# RATIONAL FUNCTIONS AND MODULAR FORMS 

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#### Abstract

There are two elementary methods for constructing modular forms that dominate in literature. One of them uses automorphic Poincare series and the other one theta functions. We start a third elementary approach to modular forms using rational functions that have certain properties regarding pole distribution and growth. We prove modularity with contour integration methods and Weil's converse theorem, without using the classical formalism of Eisenstein series and $L$-functions.


## 1. Introduction

We recall that an elliptic modular form $f$ of weight $k \in \mathbb{Z}$ for a congruence subgroup $\Gamma \subset \mathrm{SL}_{2}(\mathbb{Z})$ with multiplier system $v: \Gamma \rightarrow \mathbb{C}^{\times}$is a holomorphic function on the upper half plane $\{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau)>0\}$, which satisfies the transformation law

$$
\left.f\right|_{k} M(\tau)=v(M) f(\tau)
$$

and is additionally holomorphic in the cusps $\mathbb{Q} \cup\{\infty\}$. This means, that the Fourier expansion of $\left.f\right|_{k} M(\tau)$ is of the form $\sum_{n=0}^{\infty} a(n) q^{n / N}$ for all $M \in \mathrm{SL}_{2}(\mathbb{Z})$, where $\left.f\right|_{k} M$ denotes the usual Petersson slash operator

$$
\left.f\right|_{k}\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)(\tau)=(a d-b c)^{\frac{k}{2}}(c \tau+d)^{-k} f\left(\frac{a \tau+b}{c \tau+d}\right) .
$$

One can show that there are no non-constant modular forms for $k \leqslant 0$ and that the spaces $M_{k}(\Gamma, v)$ are finite-dimensional. A useful tool for computing the exact value of the dimensions is the Riemann-Roch formula, for more explicit details see for example [6]. Modular forms play an extraordinary important role in many fields of mathematics and physics such as number theory, geometry and string theory. Also many generalizations of the classical modular forms have been found, such as Siegel modular forms (see also [1] and [11) for matrix-valued arguments that transform under congruence subgroups of the symplectic group $S p_{n}$; and Hilbert modular forms (for a great introduction, the reader may wish to consult [10]) that transform under congruence subgroups of $\mathrm{SL}_{2}(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers of a number field $K$.
Basically, two elementary ideas for constructing modular forms dominate in literature. One of them uses so-called Poincaré series, which give in the simplest case Eisenstein series. The other one goes via Fourier analysis and quadratic forms. This leads to theta functions.
In this paper we will give a third elementary approach to modular forms, which seems to be natural in the sense that the functional equation and Fourier series are on an equal

[^0]footing. This method of construction also does not distinguish between weights $k=1,2$ and $k \geqslant 3$, as it is the case for classical Eisenstein series. It is grounded on a class of very simple functions which we will call weak functions. Here, a weak function $\omega$ is a 1-periodic meromorphic function in the entire plane, which has the following properties:
(i) All poles of $\omega$ are simple and lie in $\mathbb{Q}$.
(ii) The function $\omega$ tends to 0 rapidly as the absolute value of the imaginary part increases, so
$$
\omega(x+i y)=O\left(|y|^{-M}\right)
$$
for all $M>0$ as $|y| \rightarrow \infty$.
By Liouville's theorem one quickly sees that each weak $\omega$ is essentially just a rational function $R \in \mathbb{C}(X)$ with (only simple) poles only in roots of unity, such that $R(0)=$ $R(\infty)=0$. Here we put $\omega(z):=R(e(z))$, where $e(z):=e^{2 \pi i z}$. There are no non-trivial weak functions with poles only in $\mathbb{Z}$. We shall later see that this corresponds to the fact that there are no non-trivial cusp forms of weight $2 \leqslant k \leqslant 14, k \neq 12$, for the full modular group.
We can identify weak functions with functions on the points of finite order $\mathbb{Q} / \mathbb{Z}$ on the 1 -torus, which are zero up to finitely many exceptions. This works via its residue function $\beta_{\omega}(x):=-2 \pi i \operatorname{res}_{z=x} \omega(z)$, where the $-2 \pi i$ is a normalizing factor. It is easy to see that necessarily $\sum_{x \in \mathbb{Q} / \mathbb{Z}} \beta_{\omega}(x)=0$. We will write $\omega_{\beta}$, if $\omega$ has residue function $\beta$ and $\beta_{\omega}$, if $\beta$ is the residue function of $\omega$. If $L$ is a set, we write $L_{0}^{\mathbb{C}_{0}}$ for the vector space of all functions $f: L \rightarrow \mathbb{C}$, such that $f(x)=0$ for all but finitely many $x \in L$ and $\sum_{x \in L} f(x)=0$. For the next theorem, we consider the congruence subgroup $\Gamma_{1}\left(N_{1}, N_{2}\right)$ defined in (4.2).

