Ruprecht-Karls Universität Heidelberg
Fakultät für Mathematik und Informatik

Masterarbeit im Fach Mathematik

Infinite series representations for Dirichlet L-functions at rational arguments

Vorgelegt von

Johann Franke

Geboren am 28. September 1992 in Heidelberg

am

21. Juni 2017

Wissenschaftlicher Betreuer:

Prof. Dr. Winfried Kohnen

Abstract

With the help of transformation formulas of Dirichlet L-series we generalize some classical formulas for the values $\zeta(2N+1)$ given by Ramanujan. This will be done by constructing generalized Dirichlet series of the form $\sum_{n=1}^{\infty} a_n n^{-s/b}$ where $b>0$ is an integer, which have similar transformation properties as Dirichlet L-functions and by considering their Mellin transforms using contour integration methods.

Zusammenfassung

Mit Hilfe der Transformationsformeln der Dirichletschen L-Reihen verallgemeinern wir klassische Formeln von Ramanujan, welche die Werte $\zeta(2N+1)$ der Riemannschen Zeta-Funktion beinhalten. Dies wird mit Mellin-Transformierten von zuvor konstruierten Dirichlet-Reihen der Form $\sum_{n=1}^{\infty} a_n n^{-s/b}$, wobei $b \in \mathbb{N}$, bewerkstelligt, die ähnliche Transformationseigenschaften wie Dirichletsche L-Funktionen besitzen.

Contents

1 Introduction

The Riemann zeta function is one of the most important and also mysterious complex functions in analytic number theory. In the right half plane $\{\mathrm{Re}(s) > 1\}$ it is defined as the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

which is absolutely and uniformly convergent on compact subsets $K \subset \{\mathrm{Re}(s) > 1\}$ and hence represents a holomorphic function on $\{\mathrm{Re}(s) > 1\}$. By analytic continuation, the function $\zeta(s)$ extends to a holomorphic function on $\mathbb{C} \setminus \{1\}$ with a simple pole in $s = 1$. It is a classical mathematical problem to find closed expressions for the values $\zeta(n)$, where $n > 1$ is an integer. For even integers $n = 2k$ this problem was solved by Leonhard Euler who showed the surprising formula

$$\zeta(2k) = (-1)^{k-1} \frac{B_{2k}(2\pi)^{2k}}{2(2k)!},$$

where the B_k are the Bernoulli numbers, which are defined via the expansion

$$\frac{x}{e^x - 1} = \sum_{\nu=0}^{\infty} \frac{B_\nu}{\nu!} x^\nu.$$

This deep relationship is for example a consequence of the following important integral identity, linking the Zeta with the Gamma function, namely

$$\zeta(s)\Gamma(s) = \int_0^\infty \frac{x}{e^x - 1} x^{s-1} \frac{\mathrm{d}x}{x}, \qquad \mathrm{Re}(s) > 1,$$

and the functional equation of the zeta function. For details, see for example [8]. It can be verified easily that all the B_k are rational numbers and hence, due to Lindemann, that all numbers $\zeta(2k)$ are transcendental. However, in the case that n is an odd number greater than 1, almost nothing is known about the numbers $\zeta(n)$. In 1974, Apery showed that $\zeta(3)$ is irrational using the fast converging series

$$\zeta(3) = \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3\binom{2k}{k}},$$

and in [6] Beukers presented a different proof using modular forms. Some progress in this area was made by Zudilin, who gave a proof that an infinite number of odd zeta values is irrational, see [16]. However, there is still no proof for the irrationality of $\zeta(2n+3)$ for any single $n \geq 1$.

In 1901 the Indian mathematician S. Ramanujan suggested some interesting identities involving odd zeta values. A famous special case of these relations is the beautiful formula

$$\zeta(3) = \frac{7\pi^3}{180} - 2\sum_{n=1}^{\infty} \frac{1}{n^3(e^{2\pi n}-1)}, \tag{1.1}$$

linking the values $\zeta(3)$ and π^3 deeply. For instance, as an immediate corollary we see that at least one of the values

$$\zeta(3) \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{n^3(e^{2\pi n}-1)}$$

is transcendental. But in fact, identity (1.1) is just a special case of a much more general formula found by Ramanujan, namely

$$\begin{aligned} &\alpha^{-N}\left(\frac{1}{2}\zeta(2N+1) + \sum_{k=1}^{\infty} \frac{1}{k^{2N+1}(e^{2\alpha k}-1)}\right) \\ &= (-\beta)^{-N}\left(\frac{1}{2}\zeta(2N+1) + \sum_{k=1}^{\infty} \frac{1}{k^{2N+1}(e^{2\beta k}-1)}\right) \\ &- 2^{2N}\sum_{k=0}^{N+1}(-1)^k \frac{B_{2k}}{(2k)!}\frac{B_{2N+2-2k}}{(2N+2-2k)!}\alpha^{N+1-k}\beta^k, \end{aligned} \tag{1.2}$$

where $N > 0$ is an integer and α, β are positive real numbers such that $\alpha\beta = \pi^2$, see also [3]. In order to obtain (1.1), we have to substitute $N = 1$ and $\alpha = \beta = \pi$ into (1.2). Although this formula by Ramanujan may look curious, it can be proved quite elementary by applying residue calculus to the function

$$f_N(z) = \frac{\cot(\pi z)\cot(\pi z\tau)}{z^{2N+1}},$$

where τ is a complex number on the upper half plane. In fact, using the Laurent expansion $\cot(\pi z) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} B_{2n} \pi^{2n-1}}{(2n)!} z^{2n-1}$, we obtain

$$\mathrm{res}_{z=0} f_N(z) = 2^{2N+2}\pi^{2N}(-1)^{N+1} \sum_{\mu=0}^{N+1} \frac{B_{2\mu}}{(2\mu)!} \frac{B_{2N+2-2\mu}}{(2N+2-2\mu)!} \tau^{2\mu-1},$$

and

$$\mathrm{res}_{z=n} f_N(z) = \frac{\cot(\pi\tau|n|)}{\pi|n|^{2N+1}} = -\frac{i}{\pi|n|^{2N+1}} - 2i\frac{1}{\pi|n|^{2N+1}(e^{-2\pi i|n|\tau}-1)},$$

$$\mathrm{res}_{z=n/\tau} f_N(z) = \tau^{2N}\frac{\cot(\pi|n|/\tau)}{\pi|n|^{2N+1}} = \tau^{2N}\left(\frac{i}{\pi|n|^{2N+1}} + 2i\frac{1}{\pi|n|^{2N+1}(e^{2\pi i|n|/\tau}-1)}\right)$$

for all $n \in \mathbb{Z}\setminus\{0\}$. For all $\tau \in \mathbb{H}$ we can find a sequence of closed circles γ_{R_n} with center $m = 0$ and radius R_n, such that $R_n \to \infty$ and

$$\lim_{n\to\infty} \oint_{\gamma_{R_n}} f_N(z)\mathrm{d}z = 0.$$

Hence, the sum of all residues of f_N vanishes, which is equivalent to the identity

$$\begin{aligned}
&-\frac{1}{2}\zeta(2N+1) - \sum_{n=1}^{\infty} \frac{1}{n^{2N+1}(e^{-2\pi i n\tau}-1)} \\
&+ \tau^{2N}\left(\frac{1}{2}\zeta(2N+1) + \sum_{n=1}^{\infty} \frac{1}{n^{2N+1}(e^{2\pi i n/\tau}-1)}\right) \\
&= \frac{\pi}{4i}2^{2N+2}\pi^{2N}(-1)^N \sum_{\mu=0}^{N+1} \frac{B_{2\mu}}{(2\mu)!}\frac{B_{2N+2-2\mu}}{(2N+2-2\mu)!}\tau^{2\mu-1}.
\end{aligned}$$

By setting $\alpha = -\pi i\tau$ and $\beta = \frac{\pi i}{\tau}$, one now verifies (1.2) easily.

The aim of this master thesis is to generalize series identities for the Riemann zeta function at odd integer values of the Ramanujan type to series identities involving Dirichlet L-functions at rational points. This can be done by constructing Dirichlet series $D(s)$ which have a meromorphic continuation to the entire complex plane and possess transformation properties similar to Dirichlet L-functions, i.e. there are rational numbers $v > 0$ and w, such that for a positive integer b the function

$$f(s) = (2\pi v)^{-s}\,\Gamma(s)D(s/b)$$

fulfills a functional equation of the type $f(w-s) = \pm f(s)$ for all $s \in \mathbb{C}$. The idea of construction is quite simple: one can obtain such $D(s)$ by considering products of the form

$$L\left(\frac{s}{b},\chi\right)L\left(\frac{s+1}{b},\chi\right)\cdots L\left(\frac{s+b-1}{b},\chi\right)$$

where χ is a primitive Dirichlet character with modulus m. The method introduced will provide new series identities for Dirichlet L-functions at rational points, for example, when considering the unit character,

$$\sum_{n=1}^{\infty}\left(\sum_{d|n}\sigma_{-3}(n/d)\sigma_{-3}(d)d^{-1/2}\right)\left(e^{-2\pi\sqrt{n}}-\frac{1}{32}e^{-8\pi\sqrt{n}}\right) = A+B+C+D+E \quad (1.3)$$

where

$$\begin{aligned}
A &= \frac{511}{92160}\pi^2\zeta(3/2)\zeta(9/2),\\
B &= \frac{1}{288}\pi^3\zeta(3/2)\zeta(5/2),\\
C &= -\frac{7}{32}\zeta(3/2)\zeta(5/2)\zeta(3),\\
D &= \frac{127}{11520}\pi^3\zeta(1/2)\zeta(7/2),\\
E &= -\frac{31}{64}\zeta(1/2)\zeta(7/2)\zeta(3),
\end{aligned}$$

and $\sigma_k(n)$ is the usual divisor sum. This new identity is the case $b=2$, $k=3$ and $\tau = \frac{i}{2}$ of Theorem 3.1.5.

In chapter 2, we present some foundational material and in chapter 3 the main theorems are proved using contour integration methods. We also give some examples and present a second proof for the identity (1.2). In the last section of the thesis, we also consider the case of mixed characters.

The author is grateful to Winfried Kohnen for his support. The main results of this master thesis are new and will be published soon in the Ramanujan Journal.

2 Foundational material

In this chapter we will discuss some basic material.

2.1 The Gamma function

Definition 2.1.1. *For complex numbers* s *with* $\mathrm{Re}(s) > 0$ *we define the Gamma function by*

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} \mathrm{d}x.$$

The Gamma function has very important properties. It can be extended to a holomorphic function on $\mathbb{C} \setminus \{0, -1, -2, ...\}$ with simple poles in $s = -n \in -\mathbb{N}_0$ with residues $\mathrm{res}_{s=-n}\Gamma(s) = \frac{(-1)^n}{n!}$ and fulfills the functional equation

$$s\Gamma(s) = \Gamma(s+1).$$

In particular, we obtain the formula $\Gamma(n+1) = n!$ for all $n \geq 0$ by induction. The next theorem is an important result of Wielandt, who gave a characterization of the Gamma function.

Theorem 2.1.2 (Wielandt, 1939)**.** *Let* $V := \{s \in \mathbb{C} \mid 1 \leq \mathrm{Re}(s) < 2\}$ *and* D *a domain which contains the strip* V*. Let* $f : D \longrightarrow \mathbb{C}$ *be an analytic function which has the following properties:*

(i) f *is bounded on* V*.*

(ii) We have

$$sf(s) = f(s+1)$$

for all $s, s+1 \in D$*.*

Then we have

$$f(s) = f(1)\Gamma(s).$$

Proof. See for example [8], p. 198. $\square$

The Gamma function has some very useful properties. For instance, using residue calculus, one can show the following interesting identity holding for all complex values $s \notin \mathbb{Z}$.

