Sommersemester 2023
Vorlesung (3+1 SWS, integrierte Übungen)

Coxetergroups and buildings

1. Introduction and motivation

Two fundamental examples of buildings:

1.1. The projective plane of order 2 (the Fano plane)

Let $\mathbb{F}_q$ be the finite field of order 2.
i.e. numbers from $0, 1, \dots, q-1$ with usual operations mod q.

Let $GL_3(\mathbb{F}_q)$ be the linear group of 3×3 matrices with entries in $\mathbb{F}_q$.

We now construct a building on which this group acts. In other words: we will build a geometric object s.th. $GL_3(\mathbb{F}_q)$ is a symmetry group for this object and s.th. the object encodes information about certain subgroups of $GL_3(\mathbb{F}_q)$ – more on this at a later point in the semester.

Let $V := \mathbb{F}_q^3$ = 3-dim VS over $\mathbb{F}_q$

$\mathcal{P} := \{$ 1-dim subspaces of $V \}$ projective points

$\mathcal{L} := \{$ 2-dim subspaces of $V \}$ projective lines

Then $\mathcal{P} \sqcup \mathcal{L}$ is a projective plane (of order q).

We say a point $p \in \mathcal{P}$ is incident with a line $l \in \mathcal{L}$
benachbart
if (as linear subspaces) $p \subset l$.

The incidence graph Δ of the projective plane is given as follows:

- the vertices correspond to elements of $\mathcal{P} \cup \mathcal{L}$
- the pair $\{p, \ell\}$ is an edge if and only if p is incident with ℓ.

1.2 Rmk: (Exercise)

The projective plane over $\mathbb{F}_q$ satisfies:

a) $|\mathcal{L}| = |\mathcal{P}| = \frac{q^3-1}{q-1} = q^2+q+1$

b) Each $p \in \mathcal{P}$ is contained in $\frac{q^2-1}{q-1} = q+1$ distinct lines.

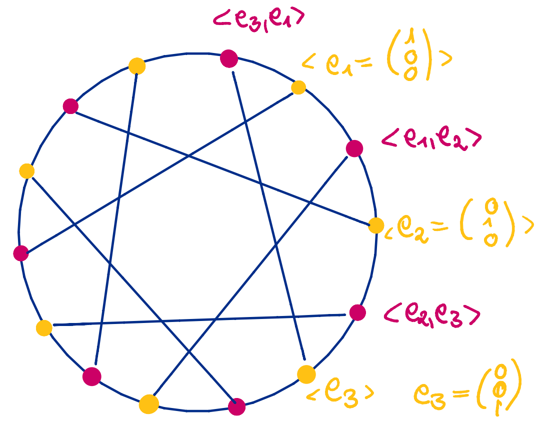
c) Each line $l \in \mathcal{L}$ contains $q+1$ distinct points.

Hence each vertex in Δ has valance $q+1$, i.e. is contained in $q+1$ edges.

1.3 Example: $q=2$, $\mathbb{F}_2 = \{0,1\}$

Δ is shown below. It contains $14 = (2^2+2+1)\cdot 2$ vertices – 7 lines and 7 points.

Heawood graph Δ:



the spherical building ass. with $GL_3(\mathbb{F}_2)$.

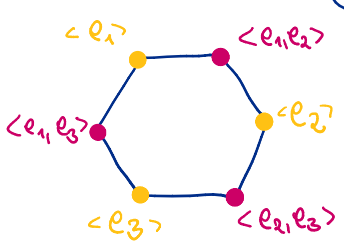
The 7 points are: $\mathcal{P} = \{ \underbrace{\langle e_i \rangle}_{3}, \underbrace{\langle e_i + e_j \rangle}_{+\,3}, \underbrace{\langle e_1+e_2+e_3 \rangle}_{+\,1}, i\neq j \in \{1,2,3\}\}$

The 7 lines are the spans of all pairs of distinct points.

Ex.1 live Exercise: 1) complete the picture.