Theorem 1.1 (cf. 4.7). Let $k \geqslant 3$ and $N_{1}, N_{2}>1$ be integers. There is a homomorphism

$$
\begin{aligned}
&\left(\mathbb{Z}\left[\frac{1}{N_{1}}\right] / \mathbb{Z}\right)_{0}^{\mathbb{C}_{0}} \otimes\left(\mathbb{Z}\left[\frac{1}{N_{2}}\right] / \mathbb{Z}\right)_{0}^{\mathbb{C}_{0}} \longrightarrow M_{k}\left(\Gamma_{1}\left(N_{1}, N_{2}\right)\right) \\
& \beta \otimes \gamma \sum_{x \in \mathbb{Q}^{\times}} x^{k-1} \gamma(x) \omega_{\beta}(x \tau) .
\end{aligned}
$$

In the case that $k=1$ and $k=2$ the map stays well-defined under the restriction that the function $z \mapsto z^{k-1} \omega_{\gamma}(z) \omega_{\beta}(z \tau)$ is removable in $z=0$.

Of course, this gives an interpretation for arbitrary functions on the global space $(\mathbb{Q} / \mathbb{Z})_{0}^{\mathbb{C}_{0}}$. We will motivate 1.1 using an approach from complex analysis. This works as follows. For each pair $\omega \otimes \eta$ we can define a holomorphic function on $\mathbb{H}$ by

$$
\vartheta_{k}(\omega \otimes \eta ; \tau)=\sum_{x \in \mathbb{Q}^{\times}} x^{k-1} \beta_{\eta}(x) \omega(x \tau) .
$$

We define the involution $\omega \mapsto \hat{\omega}$ by $\hat{\omega}(z):=\omega(-z)$. Then we can show
Theorem 1.2 (cf. 3.5). Let $k \in \mathbb{Z}$ be an integer. For all weak $\omega$ and $\eta$ we have the following transformation property.

$$
\left.\vartheta_{k}(\omega \otimes \eta ; \tau)\right|_{k} S=\vartheta_{k}(\eta \otimes-\widehat{\omega} ; \tau)+\left.g_{\omega, \eta}(\tau)\right|_{k} S,
$$

where $S:=\left(\begin{array}{cc}0 & -1 \\ 1 & 0\end{array}\right)$ and $g_{\omega, \eta}$ is a rational function which can be evaluated explicitly by

$$
g_{\omega, \eta}(\tau)=2 \pi i \operatorname{res}_{z=0}\left(z^{k-1} \eta(z) \omega(z \tau)\right) .
$$

Of course all the functions $\vartheta_{k}(\omega \otimes \eta ; \tau)$ admit Fourier series and they essentially coincide with those given by (2.4).