Theorem 2.1.3 (L. Euler, 1749)**.** *For all $s \in \mathbb{C} \setminus \mathbb{Z}$, we have the identity*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$

Proof. See for example [8], p. 204. $\square$

This identity will help us later to deal with functional equations of Dirichlet L-functions. The next theorem is an important result by Stirling, who characterized the growth behaviour of the Gamma function in terms of elementary functions.

Theorem 2.1.4 (Stirling formula)**.** *We have for all $s \in \mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leq 0\}$ the formula*

$$\Gamma(s) = \sqrt{2\pi} s^{s-\frac{1}{2}} e^{-s} e^{H(s)}$$

where $H(s)$ is analytic on $\mathbb{C} \setminus \{x \in \mathbb{R} \mid x \leq 0\}$ and converges to 0 in every segment $-\pi + \delta < \mathrm{Arg}(s) < \pi - \delta$, where $0 < \delta < \pi$.

Proof. See for example [8], p. 208. $\square$

Corollary 2.1.5. *Let $a < b$ be two real numbers and $\varepsilon > 0$ an arbitrary small real number. Then we have*

$$\Gamma(c + iT) = O(e^{-\left(\frac{\pi}{2}-\varepsilon\right)|T|})$$

uniformly in $a \leq c \leq b$, when $T \in \mathbb{R}$ and $|T| \to \infty$.

Proof. Let $c \in [a, b]$. With Stirling's formula we obtain

$$|\Gamma(c + iT)| = \left|\sqrt{2\pi}(c + iT)^{c+iT-\frac{1}{2}} e^{-c-iT} e^{H(c+iT)}\right|.$$

Since that $\sqrt{2\pi}e^{-c-iT}e^{H(c+iT)}$ is bounded for $|T| \to \infty$, there is a constant $C(a,b)$ such that

$$\leq C(a,b)\left|\exp\left(\left(c+iT-\frac{1}{2}\right)\operatorname{Log}(c+iT)\right)\right|$$

Let $T \geq 1$. Now there is a constant $K_1(a,b)$ independent from T and $\varepsilon(T) > 0$ such that $\operatorname{Re}[\operatorname{Log}(c+iT)] \leq K_1(a,b)\log(T)$ and $|\operatorname{Im}[\operatorname{Log}(c+iT)]| \geq \frac{\pi}{2} - \varepsilon(T)$ for T sufficiently large. Hence we have

$$\leq C(a,b)K_1(a,b)Te^{-\left(\frac{\pi}{2}-\varepsilon(T)\right)|T|}$$

for all sufficiently large T. Since we can choose $\varepsilon(T) \to 0$ as $|T| \to \infty$ and the constants $C(a,b)$ and $K_1(a,b)$ are independent from c, the corollary follows. □

The key to all results of this paper is a well-known formula by Gauß, which creates a relation between the Gamma function $\Gamma(s)$ and its rational transformations $\Gamma\left(\frac{s+\ell-1}{b}\right)$, where $1 \leq \ell \leq b$.

Theorem 2.1.6 (C. F. Gauß)**.** *Let b be a positive integer. Then we have*

$$\prod_{\ell=1}^{b}\Gamma\left(\frac{s+\ell-1}{b}\right) = (2\pi)^{(b-1)/2}b^{1/2-s}\Gamma(s).$$

Proof. We will use Wielandt's Theorem 2.1.2 to show the claim. Let

$$h(s) = \frac{\Gamma\left(\frac{s}{b}\right)\Gamma\left(\frac{s+1}{b}\right)\cdots\Gamma\left(\frac{s+b-1}{b}\right)b^{s-1}}{(2\pi)^{(b-1)/2}b^{-1/2}}.$$

Then h is analytic on $\mathbb{C}\setminus -\mathbb{N}_0$ and bounded on the strip $1 \leq \operatorname{Re}(s) < 2$ according to the stirling formula. It is easy to check that

$$sh(s) = b\frac{s}{b}h(s) = h(s+1),$$

so we are left to show that

$$\prod_{\ell=1}^{b-1}\Gamma\left(\frac{\ell}{b}\right) = (2\pi)^{(b-1)/2}b^{-1/2}$$

which implies $h(1) = 1$ and hence $\Gamma(s) = h(s)$ by Wielandt's Theorem. We will show this in the case where b is odd, the even case is analogous. Firstly, using Theorem 2.1.3, we obtain

$$\begin{aligned}
\prod_{\ell=1}^{b-1} \Gamma\left(\frac{\ell}{b}\right)^2 &= \prod_{\ell=1}^{(b-1)/2} \frac{\pi}{\sin(\pi\ell/b)} \cdot \prod_{\ell=1}^{(b-1)/2} \frac{\pi}{\sin(\pi(b-\ell)/b)} \\
&= \pi^{b-1} \prod_{\ell=1}^{b-1} \frac{1}{\sin(\pi\ell/b)} \\
&= (2\pi i)^{b-1} \prod_{\ell=1}^{b-1} \frac{1}{e^{\pi i\ell/b} - e^{-\pi i\ell/b}} \\
&= (2\pi i)^{b-1} \prod_{\ell=1}^{b-1} e^{-\pi i\ell/b} \cdot \prod_{\ell=1}^{b-1} \frac{1}{1 - e^{-2\pi i\ell/b}} \\
&= (2\pi i)^{b-1} e^{-\pi i b(b-1)/(2b)} \cdot \prod_{\ell=1}^{b-1} \frac{1}{1 - e^{-2\pi i\ell/b}} \\
&= (2\pi)^{b-1} \prod_{\ell=1}^{b-1} \frac{1}{1 - e^{-2\pi i\ell/b}}.
\end{aligned}$$

Now we have

$$\prod_{\ell=1}^{b-1} \frac{1}{1 - e^{-2\pi i\ell/b}} = \lim_{x\to 1} \frac{x-1}{x^b - 1} = \lim_{x\to 1} \frac{1}{bx^{b-1}} = \frac{1}{b}, \tag{2.1}$$

and hence, given that $\Gamma(x) > 0$ for all $x > 0$, we obtain

$$\prod_{\ell=1}^{b-1} \Gamma\left(\frac{\ell}{b}\right) = (2\pi)^{(b-1)/2} b^{-1/2}.$$

$\square$

2.2 Dirichlet characters

Definition 2.2.1 (Character of an abelian group)**.** *Let G be an abelian group. Then, a character χ of G is a group homomorphism $\chi : G \to \mathbb{C}^\times$, where $\mathbb{C}^\times$ is the group of complex multiplication.*

We are specially interested in the case when G is a *finite* abelian group.

Proposition 2.2.2. *Let G be a finite abelian group with neutral element e and χ be a character of G. Then $\chi(e) = 1$ and for every value $a \in G$, $\chi(a)$ is a root of untinity.*

If G is an abelian group, we define $C(G)$ to be the set of all characters of G. It is then easy to see that the set $C(G)$ is an abelian group by setting $(\chi_1\chi_2)(a) := \chi_1(a)\chi_2(a)$ for all $a \in G$. The next theorem shows, that there is an interesting connection between G and $C(G)$ in the case that G is finite.

Theorem 2.2.3. *Let G be a finite abelian group. Then G is isomorphic to $C(G)$.*

Proof. See for example [7], p. 33. □

Definition 2.2.4 (Dirichlet character)**.** *Let m be a positive integer and ψ be a character of the finite abelian group $(\mathbb{Z}/m\mathbb{Z})^\times$ of the residue classes modulo m. We then define an arithmetic function χ on $\mathbb{Z}$ by setting*

$$\chi(k) = \begin{cases} \psi(\overline{k}), & \text{if } (m,k) = 1, \\ 0, & \text{else.} \end{cases}$$

We call the function χ_f a Dirichlet character *modulo m induced by the character χ. Furthermore, we denote the unit (or trivial) character $\chi(n) = 1$ for all $n \in \mathbb{N}$ by χ_0.*

It follows easily from the previous definitions that every Dirichlet character for a given m is periodic with period m and strictly multiplicative, i.e. $\chi(ab) = \chi(a)\chi(b)$ for any $a, b \in \mathbb{Z}$. Moreover, there is a 1:1–correspondence between arithmetic functions that fulfill the above properties and characters on $(\mathbb{Z}/m\mathbb{Z})^\times$.

Definition 2.2.5 (Gauss sum)**.** *Let χ be a Dirichlet character modulo m. Then we* *define*

$$\mathcal{G}(n,\chi) = \sum_{k=0}^{m-1} \chi(k) e^{2\pi i n k/m}.$$

We also set $\mathcal{G}(\chi) := \mathcal{G}(1,\chi)$. *We call the arithmetic function* $\mathcal{G}(n,\chi)$ *the* Gauss sum *associated to* χ.

The Gauss sum is said to be *separable* for a fixed number n, if the following holds:

$$\mathcal{G}(n,\chi) = \overline{\chi}(n)\mathcal{G}(\chi),$$

where $\overline{\chi}$ is the complex conjugate of χ.

Definition 2.2.6. *A character* χ *of* $(\mathbb{Z}/m\mathbb{Z})^\times$ *is said to be* primitive, *if there is no divisor* $1 \leq d < m$ *of* m *such that* χ *factors through*

$$\chi : (\mathbb{Z}/m\mathbb{Z})^\times \xrightarrow{\pi} (\mathbb{Z}/d\mathbb{Z})^\times \xrightarrow{\chi^*} \mathbb{C}^\times \tag{2.2}$$

where π *is the natural projection and* χ^* *is a character on* $(\mathbb{Z}/d\mathbb{Z})^\times$. *The unit character* χ_0 *is primitive by definition.*

In the case that the relation (2.2) holds, we say that χ is induced by χ^*. One can show that a Dirichlet character χ modulo m is primitive if, and only if, for any divisor $1 \leq d < m$ there is some $a \equiv 1 \mod d$ with $(a,m) = 1$ such that $\chi(a) \neq 1$. Primitive Dirichlet characters have useful properties, as the next theorem shows.

Theorem 2.2.7. *Let* χ *be a primitive Dirichlet character modulo* m. *Then we have:*

(i) $G(n,\chi)$ *is seperable for any* $n \in \mathbb{N}$.

(ii) $|G(\chi)|^2 = m$, *in particular* $G(\chi) \neq 0$.

Proof. See for example [1], p. 168. □

Remark 2.2.8. *It can be shown that primitivity of* χ *and Theorem* 2.2.7 *(i) are indeed equivalences.*

Lemma 2.2.9. *Let* χ *be a primitive Dirichlet character modulo* m. *Then we have*

$$\mathcal{G}(\chi)\mathcal{G}(\overline{\chi}) = \chi(-1)m.$$

Proof. We have

$$\begin{aligned}\overline{\mathcal{G}(\chi)} &= \sum_{k=1}^{m} \overline{\chi}(k) e^{-2\pi i k/m} \\ &= \chi(-1) \sum_{k=1}^{m} \overline{\chi}(-k) e^{-2\pi i k/m} \\ &= \chi(-1)\mathcal{G}(\overline{\chi}).\end{aligned}$$

Hence, due to Theorem 2.2.7, we obtain

$$\mathcal{G}(\chi)\mathcal{G}(\overline{\chi}) = \chi(-1)\mathcal{G}(\chi)\overline{\mathcal{G}(\chi)} = \chi(-1)m.$$

This proves the lemma. $\square$

Theorem 2.2.10 (Generating function)**.** *Let χ be a primitive Dirichlet character modulo m and $|z| < 1$ be a complex number. Then we have*

$$\sum_{k=1}^{\infty} \chi(k) z^k = \frac{\mathcal{G}(\chi)}{m} \sum_{k=1}^{m} \frac{\overline{\chi}(k) z}{e^{2\pi i k/m} - z}.$$

We will need this result later for dealing with special cases of the generalized Ramanujan identities. In order to prove it, we need a lemma.

