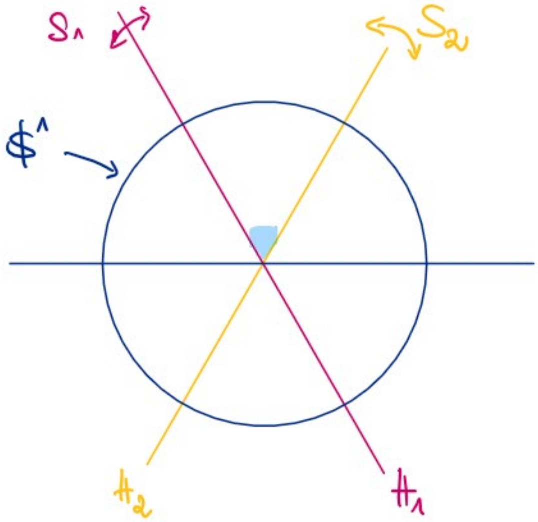
Hyperplanes correspond to pieces of circles perpendicular to the boundary.
Angle sums of triangles will be smaller than $180°$.

One can isometrically (= distance preserving) reflect $\mathbb{H}^2$ along any of these hyperplanes.

2.2 Finite dihedral groups

Consider S^1 (1-sphere centered around the origin in $\mathbb{E}^2$) and take two hyperplanes H_1, H_2 through the origin in $\mathbb{E}^2$ intersecting at an angle $\frac{\pi}{m}$.
Let s_1, s_2 be the reflections along H_1, H_2, respectively.

$\frac{\pi}{3}$



The product $s_1 \cdot s_2$ is the rotation around the origin by an angle of $2 \cdot \frac{\pi}{m}$.
So $s_1 s_2$ has finite order m (as an element of $GL_2(\mathbb{R})$.)

We refer to $W := \langle s_1, s_2 \rangle$ as the dihedral group of order $2m$ and write $W = D_{2m}$.

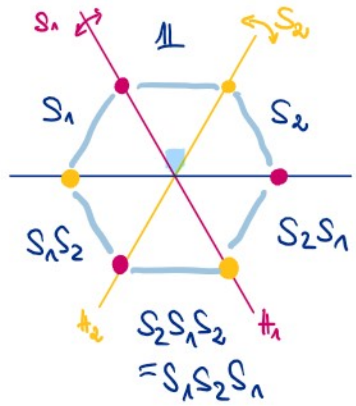
(Sometimes this is also denoted by D_m)

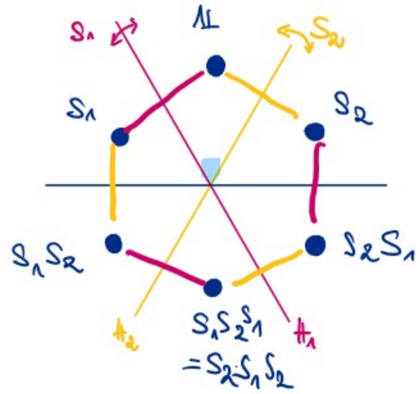
This group has the following presentation:

$$W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^m \underbrace{= \mathbb{1}}_{\text{fällt manchmal weg.}} \rangle$$

$W = \langle s_1, s_2 \mid s_1 = s_2 \quad (s_1 s_2) \quad \underbrace{= \mathbb{1}}_{\text{fällt manchmal weg.}} \rangle$

Two ways to represent this group by a graph:





Every element in W corresponds to an edge. Vertex between to edges encodes the additional generator used to write one of the two elements.

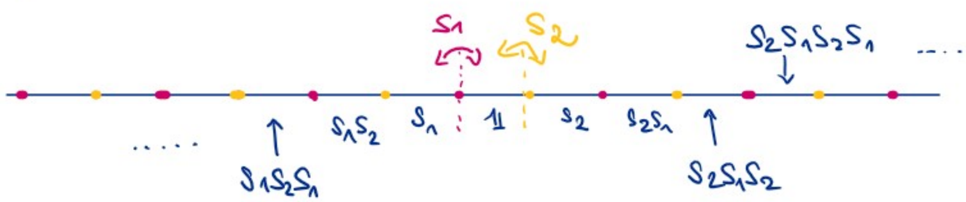
$\rightsquigarrow$ Coxeter complex of W

Every element in W corresponds to a vertex in the graph. Edges are colored according to the additional generators.

$\rightsquigarrow$ Cayleygraph of W

2.3 The infinite dihedral group

Consider $\mathbb{E}^1$ and let s_1, s_2 be the reflections along the points 0 and 1, respectively.
Their product $s_1 \cdot s_2$ is a translation by 2 units.



(*)

The group $W = \langle s_1, s_2 \rangle$ is the infinite dihedral group with presentation

$$W = D_\infty = \langle s_1, s_2 \mid s_1^2 = s_2^2 = \mathbb{1} = (s_1s_2)^\infty \rangle .$$

$$W = D_\infty = \langle s_1, s_2 \mid s_1^2 = s_2^2 = \mathbb{1} = (s_1 s_2)^\infty \rangle .$$

Again there are 2 graphs representing the group: the one in (*) and the following:

following the same rules as in the finite case.

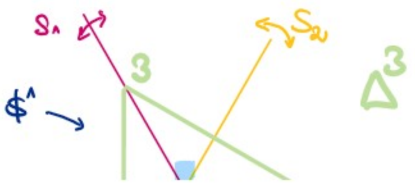
Both 2.2 and 2.3 were 1-dimensional examples.