As previously announced, we present an alternative proof of modularity. It is not based on coefficient comparison and does not use the theory of $L$-functions. It makes use of twists of the $\vartheta_{k}$ and Weil's converse theorem. For the transformation laws we use contour integration methods. This complex analytic philosophy is not new; for example, Siegel gave a short proof for the functional equation of the Dedekind eta function $\eta(\tau)=q^{\frac{1}{24}} \prod_{n=1}^{\infty}\left(1-q^{n}\right)$ using similar ideas, see [2] on p. 48 ff . They were also already used in a similar way by Berndt and Straub in [3] when deducing an interesting functional equation for the secant series

$$
\psi_{s}(\tau):=\sum_{n=1}^{\infty} \frac{\sec (n \tau)}{n^{s}}
$$

This new perspective to modular forms has several advantages. One of them is that it motivates a quite simple and natural generalization of Eisenstein series to objects with transformation properties and generalized Fourier series. Here we just go to the closure of $\mathbb{Q} / \mathbb{Z}$, which is $\mathbb{R} / \mathbb{Z}$, so we include all points of infinite order. This is done by the author in [9. Also we are able to simply interpret values of classical $L$-functions as residues of transformed rational functions, which makes the bridge to Bernoulli numbers and cotangent sums, already partially studied in [4], slightly more accessible. Indeed, the functional equation proposed in 1.2 is also true for negative integers $k$ and a wider class of meromorphic functions.
Another application is the observation, that some of the series $\vartheta_{k}$ converge exceptionally fast near the cusp $\tau=0$, where the modular form is zero. This leads to a dominated convergence theorem for Eisenstein series which is given in [8]. One can apply this result to the corresponding $L$-functions of products of Eisenstein series to get new series representations that converge in the critical strip. This could be of interest, since Dickson and Neururer have shown in [7, that, if $k \geqslant 4, N=p^{a} q^{b} N^{\prime}$ where $p^{a}, q^{b}$ are powers of primes and $N^{\prime}$ is square free, the space $M_{k}\left(\Gamma_{0}(N)\right)$ is generated by $E_{k}\left(\Gamma_{0}(N), \chi_{0,1}\right)\left(\chi_{0,1}\right.$ is the trivial character) and a subspace containing products of two Eisenstein series. A similar result for $M_{k}(p)$ and $k \geqslant 4$, where $p$ is prime, is due to Imamoglu and Kohnen [12]. For a correspondence between values of $L$-functions for products of pairs of different Eisenstein series see [5].

The paper is organized as follows. In the first section we give a short introduction to Eisenstein series. Secondly, we introduce weak functions in more detail and state some basic results we will need. In Section 3 we construct modular forms using rational functions and give examples.
Notation. As introduced we define for any set $L$ to be $L^{\mathbb{C}_{0}}$ the space of all functions $f: L \rightarrow \mathbb{C}$, that are zero everywhere except finitely many $x \in L$. The subspace $L_{0}^{\mathbb{C}_{0}} \subset L^{\mathbb{C}_{0}}$ is given by all $f$ satisfying $\sum_{x \in L} f(x)=0$.

For positive integers $N$ we write $\mathbb{F}_{N}:=\mathbb{Z} / N \mathbb{Z}$ and $\mathbb{F}_{\frac{1}{N}}:=\mathbb{Z}\left[\frac{1}{N}\right] / \mathbb{Z}$. We will identify functions $f \in \mathbb{F}_{N}^{\mathbb{C}_{0}}$ with $N$-periodic functions $f: \mathbb{Z} \rightarrow \mathbb{C}$. For integers $M$ we will set $f[M](x):=f(M x)$ when $f: \mathbb{Z} \rightarrow \mathbb{C}$. For weak functions $[M]$ will have a slightly different interpretation. We will write $S(f) \subset U$ for the set of poles of a meromorphic function $f: U \rightarrow \overline{\mathbb{C}}$.