2) What changes if we choose q to be a bigger prime? (power)

Observe that this graph is a union of hexagons:



where the vertices correspond to all linear subspaces that can be formed by a fixed basis, e.g. $\{e_1, e_2, e_3\}$

We will call these embedded hexagons <u>apartments</u>.

We will call these embedded hexagons <u>apartments</u>.
Note that S_3 acts as a group of symmetries on this sphere permuting the basis' elemements.

<u>Continuation of Example 1.3:</u>

$G := GL_3(\mathbb{F}_2)$ is a group of symmetries of Δ:

As a matrix group $GL_3(\mathbb{F}_2)$ acts on $V = \mathbb{F}_2^3$ by matrix · vector multiplication in the usual sense.
Since G only contains invertible elements we obtain that G permutes the elements of $\mathcal{P}$ and of $\mathcal{L}$.
Incidence between points and lines is preserved.

Hence: $\boxed{\text{we may view the elements in } G \text{ as color-preserving symmetries of the graph } \Delta.}$

<u>1.4 What are stabilizers of edges?</u>

Consider the standard basis $\{e_1, e_2, e_3\}$ again.
There is then an edge between $\langle e_1 \rangle$ and $\langle e_1, e_2 \rangle$ for example.

$$\rightsquigarrow \quad \{0\} \subsetneq \langle e_1 \rangle \subsetneq \langle e_1, e_2 \rangle \subsetneq V \qquad \text{std. flag}$$

the vertex stabilizers are:

$$\mathrm{Stab}_G(\langle e_1 \rangle) = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in G \right\} =: P_2$$

$$\mathrm{Stab}_G(\langle e_1, e_2 \rangle) = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \in G \right\} =: P_1$$

the <u>standard parabolic subgroups of G.</u>

Stabilizer of the edge $\underset{\langle e_1 \rangle}{\bullet} \overset{e}{\text{———}} \underset{\langle e_1, e_2 \rangle}{\bullet}$

$$\mathrm{Stab}_G(e) = P_1 \cap P_2 =: B$$

the <u>(standard) Borel subgroup of G.</u>

The orbit stabilizer theorem implies:
= Bahnensatz

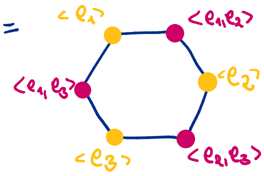
$$G/P_1 \overset{\wedge}{\underset{1:1}{=}} \mathcal{L}, \quad G/P_2 \overset{\wedge}{\underset{1:1}{=}} \mathcal{P}, \quad G/B \overset{\wedge}{\underset{1:1}{=}} \text{edges of } \Delta.$$

This implies: All vertices and edges of Δ correspond uniquely to left-cosets of P_1, P_2 or B.

Here edges gB and hB share a P_i vertex iff and only if $gP_i = hP_i$ (or, equivalently, $g^{-1}h \in P_i$).

1.5 A closer look at the 6-cycles / hexagons:

Consider again the 6-cycle corresponding to the standard basis.

A =


The pointwise stabilizer / fixator of this hexagon A is the set

$$T := \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \in G \right\}$$

As rescaling the basis vectors does not change their linear span. We call T the torus of G.

Note: Here T is very small!
But for larger q and hence larger $\mathbb{F}_q$ this subgroup grows!

The setwise stabilizer of A is the subgroup N of G consisting of all monomial matrices.
(i.e. one non-zero entry per row/column).

1.6 Properties:

a) N is the normalizer of T in G.

b) $N/T \cong \mathrm{Sym}(3)$ the symmetric group on 3 letters (represented by e_1, e_2, e_3)

c) $N/T =: W$ is generated by $\{s_1, s_2\}$, where
has 6 elements

$$s_1 := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad s_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

moreover: $W \cong D_6 := \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^3 = \mathbb{1} \rangle$

d) $P_i = B \sqcup B s_i B$ for $i = 1,2$

e) $G = \bigsqcup_{w \in W} BwB$, i.e. Bruhat decomposition of G holds

1. Rmk We will see (much later) in this lecture that this is a special case of a wider phenomenon.
$\rightsquigarrow$ see BN-pairs vs buildings.