2.4 Finite higher dimensional examples

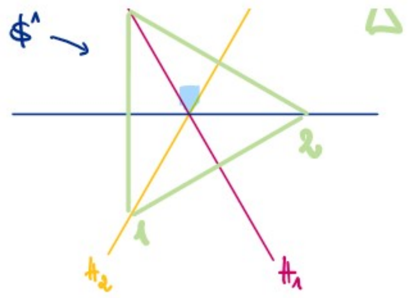
For $n \geq 2$ write Sym(n) for the symmetric group on n letters.
Let Δ^n be the regular Euclidean n-simplex, e.g. Δ^3 is an equilateral triangle in $\mathbb{R}^2$
Δ^4 a regular tetrahedron, etc.

Label the vertices of Δ^n by $1, 2, \ldots, n$.
Then we may identify Sym(n) with the group of (affine) isometries Isom(Δ^n) by viewing a transposition t_{ij} as the isometry swapping the vertices i and j of Δ^n.

Doing this we obtain a tesselation of the boundary of Δ^n by intersecting Δ^n with the reflection hyperplanes:

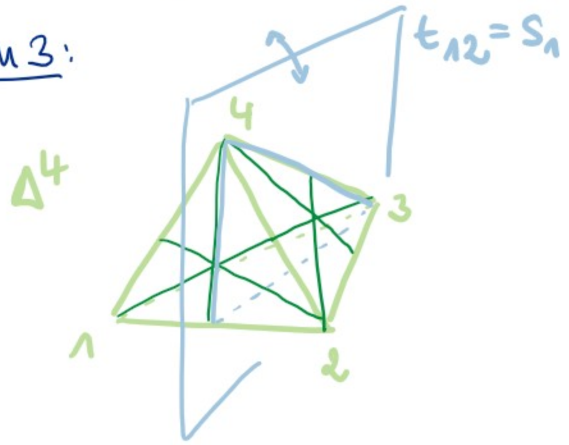


$s_1 \hat{=} t_{12}$ swaps vertices 1 and 2 of Δ^3



$s_2 \hat{=} t_{23}$ swaps vertices 2 and 3 of Δ^3

<u>dim 3:</u>



here we need 3 generators:

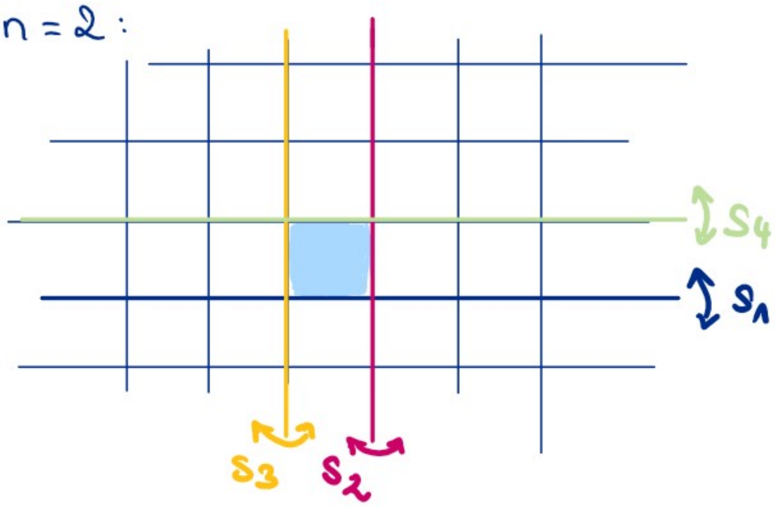
$s_1 \hat{=} t_{12}$
$s_2 \hat{=} t_{23}$
$s_3 \hat{=} t_{34}$

And we obtain $W = \langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_1 s_3)^3 = 1 \rangle$.

2.5 infinite higher dimensional examples

Consider $\mathbb{E}^n$ tiled by unit cubes.
Take one of these cubes and write $s_1, \dots, s_k$ for the reflections along the affine hyperplanes spanned by the k faces of the unit cube.

e.g. $n = 2$:



The resulting group is a Coxeter group:

$$W = \langle s_1, s_2, s_3, s_4 \mid s_i^2 = \mathbb{1} = (s_1 s_2)^2 = (s_1 s_3)^2 = (s_4 s_2)^2 = (s_4 s_3)^2 \rangle .$$

We will get to know further examples next week.

2.6. Definition

Let I be a finite indexing set, let $S = \{s_i\}_{i \in I}$.
Let $M = (m_{ij})_{i,j \in I}$ be a $|I| \times |I|$ matrix s.th.

- $m_{ii} = 1$ for all $i \in I$
- $m_{ij} = m_{ji}$ for all $i,j \in I$
- $m_{ij} \in \{2, 3, 4, \dots\} \cup \{\infty\}$ for all $i \neq j$.

Then M is called Coxeter matrix.

Every Coxeter matrix M defines a Coxeter system (W, S) as follows:

$$W = \langle S \mid (s_i s_j)^{m_{ij}} = \mathbb{1} \quad \forall i,j \in I \rangle$$

or, equivalently:

$$W = \langle S \mid s_i^2 = \mathbb{1} \text{ and } (s_i s_j)^{m_{ij}} = \mathbb{1} \text{ for all distinct } i,j \in I \rangle$$

Here we mean by $(s_i s_j)^\infty = \mathbb{1}$ that no alternating product $s_i s_j \cdots$ or $s_j s_i \cdots$ of finite length equals $\mathbb{1}$.

We call a group G a Coxeter group if there exists a Coxeter system (W, S) with $G \cong W$.

Rmk 2.7

1) We will see later that for every Coxeter group no two generators s_i, s_j for $i \neq j$ will be equal in W, the order of $s_i s_j$ will be equal to m_{ij} and generators will have order 2 and will be non-trivial elements of W.
2) All examples provided above are Coxeter groups.
3) There are Coxeter groups that are not geometric reflection groups.
   ($\rightarrow$ see [T] Section 1.3 for next week).