For any Dirichlet character $\psi$ modulo $N$ we define the Gauss sum $\mathcal{G}(\psi):=\sum_{n=0}^{N-1} \psi(n) e^{2 \pi i n / N}$. For the generalized Gauss sum it will be more convenient to use the more general notion of a discrete Fourier transform

$$
\begin{aligned}
\mathcal{F}_{N} & : \mathbb{F}_{N}^{\mathbb{C}_{0}} \sim \mathbb{F}_{N}^{\mathbb{C}_{0}} . \\
\left(\mathcal{F}_{N} f\right)(j) & :=\sum_{n=0}^{N-1} f(n) e^{-2 \pi i j n / N} .
\end{aligned}
$$

Note that we have an inverse transformation

$$
\left(\mathcal{F}_{N}^{-1} g\right)(j):=\frac{1}{N} \sum_{n=0}^{N-1} g(n) e^{2 \pi i j n / N}
$$

We use the same notation for functions $f \in \mathbb{F}_{\frac{1}{N}}^{\mathbb{C}_{0}}$ and have $\kappa_{N} \mathcal{F}_{N} f=\mathcal{F}_{N} \kappa_{N} f$. For $d \mid N$ we also use the trivial injection

$$
\begin{aligned}
& \iota_{N}^{d}:\left(\mathbb{F}_{d}\right)^{\mathbb{C}_{0}} \longrightarrow\left(\mathbb{F}_{N}\right)^{\mathbb{C}_{0}} \\
&\left(\iota_{N}^{d} f\right)(x):=\left\{\begin{array}{ll}
f\left(\frac{x d}{N}\right), & x \equiv 0 \\
0, & \text { else }
\end{array}\left(\bmod \frac{N}{d}\right)\right.
\end{aligned}
$$

for purposes of notation.
For the complex variable $z=x+i y$ we write $e(z):=e^{2 \pi i z}$ and for the complex variable $\tau$ we define $q:=e^{2 \pi i \tau}$.

We write $\chi_{0, d}$ for the principal character modulo $d$. In particular, $\chi_{0,1}$ denotes the trivial character.

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## 2. Eisenstein series for $\Gamma_{0}(M, N)$

We briefly sketch the theory of Eisenstein series associated to a pair of Dirichlet characters. For Dirichlet characters $\chi$ and $\psi$ modulo positive integers $M$ and $N$, respectively, and some integer $k \geqslant 3$ one defines the corresponding Eisenstein series for $\tau \in \mathbb{H}(=$ upper half plane) via

$$
\begin{equation*}
E_{k}(\chi, \psi ; \tau):=\sum_{(m, n) \in \mathbb{Z}^{2} \backslash\{(0,0)\}} \chi(m) \psi(n)(m \tau+n)^{-k} . \tag{2.1}
\end{equation*}
$$

This series converges absolutely and uniformly on compact subsets of the upper half plane and defines a holomorphic function in that region. One can show that (2.1) leads to a non-zero function if and only if $\chi(-1) \psi(-1)=(-1)^{k}$ and that the $E_{k}$ are modular forms of weight $k$ for the congruence subgroups

$$
\Gamma_{0}(M, N):=\left\{\left.\left(\begin{array}{ll}
a & b  \tag{2.2}\\
c & d
\end{array}\right) \in \mathrm{SL}_{2}(\mathbb{Z}) \right\rvert\, b \equiv 0 \quad(\bmod M), c \equiv 0 \quad(\bmod N)\right\}
$$

with Nebentypus character $\chi \bar{\psi}$ of $\Gamma_{0}(M, N)$. However, in the case $k=1,2$ the above series will no longer have good convergence properties, but there might be also non-trivial modular forms of weight $k=1$ and $k=2$. We can remedy this using the non-holomorphic generalization

$$
\begin{equation*}
E_{k}(\chi, \psi ; \tau, s):=\sum_{(m, n) \in \mathbb{Z}^{2} \backslash\{(0,0)\}} \chi(m) \psi(n)(m \tau+n)^{-k}|m \tau+n|^{-2 s} \tag{2.3}
\end{equation*}
$$