Lemma 2.2.11. *Let $m > 1$ be an integer and $P(X) \in \mathbb{C}[X]$ a polynomial satisfying* $\deg P < m$. *Then we have*

$$\frac{P(z)}{1 - z^m} = \frac{1}{m} \sum_{k=1}^{m} \frac{P(e^{2\pi i k/m}) e^{2\pi i k/m}}{e^{2\pi i k/m} - z}.$$

Proof. The first part of the proof is a simple application of Liouvilles theorem. Calculating the residues of $f(z) := \frac{P(z)}{1-z^m}$ at $z = e^{2\pi i \ell/m}$, $\ell = 0, 1, 2..., m-1$ we conclude that

$$z \mapsto f(z) + \sum_{k=1}^{m} \frac{P(e^{2\pi i k/m})}{z - e^{2\pi i k/m}} \prod_{j \neq k} \frac{1}{e^{2\pi i k/m} - e^{2\pi i j/m}}$$

is entire, bounded and tends to zero for $|z| \longrightarrow \infty$, hence we obtain

$$f(z) = -\sum_{k=1}^{m} \frac{P(e^{2\pi ik/m})}{z - e^{2\pi ik/m}} \prod_{j \neq k} \frac{1}{e^{2\pi ik/m} - e^{2\pi ij/m}}. \tag{2.3}$$

With a calculation analogous to (2.1) we obtain

$$\prod_{j \neq k} \frac{1}{e^{2\pi ik/m} - e^{2\pi ij/m}} = e^{2\pi ik/m} \prod_{j \neq k} \frac{1}{1 - e^{2\pi i(j-k)/m}} = \frac{e^{2\pi ik/m}}{m}$$

for all $k = 1, ..., m$. Together with (2.3) we obtain the lemma. $\square$

Now we are ready for the

Proof of Theorem 2.2.10: We have $|\chi(k)| \leq 1$ for all k and therefore it is clear that the generating function of χ is holomorphic on the open disc $\mathbb{E} = \{z \in \mathbb{C} \mid |z| < 1\}$. Let $p_\chi(z) = \sum_{\ell=1}^{m} \chi(\ell) z^{\ell-1}$. Since χ is periodic with period m, we obtain

$$\sum_{k=1}^{\infty} \chi(k) z^k = \frac{z p_\chi(z)}{1 - z^m} = \frac{z}{m} \sum_{k=1}^{m} \frac{p_\chi(e^{2\pi ik/m}) e^{2\pi ik/m}}{e^{2\pi ik/m} - z},$$

whereas the last equality follows from Lemma 2.2.11. Now we have

$$p_\chi(e^{2\pi ik/m}) e^{2\pi ik/m} = e^{2\pi ik/m} \sum_{\ell=1}^{m} \chi(\ell) e^{2\pi ik(\ell-1)/m} = \mathcal{G}(k, \chi) = \overline{\chi}(k) \mathcal{G}(\chi),$$

according to Theorem 2.2.7, since χ is primitive. $\square$

2.3 Dirichlet L-functions

Definition 2.3.1. *Let $f : \mathbb{N} \longrightarrow \mathbb{C}$ be an arithmetic function. Then we define the corresponding formal* ***Dirichlet series*** *by*

$$D(s, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

Clearly, if $f(n) = O(n^{\delta-1})$, the series $D(s, f)$ converges absolutely for all s with $\mathrm{Re}(s) > \delta$. We have the following proposition.

Proposition 2.3.2. *Let δ be a real number and $\mathcal{A}_\delta$ the set of all arithmetic functions $f : \mathbb{N} \longrightarrow \mathbb{C}$ such that $f(n) = O(n^{\delta-1})$. Then $\mathcal{A}$ has the structure of a ring whereas the multiplication is given by the* ***Dirichlet convolution***

$$f * g := \left(n \longmapsto \sum_{d|n} f(d)g(n/d) \right)$$

with neutral element $\varepsilon(n) = \delta_{n,1}$.

Let $\mathcal{D}_\delta$ be the set of all Dirichlet series $D(s, f)$ which converge absolutely on the right half-plane $\{\mathrm{Re}(s) > \delta\}$. Then $\mathcal{D}_\delta$ is a ring with pointwise addition and multiplication. We have an injective ring homomorphism

$$\varphi : \mathcal{A}_\delta \longrightarrow \mathcal{D}_\delta$$

$$f \longmapsto D(\cdot, f).$$

Proof. The ring structures are clear. The proposition follows with the above results and the Identity Theorem for Dirichlet series, which is shown in [7] on p. 26. □

We have studied Dirichlet characters χ for a given modulus m in the last section. We will now look at the Dirichlet series of the form $s \mapsto D(s, \chi)$.

Definition 2.3.3 (Dirichlet L-function)**.** *Let χ be a Dirichlet character modulo m. We call the corresponding Dirichlet series the* Dirichlet L-function *of the character χ and* *write*

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

If χ is principle, the above series converges if, and only if, we have $\mathrm{Re}(s) > 1$. *If χ is not principal, the series converges for all* $\mathrm{Re}(s) > 0$ *and converges absolutely for* $\mathrm{Re}(s) > 1$.

It can be shown that $L(s,\chi)$ has a holomorphic extension to the entire complex plane in the case that χ is not principal. In the case that χ is principal it has a holomorphic extension to $\mathbb{C} \setminus \{1\}$ with a simple pole in the point $s = 1$.

Example 2.3.4 (Riemann zeta function)**.** *An important example for a Dirichlet L-function is the so called* Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \mathrm{Re}(s) > 1,$$

which corresponds to the trivial character χ_0.

Theorem 2.3.5 (Euler product)**.** *Let χ be a Dirichlet character. Then we have for all* $\mathrm{Re}(s) > 1$*:*

$$L(s,\chi) = \prod_p (1 - \chi(p)p^{-s})^{-1},$$

where the product is extended over all primes.

Proof. Since the Dirichlet series of $L(s,\chi)$ is absolutely convergent in the half plane $\{\mathrm{Re}(s) > 1\}$ and χ is a strictly multiplicative function, the theorem follows by a general result stated in [1], p. 231. □

We are interested in the values $L(r,\chi)$, where r is a rational number. The next proposition shows, that considering primitive characters is sufficient.

Proposition 2.3.6. *Let χ be a Dirichlet character modulo m, which is induced by a primitive character χ^* modulo d. Then we have $\chi(n) = \chi^*(n)$ for all $(n,m) = 1$ and hence*

$$L(s,\chi) = L(s,\chi^*) \prod_{p|m} (1 - \chi^*(p)p^{-s}).$$

Proof. See for example [7], p. 72. □

A very important property of Dirichlet L-functions are their functional equations. They provide useful information about the behaviour of the $L(s,\chi)$ in the complex plane, for example about their zeros. Again, it is sufficient to treat the primitive case.

Theorem 2.3.7 (Functional equation of Dirichlet L-functions). *Let χ be a primitive Dirichlet character modulo m. If we set $a = (1-\chi(-1))/2$ and*

$$\xi(s,\chi) = \left(\frac{\pi}{m}\right)^{-(s+a)/2} \Gamma\left(\frac{s+a}{2}\right) L(s,\chi),$$

we have the following identity:

$$\xi(1-s,\overline{\chi}) = \frac{i^a\sqrt{m}}{\mathcal{G}(\chi)}\xi(s,\chi).$$

Alternatively, we have the equivalent relation

$$L(1-s,\overline{\chi}) = \frac{2i^a}{\mathcal{G}(\chi)}\left(\frac{2\pi}{m}\right)^{-s}\Gamma(s)\cos\left(\frac{\pi}{2}(s-a)\right) L(s,\chi).$$

Proof. See for example [7], p. 74. □

Corollary 2.3.8. *Let χ be a Dirichlet character. If χ is not principal, the function $L(s,\chi)$ can be extended to an entire function on $\mathbb{C}$. If χ is principal, $L(s,\chi)$ can be extended to a holomorphic function on $\mathbb{C}\setminus\{1\}$ with a simple pole in $s=1$.*

Theorem 2.3.9 (Special values of the Riemann zeta function). *For all integers $n \geq 1$ we have the following values for the Riemann zeta function:*

(i) $\zeta(2n) = (-1)^{n-1}\frac{B_{2n}(2\pi)^{2n}}{2(2n)!}$.

(ii) $\zeta(0) = -\frac{1}{2}$.

(iii) $\zeta(-n) = -\frac{B_{n+1}}{n+1}$. *In particular, we have $\zeta(-2n) = 0$ for all $n \geq 1$.*

(iv) $\zeta'(-2n) = \frac{1}{2}(-1)^n(2\pi)^{-n}(2n)!\zeta(2n+1)$.

Proof. The first three results can be found in [1]. The last claim follows then with the functional equation. □

Theorem 2.3.10 (Lindelöf's Theorem)**.** *Let $f(s)$ be a holomorphic function on a strip $S = \{\sigma_1 \leq \mathrm{Re}(s) \leq \sigma_2\}$ such that $f(s) = O(e^{\varepsilon|t|})$ for every $\varepsilon > 0$ in S. If*

$$f(\sigma_1 + it) = O(|t|^{\kappa_1}) \qquad \textit{and} \qquad f(\sigma_2 + it) = O(|t|^{\kappa_2})$$

then

$$f(\sigma + it) = O(|t|^{k(\sigma)})$$

uniformly for $\sigma_1 \leq \sigma \leq \sigma_2$, where $k(\sigma)$ is the linear function of σ with $k(\sigma_1) = \kappa_1$ and $k(\sigma_2) = \kappa_2$.

Proof. See for example [15], p. 180. □

Theorem 2.3.11. *Let χ be a primitive Dirichlet character and $G = \{s \in \mathbb{C} \mid 0 \leq \mathrm{Re}(s) \leq 1, \mathrm{Im}(s) \geq 1\}$. If χ is not principal there is a constant $K > 0$ such that*

$$|L(\sigma + it, \chi)| \leq K t^{\frac{1}{2} - \frac{1}{2}\sigma}$$

for all $\sigma + it \in G$. In the case $\chi = \chi_0$, we similarly obtain

$$|\zeta(\sigma + it)| \leq K t^{\frac{1}{2} - \frac{1}{2}\sigma} \log(t + 1).$$

Proof. See for example [15], p. 299. □

2.4 Mellin transforms

In this short section, we introduce a very important integral transformation.

Proposition 2.4.1 (Mellin transform). *Let $f \in \mathcal{A}_\delta$ be an arithmetic function and $D(s,f)$ its Dirichlet series. Let $P(z,f) = \sum_{n=1}^{\infty} f(n)z^n$ be the corresponding power series. Then $P(z,f)$ has a radius of convergence $R \geq 1$ and we have*

$$D(s,f)\Gamma(s) = \int_0^\infty P(e^{-x},f)x^{s-1}\mathrm{d}x$$

for all $\mathrm{Re}(s) > \max\{\delta, 0\}$.

Proof. One uses the integral of the Gamma function as a starting point. After a substitution we have $\Gamma(s)f(n)n^{-s} = \int_0^\infty f(n)e^{-nx}x^{s-1}\mathrm{d}x$. The theorem follows now by summation and applying Lebesgue's theorem. $\square$

Remark 2.4.2. *The Gamma function $\Gamma(s)$ corresponds to the neutral element $f(n) = \delta_{n,1}$.*

Theorem 2.4.3 (Mellin inversion theorem). *Let $c > 0$ be a real number. Then we have for every* $\mathrm{Re}(x) > 0$*:*

$$e^{-x} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)x^{-s}\mathrm{d}s.$$

Proof. This is a beautiful application of the Residue theorem. See for example [8], p. 424. $\square$

Theorem 2.4.4 (Inverse Mellin transformation). *Let $\delta > 0$ and $D(s,f)$ be a Dirichlet series in $\mathcal{D}_\delta$. Then we have for all $\varepsilon > 0$ and z with* $\mathrm{Re}(z) > 0$

$$P(e^{-z},f) = \frac{1}{2\pi i}\int_{\delta+\varepsilon-i\infty}^{\delta+\varepsilon+i\infty} \Gamma(s)D(s,f)z^{-s}\mathrm{d}s.$$

Proof. See for example [14], p. 46. $\square$

2.5 Eisenstein series and Eichler integrals

We briefly sketch the theory of modular forms in this section. As usual, we denote $\Gamma[1] = \mathrm{SL}_2(\mathbb{Z})$ as the full modular group and for any $N \geq 1$

$$\Gamma[N] := \left\{ M \in \Gamma[1] \middle| M \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}.$$

We call a subgroup $\Lambda \subset \Gamma[1]$ a congruence subgroup, if there is an integer N such that $\Gamma[N] \subset \Lambda$. For any complex function $f : \mathbb{H} \longrightarrow \mathbb{C}$ we define $(f|_k M)(\tau) := (c\tau + d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right)$, where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma[1]$ and k is an integer.

