Higher Teichmüller theory via Higgs bundles

This seminar is intended to give an overview on recent developments in higher Teichmüller theory using Higgs bundles. The moduli space of G-Higgs bundles on a surface is homeomorphic to the G-character variety by the celebrated Non-Abelian-Hodge Theory. This allows the study of topological properties of character varieties via Higgs bundles. In the recent years, this was used to establish the classification of components of geometrically rich representations, so-called higher Teichmüller components.

Location INF 205, Seminarraum 4 **Time** Friday, 9:15-11:45.
Contact jhorn (at) mathi.uni-heidelberg.de

This collection of topics is thought as a guide through the seminar. Not all talks in this list are necessary. Concerning the individual talks the bullet points should be rather understood as an inspiration than strict guidelines.

1. Introduction to Holomorphic Vector Bundles (Arnaud)

- topological $\mathbb{C}$-vector bundles on a surface, classification by degree
- holom. vector bundles, stability, the moduli space of holomorphic vector bundles
- Narasimhan-Seshadri theorem, Idea of Donaldsons proof.

1 talk, references: [Sch15], [NS65], [Don83].

2. Introduction to Higgs bundles and Non-Abelian Hodge (Levin)

- $\mathrm{GL}(n, \mathbb{C})$-Higgs bundles, stability, moduli space.

- Abelian Hodge theory $G = \mathrm{GL}(1, \mathbb{C})$.
- NAH: Hitchin equation, from Higgs bundles to flat connections, harmonic maps to symmetric spaces, from flat connections to Higgs bundles.
- The uniformizing $\mathrm{SL}(2, \mathbb{C})$-Higgs bundle

2 talks, references: [GX08], [Wen16], [Hit87].

3. Higgs for real Lie groups (Jonas)

- Holomorphic prinicipal G-bundles, vector bundles associated by representations
- Definition of G-Higgs bundle, Examples, associated bundles under standard representation of matrix Lie groups
- stability conditions, Hitchin equation.

1 talk, references: [Col19], [GGR09].

4. Lie theory of $\mathrm{SL}(2, \mathbb{C})$ embeddings (Oscar)

- representation theory of $\mathrm{SL}(2, \mathbb{C})$
- Jacobson-Morozov theorem

0-1 talk, references: [FH91], [Kna02], [Col19].

5. The Hitchin component - the first higher Teichmüller component

- Hitchin map, properties
- split real forms, principal embedding
- constructing section, reality

1 talk, references: [Hit92], [Col19].

6. Geometric Properties of Hitchin representations

- Anosov property

- discrete, faithful
- positivity

1 talk, references: [Lab06],[BIW14].

7. Higgs bundles for hermitian Lie groups

- Examples $G = \mathrm{Sp}(2n, \mathbb{R}), \mathrm{SU}(p, q)$, Toledo number, Milnor wood inequality
- Morse theory, component count
- Cayley correspondence

2 talks, references: [BGG06], [BGR17].

8. Geometric Properties of maximal representations

- Anosov property
- discrete, faithful
- positivity

1 talk, references: [BIW14] [Poz].

9. Higher Teichmüller components in $\mathrm{SO}(p, q)$-character varieties

- topological invariants, $\mathbb{C}^*$-fixed points
- local minima of Hitchin function, exotic components

1 talk, references: [Col19], [Apa+19].

10. Positive Anosov representations

- notion of positivity, examples: Hitchin representations, maximal representations
- Classification, conjecture: closeness property

1 talk, references: [GW18].

11. Magical nilpotents and higher Teichmüller components

- notion of magical nilpotent, Classification
- Cayley correspondence

1 talk, references: to appear: Bradlow, Collier, Gracia-Prada, Gothen, Oliviera

References

[Apa+19] Marta Aparicio-Arroyo, Steven Bradlow, Brian Collier, Oscar Garcıa-Prada, Peter B. Gothen, and André Oliveira. "SO(p,q)-Higgs bundles and higher Teichmüller components." English. In: *Invent. Math.* 218.1 (2019), pp. 197–299.

[BGR17] Olivier Biquard, Oscar Garcia-Prada, and Roberto Rubio. "Higgs bundles, the Toledo invariant and the Cayley correspondence." English. In: *J. Topol.* 10.3 (2017), pp. 795–826.

[BGG06] Steven B. Bradlow, Oscar Garcia-Prada, and Peter B. Gothen. "Maximal surface group representations in isometry groups of classical Hermitian symmetric spaces." English. In: *Geom. Dedicata* 122 (2006), pp. 185–213.

[BIW14] Marc Burger, Alessandra Iozzi, and Anna Wienhard. "Higher Teichmüller spaces: from SL(2, $\mathbb{R}$) to other Lie groups." English. In: *Handbook of Teichmüller theory. Volume IV.* Zürich: European Mathematical Society (EMS), 2014, pp. 539–618.

[Col19] Brian Collier. "Studying deformations of Fuchsian representations with Higgs bundles." English. In: *SIGMA, Symmetry Integrability Geom. Methods Appl.* 15 (2019), paper 010, 32.

[Don83] S. K. Donaldson. "A new proof of a theorem of Narasimhan and Seshadri." English. In: *J. Differ. Geom.* 18 (1983), pp. 269–277.

[FH91] William Fulton and Joe Harris. *Representation theory. A first course.* English. Vol. 129. New York etc.: Springer-Verlag, 1991, pp. xv + 551.

[GGR09] O. Garcia-Prada, P. B. Gothen, and I. M. i Riera. "The Hitchin-Kobayashi correspondence, Higgs pairs and surface group representations". In: *ArXiv e-prints* (2009).

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[GW18] Olivier Guichard and Anna Wienhard. "Positivity and higher Teichmüller theory." English. In: *European congress of mathematics. Proceedings of the 7th ECM (7ECM) congress, Berlin, Germany, July 18–22, 2016*. Zürich: European Mathematical Society (EMS), 2018, pp. 289–310.

[Hit87] N. J. Hitchin. "The Self-Duality Equations on a Riemann Surface". In: *Proceedings of the London Mathematical Society* s3-55.1 (1987), pp. 59–126.

[Hit92] N.J. Hitchin. "Lie groups and teichmüller space". In: *Topology* 31.3 (1992), pp. 449–473.

[Kna02] Anthony W. Knapp. *Lie groups beyond an introduction. 2nd ed.* English. 2nd ed. Vol. 140. Boston, MA: Birkhäuser, 2002, pp. xviii + 812.

[Lab06] François Labourie. "Anosov flows, surface groups and curves in projective space". In: *Invent. Math.* 165.1 (2006), pp. 51–114.

[NS65] M.S. Narasimhan and C.S. Seshadri. "Stable and unitary vector bundles on a compact Riemann surface." English. In: *Ann. Math. (2)* 82 (1965), pp. 540–567.

[Poz] Beatrice Pozzetti. "Notes on Maximal Representations". In: ().

[Sch15] Florent Schaffhauser. "Differential geometry of holomorphic vector bundles on a curve". In: *arXiv e-prints*, arXiv:1509.01734 (2015), arXiv:1509.01734.

[Wen16] Richard A. Wentworth. "Higgs bundles and local systems on Riemann surfaces." English. In: *Geometry and quantization of moduli spaces. Based on 4 courses, Barcelona, Spain, March – June 2012*. Basel: Birkhäuser/Springer, 2016, pp. 165–219.