Proposition 2.5.1. *Let f be a holomorphic function on $\mathbb{H}$ which is periodic with period N. Then we can expand f in a Fourier series*

$$f(\tau) = \sum_{m=-\infty}^{\infty} a(m) q^{m/N}, \qquad q = e^{2\pi i \tau},$$

which converges uniformly on every compact subset of $\mathbb{H}$. We say that f is holomorphic at the point $i\infty$, if $a(n) = 0$ for all $n < 0$.

Proof. This is routine, see for example [8], p. 148. □

Definition 2.5.2 (Modular form)**.** *Let f be a holomorphic function on $\mathbb{H}$ and Λ a congruence subgroup. Then we say that f is a* modular form of weight k *for the group Λ, if the following conditions are satisfied:*

(i) For every $M \in \Lambda$, we have $f|_k M = \psi(M) f$ for an abelian character ψ with $|\psi(M)| = 1$, which only depends on the matrix M.

(ii) For every $M \in \Gamma[1]$, the function $f|_k M$ is holomorphic in $i\infty$.

Example 2.5.3. *A famous non-trivial example of a modular form of weight 12 for the full modular group is the so called* discriminant Δ*, which is defined by*

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

A very short and elegant proof for the modularity of this expansion is given in [12].

Definition 2.5.4 (Eisenstein series of weight $k \geq 3$)**.** *Let χ and ψ be Dirichlet characters of modulus m and ℓ respectively. Then we define for any integer $k \geq 3$:*

$$E_k(\tau, \chi, \psi) = \sum_{(p,q) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{\chi(p)\psi(q)}{(p\tau + q)^k}.$$

It can be shown that the E_k are indeed holomorphic functions on the upper half-plane $\mathbb{H}$. We want to study their invariance properties under particular transformations.

Definition 2.5.5. *Let m and ℓ be positive integers. Then we define:*

$$\Gamma_0(m, \ell) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| b \equiv 0 \mod \ell,\ c \equiv 0 \mod m \right\}.$$

It is clear that the groups $\Gamma_0(m, \ell)$ are congruence subgroups of $\Gamma[1]$. The next theorem shows, that the E_k are modular forms for the subgroups $\Gamma_0(m, \ell)$.

Theorem 2.5.6. *Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(m, \ell)$ and χ and ψ be characters modulo m and ℓ respectively. Then we have*

$$(c\tau + d)^{-k} E_k\left(\frac{a\tau + b}{c\tau + d}, \chi, \psi\right) = \chi(d)\overline{\psi}(d) E_k(\tau, \chi, \psi).$$

Proof. See for example [13], p. 269. □

Theorem 2.5.7 (Fourier expansion of E_k)**.** *Assume $k \geq 3$. Let χ and ψ be Dirichlet characters modulo m and modulo ℓ, respectively, satisfying $\chi(-1)\psi(-1) = (-1)^k$. Let ℓ_ψ be the conductor of ψ, and ψ^0 the primitive character associated with ψ. Then*

$$E_k(\tau, \chi, \psi) = a_{k,\chi,\psi}(0) + A_{k,\chi,\psi} \sum_{n=1}^{\infty} a_{k,\chi,\psi}(n) q^{n/\ell},$$

where

$$A_{k,\chi,\psi} = 2(-2\pi i)^k \frac{\mathcal{G}(\psi^0)}{\ell^k (k-1)!},$$

$$a_{k,\chi,\psi}(0) = \begin{cases} 2L(k,\psi), & \text{if } \chi \text{ is principal} \\ 0, & \text{else}, \end{cases}$$
$$a_{k,\chi,\psi}(n) = \sum_{c|n} \chi\left(\frac{n}{c}\right) c^{k-1} \sum_{d|(g,c)} d\mu\left(\frac{g}{d}\right) \psi^0\left(\frac{g}{d}\right) \overline{\psi^0}\left(\frac{c}{d}\right).$$

Here, $g = \frac{\ell}{\ell_\psi}$ and μ the Möbius function. In particular, $E_k(\tau,\chi,\psi)$ is holomorphic at $i\infty$.

Proof. See for example [13], p. 270. □

Remark 2.5.8. *We observe that if ψ is primitive, the above formula simplifies to*

$$a_{k,\chi,\psi}(n) = \sum_{c|n} c^{k-1}\overline{\psi}(c)\chi\left(\frac{n}{c}\right).$$

These coefficients play an important role in this thesis. They will appear in the Dirichlet expansion of products of Dirichlet L-functions.

Definition 2.5.9 (Eichler integral)**.** *Let f be a modular form of weight k for the congruence subgroup Λ such that $f(i\infty) = 0$. Then we define the corresponding* Eichler integral *of f as a function on $\mathbb{H}$ by*

$$\tilde{f}(\tau) = \frac{1}{2\pi i} \int_{i\infty}^{\tau} f(z)(z-\tau)^{k-2}\mathrm{d}z.$$

More generally, if $f(i\infty) = c$, we define

$$\tilde{f}(\tau) = \frac{1}{2\pi i} \int_{i\infty}^{\tau} (f(z)-c)(z-\tau)^{k-2}\mathrm{d}z$$

in the same way as above.

3 Generalized Ramanujan identities

3.1 Rationally scaled Dirichlet series with transformation properties analogous to Dirichlet L-functions

As usual we let $q = e^{2\pi i\tau}$ for any complex variable τ in the upper half plane $\mathbb{H}$.

Definition 3.1.1. *Let χ be a Dirichlet character modulo m. We define*

$$\psi_k(s,\chi) = \left(\frac{2\pi}{m}\right)^{-s} \Gamma(s) L(s,\chi) L(s+k,\overline{\chi}).$$

For real numbers u, v, we denote by $\sigma_{u,v}(n,\chi)$ the n-th coefficient of the Dirichlet series $L(s+u,\chi)L(s+u-v,\overline{\chi})$, provided $\mathrm{Re}(s) > \max\{1-u, 1-u+v\}$. For instance in the case that $\chi = \chi_0$ is principal, we just have $\sigma_{0,k}(n,\chi_0) = \sigma_k(n)$ where σ_k is the divisor sum.

Proposition 3.1.2. *We have*

$$\sigma_{u,v}(n,\chi) = n^{-u} \sum_{d|n} d^v \overline{\chi}(d) \chi\left(\frac{n}{d}\right). \tag{3.1}$$

Proof. This is a simple application of Proposition 2.3.2. The coefficients of $D_1(s) = L(s+u,\chi)$ and $D_2(s) = L(s+u-v,\overline{\chi})$ are given by $a_1(n) = n^{-u}\chi(n)$ and $a_2(n) = n^{v-u}\overline{\chi}(n)$ respectively. We obtain $\sigma_{u,v} = a_1 * a_2$ and therefore

$$\sigma_{u,v}(n,\chi) = \sum_{d|n} a_2(d) a_1\left(\frac{n}{d}\right) = \sum_{d|n} \overline{\chi}(d) d^{v-u} \left(\frac{n}{d}\right)^{-u} \chi\left(\frac{n}{d}\right) = n^{-u} \sum_{d|n} d^v \overline{\chi(d)} \chi\left(\frac{n}{d}\right).$$

□

It is clear that for any integers $b, k > 0$ the product $L(s,\chi)L(s+1/b,\chi)\cdots L(s+(b-1)/b,\chi)L(s+k,\overline{\chi})\cdots L(s+k+(b-1)/b,\overline{\chi})$ is generated by the Dirichlet convolution function

$$\lambda_{k,b}(n,\chi) := (\sigma_{0,-k}(.,\chi) * \cdots * \sigma_{(b-1)/b,-k}(.,\chi))(n). \tag{3.2}$$

Definition 3.1.3. *Let $b > 0$ be an integer and $k > 0$ be odd. Let χ be a primitive character modulo m. Then we define*

$$M_{k,b}(\tau,\chi) = \sum_{n=1}^{\infty} \lambda_{k,b}(n,\chi) q^{bn^{1/b}/m}.$$

Obviously, in the case $b = 1$ the series $M_{k,b}(\tau,\chi)$ are regular Fourier series. Then we have the following theorem.

Theorem 3.1.4. *Let χ be a primitive character modulo m. We can write $M_{k,1}(\tau,\chi)$ in terms of an Eichler integral. In fact we have*

$$M_{k,1}(\tau,\chi) = \frac{km}{4\pi i \mathcal{G}(\overline{\chi})} \int_{i\infty}^{\tau} [E_{k+1}(z,\overline{\chi},\overline{\chi}) - 2L(k+1,\overline{\chi})\delta_{1,m}](z-\tau)^{k-1}\mathrm{d}z.$$

Proof. We have $\overline{\chi}(-1)\overline{\chi}(-1) = 1 = (-1)^{k+1}$. Let $\rho(z) = E_{k+1}(z,\overline{\chi},\overline{\chi}) - 2L(k+1,\overline{\chi})\delta_{1,m}$ and $A_{k+1,\overline{\chi},\overline{\chi}} \cdot a_{k+1,\overline{\chi},\overline{\chi}}(n)$ the Fourier coefficients of $\rho(z)$ in Theorem 2.5.7, where

$$A_{k+1,\overline{\chi},\overline{\chi}} = 2(-2\pi i)^{k+1}\frac{\mathcal{G}(\overline{\chi})}{m^{k+1}k!}.$$

We then have

$$\begin{aligned}
\int_{i\infty}^{\tau} \rho(z)(z-\tau)^{k-1}\mathrm{d}z &= \int_{i\infty}^{\tau} A_{k+1,\overline{\chi},\overline{\chi}} \sum_{n=1}^{\infty} a_{k+1,\overline{\chi},\overline{\chi}}(n) e^{2\pi i z n/m}(z-\tau)^{k-1}\mathrm{d}z \\
&= \int_{i\infty}^{0} A_{k+1,\overline{\chi},\overline{\chi}} \sum_{n=1}^{\infty} a_{k+1,\overline{\chi},\overline{\chi}}(n) e^{2\pi i \tau n/m} e^{2\pi i z n/m} z^{k-1}\mathrm{d}z \\
&= A_{k+1,\overline{\chi},\overline{\chi}} \sum_{n=1}^{\infty} a_{k+1,\overline{\chi},\overline{\chi}}(n) e^{2\pi i \tau n/m}(-i^k) \int_0^{\infty} e^{-2\pi x n/m} x^{k-1}\mathrm{d}x
\end{aligned}$$

$$= -A_{k+1,\overline{\chi},\overline{\chi}} \sum_{n=1}^{\infty} a_{k+1,\overline{\chi},\overline{\chi}}(n) e^{2\pi i \tau n/m} \cdot \left(\frac{mi}{2\pi n}\right)^k (k-1)!$$

Now we have $n^{-k} a_{k+1,\overline{\chi},\overline{\chi}}(n) = \sum_{d|n} \left(\frac{d}{n}\right)^k \chi(d)\overline{\chi}\left(\frac{n}{d}\right) = \lambda_{k,1}(n,\chi)$. Since k is odd, we obtain

$$= 4\pi i \frac{\mathcal{G}(\overline{\chi})}{km} \sum_{n=1}^{\infty} \lambda_{k,1}(n,\chi) e^{2\pi i \tau n/m}. \qquad \square$$

Such integrals are deeply related to zeta values and are linked to Beukers' proof [6] of the irrationality of $\zeta(3)$. Beukers considers various integrals of holomorphic modular forms that themselves satisfy certain functional equations.
For more about transcendent values of Eichler integrals see [10].
Our purpose is to show the next two theorems which make an interesting statement about the coefficients $\lambda_{k,b}$.

Theorem 3.1.5. *Let $k > 1$ be an odd integer and*

$$\gamma_\ell = (2\pi b)^{-\ell} b(\ell-1)! \prod_{b-\ell+1 \neq j=1}^{b} \zeta\left(\frac{\ell+j-1}{b}\right) \prod_{j=1}^{b} \zeta\left(\frac{\ell+j-1}{b} + k\right)$$

if $1 \leq \ell \leq b$,

$$\gamma_\ell = (2\pi b)^{-\ell} \frac{(-1)^\ell}{(-\ell)!} \zeta'(1-k) \prod_{(1-k)b+1-\ell \neq j=1}^{b} \zeta\left(\frac{\ell+j-1}{b}\right) \zeta\left(\frac{\ell+j-1}{b} + k\right)$$

if $1 - bk \leq \ell \leq b - bk$ and

$$\gamma_\ell = (2\pi b)^{-\ell} \frac{(-1)^\ell}{(-\ell)!} \prod_{j=1}^{b} \zeta\left(\frac{\ell+j-1}{b}\right) \zeta\left(\frac{\ell+j-1}{b} + k\right)$$

else. Then we have the modular identity

$$M_{k,b}(\tau,\chi_0) - (-1)^A (-i\tau)^{bk-1} M_{k,b}(-1/\tau,\chi_0) = \sum_{\ell=1-bk-b}^{b} \gamma_\ell (-i\tau)^{-\ell},$$

where $A = b(k - \chi_0(-1))/2 = b(k-1)/2$.

Theorem 3.1.6. *Let k be a positive odd integer and $\chi \neq \chi_0$ be primitive. Let also*

$$\gamma_\ell = \left(\frac{2\pi b}{m}\right)^{-\ell} \frac{(-1)^\ell}{(-\ell)!} \prod_{j=1}^{b} L\left(\frac{\ell+j-1}{b}, \chi\right) L\left(\frac{\ell+j-1}{b} + k, \overline{\chi}\right)$$

if $1 - bk - b \leq \ell \leq 0$ and $\gamma_\ell = 0$ otherwise. Then we have the modular identity

$$M_{k,b}(\tau, \chi) - (-1)^B (-i\tau)^{bk-1} M_{k,b}(-1/\tau, \chi) = \sum_{\ell=1-bk-b}^{b} \gamma_\ell (-i\tau)^{-\ell},$$

where $B = b(k - \chi(-1))/2$.

In the case $b = 1$ both theorems are well-known and lead directly to identity (1.2) suggested by Ramanujan. More precisely, using Theorem 3.1.5 we obtain an identity by Grosswald (see also [9]) for zeta values at odd arguments $k > 1$:

$$\begin{aligned} M_{k,1}(\tau, \chi_0) - \tau^{k-1} M_{k,1}(-1/\tau, \chi_0) &= \frac{1}{2}\zeta(k)(\tau^{k-1} - 1) \\ &\quad + \frac{(2\pi i)^k}{2\tau} \sum_{j=0}^{\frac{k-1}{2}+1} \tau^{k+1-2j} \frac{B_{2j}}{(2j)!} \frac{B_{2k+1-2j}}{(2k+1-2j)!}. \end{aligned}$$

Putting $\tau = \beta i/\pi$, with some $\beta > 0$, gives (1.2). The details of this calculation are presented at the beginning of the next section.
The cases $b > 1$ are new results and lead to series identities for products of Dirichlet L-functions at rational arguments.
Theorem 3.1.5 and Theorem 3.1.6 are proved in Theorem 3.1.12 in this section. In Section 3 we give some corollaries and examples. We start with the following proposition.

Proposition 3.1.7. *Let k be an odd positive integer and χ be a primitive character. Then we have the reflection formula*

$$\psi_k(1-k-s, \chi) = (-1)^{(k-\chi(-1))/2} \psi_k(s, \chi).$$

Proof. Let $a = (1 - \chi(-1))/2$. With Theorem 2.3.7 we obtain

$$\psi_k(1-k-s, \chi) = \left(\frac{2\pi}{m}\right)^{s+k-1} \Gamma(1-k-s) L(1-k-s, \chi) L(1-s, \overline{\chi})$$

$$= \left(\frac{2\pi}{m}\right)^{s+k-1} \Gamma(1-k-s)\frac{2i^a}{\mathcal{G}(\overline{\chi})}\Gamma(s+k)\cos\left(\frac{\pi}{2}(k+s-a)\right)$$
$$\times \left(\frac{2\pi}{m}\right)^{-s-k} L(s+k,\overline{\chi}) \cdot \frac{2i^a}{\mathcal{G}(\chi)}\Gamma(s)\cos\left(\frac{\pi}{2}(s-a)\right)\left(\frac{2\pi}{m}\right)^{-s} L(s,\chi).$$

With the help of Theorem 2.1.3 and since $\mathcal{G}(\chi)\mathcal{G}(\overline{\chi}) = \chi(-1)m = (-1)^a m$ this simplifies to

$$= 2(-1)^a \left(\frac{2\pi}{m}\right)^{-s} \frac{\cos\left(\frac{\pi}{2}(k+s)\right)\cos\left(\frac{\pi}{2}s\right)}{\sin\left(\pi(s+k)\right)}\Gamma(s)L(s,\chi)L(s+k,\overline{\chi}).$$

Hence we are left to show

$$R(s) = \frac{\cos\left(\frac{\pi}{2}(k+s)\right)\cos\left(\frac{\pi}{2}s\right)}{\sin(\pi(k+s))} = \frac{1}{2}(-1)^{(k-1)/2}$$

for odd integers k to show the proposition. This is an easy application of Liouvilles theorem, since $R(s)$ is periodic with period 1, bounded on $\{s \in \mathbb{C} \mid |\mathrm{Im}(s)| \geq 1\}$ and we have

$$\lim_{s\to 0} R(s) = \frac{-\frac{\pi}{2}\sin\left(\frac{\pi}{2}(k+s)\right)\cos\left(\frac{\pi}{2}s\right) - \frac{\pi}{2}\cos\left(\frac{\pi}{2}(k+s)\right)\sin\left(\frac{\pi}{2}s\right)}{\pi\cos(\pi(k+s))} = \frac{1}{2}(-1)^{(k-1)/2}$$

with L'Hospital. Hence R is entire and bounded with $R(0) = \frac{1}{2}(-1)^{(k-1)/2}$. □

Definition 3.1.8. *Let $b > 0$ be an integer and $\varepsilon > 0$ an arbitrary real number. Then we define $I_{k,b}(\tau,\chi)$ to be the Mellin integral*

$$I_{k,b}(\tau,\chi) = \frac{m^{-(b-1)/2}b^{-1/2}}{2\pi i}\int_{b+\varepsilon-i\infty}^{b+\varepsilon+i\infty}\prod_{j=1}^{b}\psi_k\left(\frac{s+j-1}{b},\chi\right)(-i\tau)^{-s}\mathrm{d}s.$$

It follows quickly from Stirling's formula and the polynomial growth of Dirichlet L-functions along the imaginary axis and $\mathrm{Re}\left(\frac{s+j-1}{b}\right) > 1$ for all $\mathrm{Re}(s) > b$ and $j \geq 1$ that $I_{k,b}$ is indeed a holomorphic function in τ on the upper half plane. Our aim is to identify functional equations of the $I_{k,b}$ of the modular type. The next step is to take a closer look at the product of the ψ_k.

Proposition 3.1.9. *Let k be a positive odd integer and χ be a primitive character. The function $\Phi_{k,b}(s,\chi) = \prod_{j=1}^{b} \psi_k\left(\frac{s+j-1}{b},\chi\right)$ has the following properties:*

(i) For all $\mathrm{Re}(s) > b$ *one has*

$$\Phi_{k,b}(s,\chi) = \Gamma(s)\left(\frac{2\pi b}{m}\right)^{-s} m^{(b-1)/2} b^{1/2} \sum_{n=1}^{\infty} \lambda_{k,b}(n,\chi) n^{-s/b},$$

where the coefficients $\lambda_{k,b}$ are given in (3.2).

(ii) It extends to a holomorphic function on $\mathbb{C}\setminus\{\ell \in \mathbb{Z} \mid 1-bk-b \le \ell \le b\}$ with possible simple poles or removable singularities for all $s \in \{\ell \in \mathbb{Z} \mid 1-kb-b \le \ell \le b\}$.

(iii) It satisfies the functional equation $\Phi_{k,b}(1-bk-s,\chi) = (-1)^{b(k-\chi(-1))/2}\Phi_{k,b}(s,\chi)$.

(iv) Let $R_\ell := \mathrm{res}_{s=\ell}\Phi_{k,b}(s,\chi)$ and $A(s) := (2\pi b/m)^{-s} m^{(b-1)/2} b^{1/2}$. In the case $\chi \neq \chi_0$, we obtain

$$R_\ell = A(\ell)\frac{(-1)^\ell}{(-\ell)!}\prod_{j=1}^{b} L\left(\frac{\ell+j-1}{b},\chi\right) L\left(\frac{\ell+j-1}{b}+k,\overline{\chi}\right)$$

if $1-bk-b \le \ell \le 0$ and $R_\ell = 0$ if $1 \le \ell \le b$. In the case that χ is principal and $k > 1$, we obtain

$$R_\ell = A(\ell) b(\ell-1)! \prod_{b-\ell+1 \neq j=1}^{b} \zeta\left(\frac{\ell+j-1}{b}\right) \prod_{j=1}^{b} \zeta\left(\frac{\ell+j-1}{b}+k\right)$$

if $1 \le \ell \le b$,

$$R_\ell = A(\ell)\frac{(-1)^\ell}{(-\ell)!}\zeta'(1-k) \prod_{(1-k)b+1-\ell \neq j=1}^{b} \zeta\left(\frac{\ell+j-1}{b}\right)\zeta\left(\frac{\ell+j-1}{b}+k\right)$$

if $1-bk \le \ell \le b-bk$ and

$$R_\ell = A(\ell)\frac{(-1)^\ell}{(-\ell)!}\prod_{j=1}^{b} \zeta\left(\frac{\ell+j-1}{b}\right)\zeta\left(\frac{\ell+j-1}{b}+k\right)$$

else.

Proof.

(i) To show this, one uses the well-known product formula

$$\prod_{j=1}^{b} \Gamma\left(\frac{s+j-1}{b}\right) = \Gamma(s)(2\pi)^{(b-1)/2} b^{1/2-s},$$

which was shown in Theorem 2.1.6. By definition we have

$$\sum_{n=1}^{\infty} \lambda_{k,b}(n,\chi) n^{-s} = \prod_{j=1}^{b} L\left(s + \frac{j-1}{b}, \chi\right) L\left(s + \frac{j-1}{b} + k, \overline{\chi}\right).$$

We obtain for all s with $\mathrm{Re}(s) > b$:

$$\begin{aligned}
&\prod_{j=1}^{b} \Gamma\left(\frac{s+j-1}{b}\right) \left(\frac{2\pi}{m}\right)^{-\frac{s+j-1}{b}} L\left(\frac{s+j-1}{b}, \chi\right) L\left(\frac{s+j-1}{b} + k, \overline{\chi}\right) \\
&= \Gamma(s) \left(\frac{2\pi}{m}\right)^{-s-(b-1)/2} (2\pi)^{(b-1)/2} b^{1/2-s} \sum_{n=1}^{\infty} \lambda_{k,b}(n,\chi) n^{-s/b} \\
&= \Gamma(s) \left(\frac{2\pi b}{m}\right)^{-s} m^{(b-1)/2} b^{1/2} \sum_{n=1}^{\infty} \lambda_{k,b}(n,\chi) n^{-s/b}.
\end{aligned}$$

(ii) This is immediate. If χ is principal, we additionally have to consider the cases $(s+j-1)/b = 1$ and $(s+j-1)/b + k = 1$ given that $\zeta(s)$ has a simple pole at $s = 1$. In these cases we have the equivalences $s = b - j + 1 \le b$ and $s = b(1-k) - j + 1 \ge 1 - bk - b$, respectively.

(iii) Using Proposition 3.1.7 we find

$$\begin{aligned}
\Phi_{k,b}(1 - bk - s, \chi) &= \prod_{j=1}^{b} \psi_k\left(\frac{1 - bk - s + j - 1}{b}, \chi\right) \\
&= \prod_{j=1}^{b} \psi_k\left(1 - k - \frac{s + b - j}{b}, \chi\right) \\
&= (-1)^{b(k-\chi(-1))/2} \prod_{j=1}^{b} \psi_k\left(\frac{s+b-j}{b}, \chi\right) \\
&= (-1)^{b(k-\chi(-1))/2} \Phi_{k,b}(s, \chi).
\end{aligned}$$

(iv) Let $\chi \neq \chi_0$. Since $L(s,\chi)$ and e^s are entire functions and $\operatorname{res}_{s=\ell}\Gamma(s) = \frac{(-1)^\ell}{(-\ell)!}$ for $\ell \leq 0$, we obtain

$$R_\ell = \operatorname{res}_{s=\ell}\left(A(s)\Gamma(s)\prod_{j=1}^{b} L\left(\frac{s+j-1}{b},\chi\right) L\left(\frac{s+j-1}{b}+k,\overline{\chi}\right)\right)$$
$$= A(\ell)\frac{(-1)^\ell}{(-\ell)!}\prod_{j=1}^{b} L\left(\frac{\ell+j-1}{b},\chi\right) L\left(\frac{\ell+j-1}{b}+k,\overline{\chi}\right)$$

in the case that $1-bk-b \leq \ell \leq 0$ and $R_\ell = 0$ else.

Let now $\chi = \chi_0$. Then we have $L(s,\chi) = \zeta(s)$. We will distinguish between the different cases.

Case 1. $1 \leq \ell \leq b$. In this case, the only pole which occurs in the term of $\Phi_{k,b}$ is the one contributed by $\zeta\left(\frac{s+j-1}{b}\right)$ if $j = b-\ell+1$. Moreover, the pole has residue $\operatorname{res}_{s=\ell}\zeta\left(\frac{s+b-\ell}{b}\right) = b$, hence

$$R_\ell = \operatorname{res}_{s=\ell}\left(A(s)\Gamma(s)\prod_{j=1}^{b}\zeta\left(\frac{s+j-1}{b}\right)\zeta\left(\frac{s+j-1}{b}+k\right)\right)$$
$$= A(\ell)b(\ell-1)!\prod_{b-\ell+1\neq j=1}^{b}\zeta\left(\frac{\ell+j-1}{b}\right)\prod_{j=1}^{b}\zeta\left(\frac{\ell+j-1}{b}+k\right)$$

in this case.

Case 2. $1-bk \leq \ell \leq b-bk$. Since k is odd, we obtain $\lim_{s\to\ell}\zeta\left(\frac{s+j-1}{b}\right)\zeta\left(\frac{s+j-1}{b}+k\right) = \zeta'(1-k)$ for $j = b-bk-\ell+1$ which can only occur if $1-bk \leq j \leq b-bk$. On the other side, we have $\ell \leq 0$ and therefore the zeta function will contribute a simple pole. Hence

$$R_\ell = \operatorname{res}_{s=\ell}\left(A(s)\Gamma(s)\prod_{j=1}^{b}\zeta\left(\frac{s+j-1}{b}\right)\zeta\left(\frac{s+j-1}{b}+k\right)\right)$$
$$= A(\ell)\frac{(-1)^\ell}{(-\ell)!}\zeta'(1-k)\prod_{(1-k)b+1-\ell\neq j=1}^{b}\zeta\left(\frac{\ell+j-1}{b}\right)\zeta\left(\frac{\ell+j-1}{b}+k\right).$$

Case 3. If Case 1. and 2. do not apply, we clearly have $\ell \leq 0$ but also no singularities from the zeta terms. The result follows analogous to the case $\chi \neq \chi_0$. $\square$

Lemma 3.1.10. *Let $y > 0$ and $a < b$ be real numbers. Then we have*

$$\lim_{T\to\infty} \int_{a\pm iT}^{b\pm iT} \Phi_{k,b}(s,\chi)y^{-s}\mathrm{d}s = 0.$$

Proof. This follows directly with Corollary 2.1.5, Theorem 2.3.7 and Theorem 2.3.11. □

Remark 3.1.11. *The case $\chi = \chi_0$ and $k = 1$ leads to $f(s) = (2\pi)^{-s}\Gamma(s)\zeta(s)\zeta(s+1)$, which has a double pole at $s = 0$ and is related to the Dedekind η-function $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ by the formula $\exp I_{1,1}(\tau,\chi_0) = q^{-1/24}\eta(\tau)$. One can use the techniques of the next theorem to show the functional equation $\eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau)$. For more details, see also [13] on p. 129, where the complete proof (of Weil) is presented.*

From now on, if χ is principal, we will assume that $k > 1$.

Theorem 3.1.12. *Let k be a positive odd integer and χ be a primitive character. Then the functions $I_{k,b}$ are modular in the sense that they fulfill the functional equation*

$$(-1)^{b(k-\chi(-1))/2}(-i\tau)^{bk-1}I_{k,b}\left(-\frac{1}{\tau},\chi\right) = I_{k,b}(\tau,\chi) - \sum_{\ell=1-bk-b}^{b}\gamma_\ell(-i\tau)^{-\ell}$$

where the γ_ℓ are defined in Theorem 3.1.5 and Theorem 3.1.6, respectively. We have the following generalized Fourier expansion

$$I_{k,b}(\tau,\chi) = \sum_{n=1}^{\infty}\lambda_{k,b}(n,\chi)q^{bn^{1/b}/m},$$

and in particular, $M_{k,b} = I_{k,b}$.

Proof. Let $y > 0$ be real and $\tau = iy$. We will use Mellin's inversion theorem to show the theorem. One can consider the closed contour integral

$$W_{k,b}(iy,\chi) = \frac{m^{-(b-1)/2}b^{-1/2}}{2\pi i}\oint_\gamma \Phi_{k,b}(s,\chi)y^{-s}\mathrm{d}s$$

where γ is the rectangle with vertices at $b+\varepsilon \pm iR$ and $1-bk-b-\varepsilon \pm iR$, integrated anticlockwise, R running to $+\infty$. With the help of Proposition 3.1.9 (note that $\gamma_\ell =$

$m^{-(b-1)/2}b^{-1/2}R_\ell)$, residue calculus leads us to

$$W_{k,b}(iy,\chi) = \sum_{\ell=1-bk-b}^{b} \gamma_\ell y^{-\ell}.$$

On the other hand, as R goes to infinity, the integrals from $b+\varepsilon+iR$ to $1-bk-b-\varepsilon+iR$ and $1-bk-b-\varepsilon-iR$ to $b+\varepsilon-iR$ respectively will vanish according to Lemma 3.1.10, therefore we have

$$I_{k,b}(iy,\chi) - \frac{m^{-(b-1)/2}b^{-1/2}}{2\pi i} \int\limits_{1-bk-b-\varepsilon-i\infty}^{1-bk-b-\varepsilon+i\infty} \Phi_{k,b}(s,\chi)y^{-s}\mathrm{d}s = \sum_{\ell=1-bk-b}^{b} \gamma_\ell y^{-\ell}.$$

If we now make the substitution $s \longmapsto 1-bk-s$ into the above integral, we obtain

$$\int\limits_{1-bk-b-\varepsilon-i\infty}^{1-bk-b-\varepsilon+i\infty} \Phi_{k,b}(s,\chi)y^{-s}\mathrm{d}s = y^{bk-1}(-1)^{b(k-\chi(-1))/2} \int\limits_{b+\varepsilon-i\infty}^{b+\varepsilon+i\infty} \Phi_{k,b}(s,\chi)y^{s}\mathrm{d}s$$

and this leads to

$$I_{k,b}(iy,\chi) - y^{bk-1}(-1)^{b(k-\chi(-1))/2} I_{k,b}\left(\frac{i}{y},\chi\right) = \sum_{\ell=1-bk-b}^{b} \gamma_\ell y^{-\ell}.$$

The claim now follows by analytic continuation. The Fourier expansion part follows with

$$I_{k,b}(\tau,\chi) = \sum_{n=1}^{\infty} \lambda_{k,b}(n,\chi) \frac{1}{2\pi i} \int\limits_{b+\varepsilon-i\infty}^{b+\varepsilon+i\infty} \Gamma(s)\left(-\frac{2\pi b n^{1/b}\tau i}{m}\right)^{-s} \mathrm{d}s = \sum_{n=1}^{\infty} \lambda_{k,b}(n,\chi) q^{bn^{1/b}/m}.$$

The absolute convergence of the series justifies the switching of sum and integral by Lebesgue's theorem. □

It is plain that Theorem 3.1.5 and Theorem 3.1.6 are immediate corollaries.

3.2 Series identities for values of L-functions at rational points

If we consider the cases $b = 1$, $k = 2N + 1$ for some integer $N > 0$ and $\chi = \chi_0$ we can deduce the identity (1.2) by setting $\pi i\tau = -\alpha$ and $-\pi i/\tau = -\beta$. If τ/i is a positive real number, then $\alpha, \beta > 0$ and $\alpha\beta = \pi^2$. First we obtain $\lambda_{2N+1,1}(n, \chi_0) = \sigma_{-2N-1}(n)$ and it follows that

$$\sum_{n=1}^{\infty} \sigma_{-2N-1}(n) \left(e^{-2\alpha n} - (-1)^N \left(\frac{\alpha}{\beta}\right)^N e^{-2\beta n} \right) = \sum_{\ell=-2N-1}^{1} \gamma_\ell \left(\frac{\alpha}{\beta}\right)^{-\ell/2}. \tag{3.3}$$

We use the identities $\zeta(2n) = (-1)^{n+1}\frac{(2\pi)^{2n}B_{2n}}{2(2n)!}$, $\zeta(1-n) = -\frac{B_n}{n}$ as well as $\zeta'(-2n) = \frac{1}{2}(-1)^n(2\pi)^{-2n}(2n)!\zeta(2n+1)$ for all $n \geq 1$ given in Theorem 2.3.9 to transform the right hand side of (3.3) into

$$\zeta(2N+1)\left(\frac{(-1)^N}{2}\left(\frac{\alpha}{\beta}\right)^N - \frac{1}{2}\right) + \frac{(2\pi)^{2N+1}}{2}\sum_{\ell=0}^{N+1}(-1)^{N-\ell+1}\frac{B_{2\ell}}{(2\ell)!}\frac{B_{2N+2-2\ell}}{(2N+2-2\ell)!}\left(\frac{\alpha}{\beta}\right)^{(2\ell-1)/2}.$$

The left hand side is a Lambert sum, hence it equals to

$$\sum_{n=1}^{\infty} \frac{1}{n^{2N+1}(e^{2\alpha n}-1)} - (-1)^N \left(\frac{\alpha}{\beta}\right)^N \sum_{n=1}^{\infty} \frac{1}{n^{2N+1}(e^{2\beta n}-1)}.$$

The equation (1.2) follows now immediately.
The next corollary refers to the general case, that is when χ is not principal.

Corollary 3.2.1. *Let k be a positive odd integer and $\chi \neq \chi_0$ be a primitive character. We define for all τ on the upper half plane*

$$F_k(\tau, \chi) = \frac{\mathcal{G}(\chi)}{m} \sum_{n=1}^{\infty} \sum_{\ell=1}^{m} \frac{\overline{\chi}(\ell n) q^{n/m}}{n^k (e^{2\pi i \ell/m} - q^{n/m})}.$$

Then the following identity holds.

$$(-1)^{(k-\chi(-1))/2}(-i\tau)^{k-1}F_k\left(-\frac{1}{\tau},\chi\right) = F_k(\tau,\chi) - \sum_{\ell=0}^{k}\frac{(-1)^\ell}{\ell!}L(-\ell,\chi)L(k-\ell,\overline{\chi})\left(-\frac{2\pi i\tau}{m}\right)^\ell.$$

Proof. With the help of the identity

$$\sum_{n=1}^{\infty}\chi(n)q^n = \frac{\mathcal{G}(\chi)}{m}\sum_{\ell=1}^{m}\frac{\overline{\chi}(\ell)q}{e^{2\pi i\ell/m}-q}$$

which was shown in Theorem 2.2.10, one obtains

$$\begin{aligned} M_{k,1}(\tau,\chi) &= \sum_{n=1}^{\infty}\sigma_{0,-k}(n,\chi)q^{n/m} \\ &= \sum_{n=1}^{\infty}\sum_{d|n}d^{-k}\overline{\chi}(d)\chi(n/d)q^{n/m} \\ &= \sum_{n=1}^{\infty}\sum_{a=1}^{\infty}\frac{\overline{\chi}(a)}{a^k}\chi(n)q^{an/m} \\ &= \frac{\mathcal{G}(\chi)}{m}\sum_{a=1}^{\infty}\frac{\overline{\chi}(a)}{a^k}\sum_{\ell=1}^{m}\frac{\overline{\chi}(\ell)q^{a/m}}{e^{2\pi i\ell/m}-q^{a/m}} \\ &= \frac{\mathcal{G}(\chi)}{m}\sum_{n=1}^{\infty}\sum_{\ell=1}^{m}\frac{\overline{\chi}(\ell n)q^{n/m}}{n^k(e^{2\pi i\ell/m}-q^{n/m})}. \end{aligned}$$

The corollary follows now from Theorem 3.1.6. □

Example 3.2.2. *Let $k=3$, $b=1$ and χ_5 be the Legendre symbol with modulus 5. Then we obtain the series identity*

$$L(2,\chi_5) = \frac{5\sqrt{5}}{2\pi}\sum_{n=1}^{\infty}\frac{\chi_5(n)}{n^3}\left[\frac{1}{e^{2\pi n/5}\zeta_5-1} - \frac{1}{e^{2\pi n/5}\zeta_5^2-1} - \frac{1}{e^{2\pi n/5}\zeta_5^3-1} + \frac{1}{e^{2\pi n/5}\zeta_5^4-1}\right].$$

Here, $\zeta_5 = \exp(2\pi i/5)$.

Proof. One uses the substitution $\tau = i$ in the formula of Corollary 3.2.1, $\mathcal{G}(\chi_5) = \sqrt{5}$, $L(-1,\chi_5) = -\frac{2}{5}$ and $L(-2n,\chi_5) = 0$ for all $n \geq 0$ to find the identity. □

Example 3.2.3. *Let $k = 1$, $b = 3$ and χ_7 be the Legendre symbol with conductor 7. Then the coefficients $\lambda_{1,3}(n, \chi_7)$ are given by*

$$\lambda_{1,3}(n,\chi_7) = \chi_7(n)n^{-2/3}\sum_{d|n} d^{1/3}\sigma_{-1}\left(\frac{n}{d}\right)\sum_{c|d} c^{1/3}\sigma_{-1}(c)\sigma_{-1}\left(\frac{d}{c}\right).$$

If we now set

$$A = \frac{9}{49}\pi^2 L(-2/3,\chi_7)L(-1/3,\chi_7)L(1/3,\chi_7)L(2/3,\chi_7)L(1,\chi_7),$$

$$B = -\frac{3}{7}\pi L(-1/3,\chi_7)L(1/3,\chi_7)L(2/3,\chi_7)L(1,\chi_7)L(4/3,\chi_7),$$

and

$$C = \frac{1}{2}L(1/3,\chi_7)L(2/3,\chi_7)L(1,\chi_7)L(4/3,\chi_7)L(5/3,\chi_7),$$

we obtain

$$\sum_{n=1}^{\infty}\lambda_{1,3}(n,\chi_7)e^{-6\pi\sqrt[3]{n}/7} = A + B + C.$$

Proof. Since χ_7 is real, formula (3.1) reduces to

$$\sigma_{u,v}(n,\chi_7) = n^{-u}\chi_7(n)\sigma_v(n).$$

Hence

$$\begin{aligned}\lambda_{1,3}(n,\chi_7) &= (\sigma_{0,-1}(.,\chi_7) * \sigma_{1/3,-1}(.,\chi_7) * \sigma_{2/3,-1}(.,\chi_7))(n)\\ &= \sum_{d|n}\sum_{c|d}\chi_7(c)\sigma_{-1}(c)(d/c)^{-1/3}\chi_7(d/c)\sigma_{-1}(d/c)(n/d)^{-2/3}\chi_7(n/d)\sigma_{-1}(n/d)\\ &= \chi_7(n)n^{-2/3}\sum_{d|n}\sum_{c|d}\sigma_{-1}(c)\sigma_{-1}(d/c)\sigma_{-1}(n/d)(dc)^{1/3}.\end{aligned}$$

To get the series identity, substitute $\tau = i$ into the formula of Theorem 3.1.6. One may use $L(0,\chi_7) = 1$. □

3.3 Mixed characters

There is no difficulty to generalize the introduced method to the case of mixed characters. To be exact, one obtains similar results for primitive characters χ and ψ with modulus m and ℓ respectively by generalizing the function $\psi_k(s,\chi)$ to a function $\psi_k(s,\chi,\psi)$.

Definition 3.3.1. *Let χ and ψ be primitive characters modulo m and ℓ respectively. We define*

$$\psi_k(s,\chi,\psi) = \left(\frac{2\pi}{m}\right)^{-s} \Gamma(s) L(s,\chi) L(s+k,\overline{\psi}).$$

The functions $\psi_k(s,\chi,\psi)$ behave similarly to $\psi_k(s,\chi)$, as the next theorem shows.

Theorem 3.3.2. *Let $k \geq 1$ be an odd integer and $\chi(-1) = \psi(-1)$. Then we have*

$$\psi_k(1-k-s,\chi,\psi) = \frac{m}{\mathcal{G}(\overline{\chi})\mathcal{G}(\psi)}(-1)^{(k-\chi(-1))/2}\psi_k(s,\psi,\chi).$$

Similarly, let $\chi(-1) = -\psi(-1)$ and $k > 1$ and $k > 1$ even. Then we obtain

$$\psi_k(1-k-s,\chi,\psi) = \frac{im}{\mathcal{G}(\overline{\chi})\mathcal{G}(\psi)}(-1)^{\frac{k}{2}}\psi_k(s,\psi,\chi).$$

Proof. The proof is nearly analogous to Proposition 3.1.7. At the beginning, let us consider the first case. Let $a = (1-\chi(-1))/2 = (1-\psi(-1))/2$. With Theorem 2.3.7 we obtain

$$\begin{aligned}
\psi_k(1-k-s,\chi,\psi) &= \left(\frac{2\pi}{m}\right)^{s+k-1} \Gamma(1-k-s) L(1-k-s,\chi) L(1-s,\overline{\psi}) \\
&= \left(\frac{2\pi}{m}\right)^{s+k-1} \Gamma(1-k-s) \frac{2i^a}{\mathcal{G}(\overline{\chi})} \Gamma(s+k) \cos\left(\frac{\pi}{2}(k+s-a)\right) \\
&\times \left(\frac{2\pi}{m}\right)^{-s-k} L(s+k,\overline{\chi}) \cdot \frac{2i^a}{\mathcal{G}(\psi)} \Gamma(s) \cos\left(\frac{\pi}{2}(s-a)\right) \left(\frac{2\pi}{\ell}\right)^{-s} L(s,\psi) \\
&= \frac{2}{\pi} \frac{m(-1)^a}{\mathcal{G}(\overline{\chi})\mathcal{G}(\psi)} \left(\frac{2\pi}{\ell}\right)^{-s} J(s) L(s,\psi) L(s+k,\overline{\chi})
\end{aligned}$$

where $J(s) = \cos\left(\frac{\pi}{2}(k+s-a)\right)\cos\left(\frac{\pi}{2}(s-a)\right)\Gamma(1-k-s)\Gamma(s+k)$, which simplifies to $J(s) = \frac{\pi}{2}(-1)^{(k-1)/2}$ (analogous to Proposition 3.1.7) since k is assumed to be odd, hence

$$= \frac{m}{\mathcal{G}(\overline{\chi})\mathcal{G}(\psi)}(-1)^{(k-\chi(-1))/2}\left(\frac{2\pi}{\ell}\right)^{-s} L(s,\psi)L(s+k,\overline{\chi}),$$

which shows the first part of the theorem. To show the second part, let $a = (1-\chi(-1))/2 = 1-(1-\psi(-1))/2$. We obtain

$$\begin{aligned}
\psi_k(1-k-s,\chi,\psi) &= \left(\frac{2\pi}{m}\right)^{s+k-1}\Gamma(1-k-s)L(1-k-s,\chi)L(1-s,\overline{\psi}) \\
&= \left(\frac{2\pi}{m}\right)^{s+k-1}\Gamma(1-k-s)\frac{2i^a}{\mathcal{G}(\overline{\chi})}\Gamma(s+k)\cos\left(\frac{\pi}{2}(k+s-a)\right) \\
&\times \left(\frac{2\pi}{m}\right)^{-s-k} L(s+k,\overline{\chi})\cdot\frac{2i^{1-a}}{\mathcal{G}(\psi)}\Gamma(s)\cos\left(\frac{\pi}{2}(s-1+a)\right)\left(\frac{2\pi}{\ell}\right)^{-s} L(s,\psi) \\
&= \frac{2}{\pi}\frac{im}{\mathcal{G}(\overline{\chi})\mathcal{G}(\psi)}\left(\frac{2\pi}{\ell}\right)^{-s} K(s)L(s,\psi)L(s+k,\overline{\chi})
\end{aligned}$$

where $K(s) = \cos\left(\frac{\pi}{2}(s-1+a)\right)\cos\left(\frac{\pi}{2}(s+k-a)\right)\Gamma(1-k-s)\Gamma(s+k)$, which simplifies to $K(s) = \frac{\pi}{2}(-1)^{\frac{k}{2}}$ (analogous to Proposition 3.1.7) since k is assumed to be even, hence

$$= \frac{im}{\mathcal{G}(\overline{\chi})\mathcal{G}(\psi)}(-1)^{\frac{k}{2}}\left(\frac{2\pi}{\ell}\right)^{-s} L(s,\psi)L(s+k,\overline{\chi}),$$

which proves the theorem. $\square$

One can use Theorem 3.3.2 to show other relations in the spirit of (1.2), for instance the following sum of level 4 by Ramanujan (see also [3])

$$\begin{aligned}
\alpha^{-N+\frac{1}{2}}\left(\frac{1}{2}L(2N,\chi_{-4}) + \sum_{k=1}^{\infty}\frac{\chi_{-4}(k)}{k^{2N}(e^{\alpha k}-1)}\right) &= \frac{(-1)^N\beta^{-N+\frac{1}{2}}}{2^{2N+1}}\sum_{k=1}^{\infty}\frac{\operatorname{sech}(\beta k)}{k^{2N}} \qquad (3.4) \\
+\frac{1}{4}\sum_{k=0}^{N}\frac{(-1)^k}{2^{2k}}\frac{E_{2k}}{(2k)!}\frac{B_{2N-2k}}{(2N-2k)!}\alpha^{N-k}\beta^{k+\frac{1}{2}}, &
\end{aligned}$$

which is valid for all positive values α,β satisfying $\alpha\beta = \pi^2$, where N is any positive integer and χ_{-4} is the character $\chi_{-4}(n) = 0$ for n even and $\chi_{-4}(n) = (-1)^{(n-1)/2}$ else.

Here, the E_n are the Euler numbers characterized by the expansion

$$\operatorname{sech}(z) = \sum_{n=0}^{\infty} E_n \frac{z^n}{n!}, \qquad |z| < \pi.$$

Identity (3.4) can be derived using Theorem 3.3.2 with the characters $\chi = \chi_0$ and $\psi = \chi_{-4}$. Let $k = 2N$. One consideres the closed contour integral

$$W_{2N}(iy, R) = \frac{1}{2\pi i} \oint_{\gamma} \psi_{2N}(s, \chi_0, \chi_{-4}) y^{-s} \mathrm{d}s$$

where γ is the rectangle with vertices $1 + \varepsilon \pm iR$ and $-2N - \varepsilon \pm iR$ respectively and $\tau = iy$ where $y > 0$ is a real number. As R goes to $+\infty$, we obtain

$$\lim_{R\to\infty} W_{2N}(iy, R) = \frac{1}{2\pi i} \int\limits_{1+\varepsilon-i\infty}^{1+\varepsilon+i\infty} \psi_{2N}(s, \chi_0, \chi_{-4}) y^{-s} \mathrm{d}s - \frac{1}{2\pi i} \int\limits_{-2N-\varepsilon-i\infty}^{-2N-\varepsilon+i\infty} \psi_{2N}(s, \chi_0, \chi_{-4}) y^{-s} \mathrm{d}s$$

as usually with Stirling's and Lindelöf's Theorem. This equals to

$$\frac{1}{2\pi i} \int\limits_{1+\varepsilon-i\infty}^{1+\varepsilon+i\infty} \psi_{2N}(s, \chi_0, \chi_{-4}) y^{-s} \mathrm{d}s - \frac{y^{2N-1}}{2\pi i} \frac{i(-1)^N}{2i} \int\limits_{1+\varepsilon-i\infty}^{1+\varepsilon+i\infty} \psi_{2N}(s, \chi_{-4}, \chi_0) y^{s} \mathrm{d}s$$

since $\mathcal{G}(\chi_0) = 1$ and $\mathcal{G}(\chi_{-4}) = 2i$. Using Mellin's inversion theorem, we can write both integrals as a Fourier series, hence they equal

$$\begin{aligned}
&\sum_{\nu=1}^{\infty} \sum_{\mu=1}^{\infty} \frac{\chi_{-4}(\nu)}{\nu^{2N}} e^{-2\pi y \nu \mu} - \frac{(-1)^N y^{2N-1}}{2} \sum_{\nu=1}^{\infty} \sum_{\mu=1}^{\infty} \frac{\chi_{-4}(\mu)}{\nu^{2N}} e^{-\pi \nu \mu/(2y)} \\
=&\sum_{\nu=1}^{\infty} \frac{\chi_{-4}(\nu)}{\nu^{2N}(e^{2\pi y \nu} - 1)} - (-1)^N y^{2N-1} \sum_{\nu=1}^{\infty} \frac{\operatorname{sech}(\pi \frac{\nu}{2y})}{\nu^{2N}}
\end{aligned}$$

where $q = e^{-2\pi y}$ and this remains valid, when yi is any number from the upper half-plane. On the other hand, we have

$$W_{2N}(iy, R) = \sum_{\ell=-2N}^{1} \operatorname{res}_{s=\ell} \psi_k(s, \chi_0, \chi_{-4}) y^{-s}$$

and

$$L(2\nu + 1, \chi_{-4}) = \frac{1}{4} \cdot \frac{(-1)^\nu E_{2\nu} \pi^{2\nu+1}}{2^{2\nu}(2\nu)!},$$

which is given in [4] on p. 125, we obtain

$$\begin{aligned}
&\sum_{\ell=-2N}^{-1} \operatorname{res}_{s=\ell}(2\pi y)^{-s}\Gamma(s)\zeta(s)L(s+2N,\chi_{-4})\\
=&\sum_{\ell=-2N}^{-1}(2\pi y)^{-\ell}\frac{(-1)^\ell}{(-\ell)!}\zeta(\ell)L(\ell+2N,\chi_{-4})\\
=&\frac{1}{4}\sum_{\substack{\ell=-2N\\ \ell\text{ odd}}}^{-1}(2\pi y)^{-\ell}\frac{(-1)^\ell}{(-\ell)!}\frac{-B_{1-\ell}}{1-\ell}\frac{(-1)^{\frac{\ell+2N-1}{2}}\pi^{2N+\ell}E_{\ell+2N-1}}{(\ell+2N-1)!}\\
=&\frac{1}{4}\sum_{\substack{f=1\\ f\text{ odd}}}^{2N}(2\pi y)^{f}\frac{(-1)^{f-1}}{f!}\frac{B_{1+f}}{(1+f)!}\frac{(-1)^{\frac{2N-f-1}{2}}\pi^{2N-f}E_{2N-f-1}}{(2N-f-1)!}\\
=&\frac{1}{4}\sum_{\nu=0}^{N-1}(2\pi y)^{2\nu+1}\frac{B_{2\nu+2}}{(2\nu+2)!}\frac{E_{2N-2\nu-2}}{(2N-2\nu-2)!}\frac{(-1)^{N-\nu-1}}{4^{N-\nu-1}}\pi^{2N-2\nu-1}\\
=&\frac{1}{4}\sum_{\mu=0}^{N-1}(2\pi y)^{2N-2\mu-1}\frac{B_{2N-2\mu}}{(2N-2\mu)!}\frac{E_{2\mu}}{(2\mu)!}\frac{(-1)^{\mu}}{4^{\mu}}\pi^{2\mu+1}\\
=&\frac{\pi^{2N}}{4}\sum_{\mu=0}^{N-1}(2y)^{2N-2\mu-1}\frac{B_{2N-2\mu}}{(2N-2\mu)!}\frac{E_{2\mu}}{(2\mu)!}\frac{(-1)^{\mu}}{4^{\mu}}
\end{aligned}$$

and hence

$$\begin{aligned}
W_{2N}(iy,R)=&\frac{L(2N+1,\chi_{-4})}{2\pi y}-\frac{1}{2}L(2N,\chi_{-4})+\frac{\pi^{2N}}{4}\sum_{\ell=0}^{N-1}\frac{(-1)^\ell}{2^{2\ell}}\frac{E_{2\ell}}{(2\ell)!}\frac{B_{2N-2\ell}}{(2N-2\ell)!}(2y)^{2N-2\ell-1}\\
=&-\frac{1}{2}L(2N,\chi_{-4})+\frac{\pi^{2N}}{4}\sum_{\ell=0}^{N}\frac{(-1)^\ell}{2^{2\ell}}\frac{E_{2\ell}}{(2\ell)!}\frac{B_{2N-2\ell}}{(2N-2\ell)!}(2y)^{2N-2\ell-1}.
\end{aligned}$$

Putting everything together leads us to

$$\begin{aligned}
&\sum_{\nu=1}^{\infty}\frac{\chi_{-4}(\nu)}{\nu^{2N}(e^{2\pi y\nu}-1)}-(-1)^N y^{2N-1}\sum_{\nu=1}^{\infty}\frac{\operatorname{sech}(\pi\frac{\nu}{2y})}{\nu^{2N}}\\
=&-\frac{1}{2}L(2N,\chi_{-4})+\frac{\pi^{2N}}{4}\sum_{\ell=0}^{N}\frac{(-1)^\ell}{2^{2\ell}}\frac{E_{2\ell}}{(2\ell)!}\frac{B_{2N-2\ell}}{(2N-2\ell)!}(2y)^{2N-2\ell-1}
\end{aligned}$$

and with $\alpha = 2\pi y$ and $\beta = \frac{\pi}{2y}$

$$\frac{1}{2}L(2N, \chi_{-4}) + \sum_{\nu=1}^{\infty} \frac{\chi_{-4}(\nu)}{\nu^{2N}(e^{\alpha\nu} - 1)}$$
$$= \frac{(-1)^N}{2^{2N-1}} \alpha^{N-\frac{1}{2}} \beta^{-N+\frac{1}{2}} \sum_{\nu=1}^{\infty} \frac{\operatorname{sech}(\beta\nu)}{\nu^{2N}} + \frac{\alpha^N \beta^N}{4} \sum_{\ell=0}^{N} \frac{(-1)^\ell}{2^{2\ell}} \frac{E_{2\ell}}{(2\ell)!} \frac{B_{2N-2\ell}}{(2N-2\ell)!} \alpha^{N-\ell-\frac{1}{2}} \beta^{\frac{1}{2}+\ell-N},$$

respectively. Obviously, we have $\alpha\beta = \pi^2$. Finally, multiplying with $\alpha^{\frac{1}{2}-N}$ shows (3.4).

Similarly, one can use the methods of this master thesis to show series representations for Dirichlet L-functions at rational arguments for mixed characters.

Bibliography

[1] T. M. Apostol, *Introduction to analytic number theory*, Undergraduate Text in Mathematics, 1976, Springer.

[2] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, Second Edition, Graduate Text in Mathematics, 1997, Springer.

[3] B. Berndt, *Modular transformations and generalizations of several formulae of Ramanujan*, Rocky Mountain Journal of Mathematics, Volume 7, Number 1, 1977.

[4] B. Berndt, *Ramanujan's Notebooks Part I*, Springer-Verlag, 1985.

[5] B. Berndt, A. Straub, *On a secant Dirichlet series and Eichler integrals of Eisenstein series*, *Math. Z.*, 284(3):827−852, 2016.

[6] F. Beukers, *Irrationality proofs using modular forms*, *Astérisque* 147 - 148, 271 - 283, 1987.

[7] J. Brüdern, *Einführung in die analytische Zahlentheorie*, Springer-Verlag Berlin Heidelberg, 1995.

[8] E. Freitag, R. Busam, *Funktionentheorie 1*, 4. Auflage, Springer (2006)

[9] E. Grosswald, *Die Werte der Riemannschen an ungeraden Argumentstellen*, Nachr. Akad. Wiss. Göttingen Math.-Physik. Kl. 2, 9-13, 1970.

[10] S. Gun, M. R. Murty, P. Rath, *Transcendental values of certain Eichler integrals*, Bull. Lond. Math. Soc., 43(5):939−952, 2011.

[11] M. Koecher, A. Krieg, *Elliptische Funktionen und Modulformen*, Springer-Verlag Berlin Heidelberg New York, 1998.

[12] W. Kohnen, *A very simple proof for the q-product expansion of the Δ-function*, The Ramanujan Journal, August 2010, Volume 10, p. 71-73.

[13] T. Miyake, *Modular forms*, Springer Monographs in Mathematics, 1989, Springer.

[14] E. C. Titchmarsh, *Introduction to the theory of Fourier integrals*, Oxford at the clarendon press, 1948

[15] E. C. Titchmarsh, *The theory of functions*, Oxford university press, Second edition, 1939.

[16] W. Zudilin, *Irrationality of values of the Riemann zeta function*, Izvestiya: Mathematics 66:3 489–542

Selbstständigkeitserklärung

Hiermit versichere ich Eides statt, dass ich die vorliegende Arbeit selbständig und ohne Benutzung anderer als der angegebenen Quellen und Hilfsmittel angefertigt habe.

Ort, Datum

Johann Franke