and analytic continuation in $s$. As a result, the functions $E_{k}$ keep their modularity properties when considering the weights $k=1,2$. In this situation it is reasonable to define $E_{k}$ over their Fourier expansion. For a very detailed presentation of this Hecke trick see 13] on p. 274 ff .
Every Eisenstein series admits a Fourier series. The coefficients are well-known and given by

$$
\begin{equation*}
2 L(\psi, k) \chi(0)+\frac{2(-2 \pi i)^{k}}{N^{k}(k-1)!} \sum_{m=1}^{\infty}\left(\sum_{d \mid m} d^{k-1}\left(\mathcal{F}_{N} \psi\right)(-d) \chi\left(\frac{m}{d}\right)\right) q^{m / N} \tag{2.4}
\end{equation*}
$$

where $L(\psi, s)$ is the corresponding Dirichlet $L$-function. Note that in the case that $\psi$ is primitive one has $\left(\mathcal{F}_{N} \psi\right)(a)=\overline{\psi(a)}\left(\mathcal{F}_{N} \psi\right)(1)$ and one obtains the simpler expression $\sum_{d \mid n} d^{k-1} \bar{\psi}(d) \chi(n / d)$ for the coefficients up to a constant.

## 3. Weak functions

Let $\omega$ be a 1-periodic meromorphic function on $\mathbb{C}$ such that all poles of $\omega$ lie in $\mathbb{Q}$. We also want $\omega$ to be of rapid decay as the imaginary part of its arguments runs to $\pm \infty$. If we further assume that all poles are simple, it is an easy consequence from Liouville's theorem that such an $\omega$ is given by

$$
\omega(z)=\sum_{x \in \mathbb{Q} / \mathbb{Z}} \beta_{\omega}(x) h_{x}(z)
$$

where $h_{x}(z):=e(z) /(e(x)-e(z))$ with some $\beta_{\omega} \in(\mathbb{Q} / \mathbb{Z})_{0}^{\mathbb{C}_{0}}$. We call such $\omega$ weak functions. The level of $\omega$ is defined as the smallest positive integer $N$ such that $\omega(z / N)$ only has poles at integers. It is obvious that the set of all weak functions with level $d$ such that $d \mid N$ form a finite dimensional vector space over the complex numbers, which we will denote by $W_{N}$. The global vector space of all weak functions will be denoted as $W_{\infty}$.

Especially when going over to Fourier series it will be useful to identify functions in $\mathbb{F}_{\frac{1}{N}}^{\mathbb{C}_{0}}$ with those in $\mathbb{F}_{N}^{\mathbb{C}_{0}}$ via the obvious map

$$
\begin{gathered}
\kappa_{N}: \mathbb{F}_{\frac{1}{N}}^{\mathbb{C}_{0}} \xrightarrow{\sim} \mathbb{F}_{N}^{\mathbb{C}_{0}} \\
\left(\kappa_{N} f\right)(x):=f\left(\frac{x}{N}\right) .
\end{gathered}
$$

Remark 3.1. We have $W_{1}=0$, since all weak functions with level 1 are multiples of $\cot (\pi z)$, which does not satisfy the growth condition. This elementary fact also has an interpretation using modular forms, see 4.8.

For a non-principal Dirichlet character $\chi$ modulo $N$ we write

$$
\omega_{\chi}(z):=\sum_{j \in \mathbb{F}_{N}} \chi(j) h_{j / N}(z) .
$$

Let $\mathfrak{C}_{L}$ be the group of all Dirichlet characters modulo $L$. We define the principal part of $W_{N}$ by

$$
\mathfrak{P}_{N}:=\left\{\omega \in W_{N} \mid \omega=\sum_{d \mid N} c_{d} \sum_{j \in \mathbb{F}_{d}} \chi_{0, d}(j) h_{j / d}\right\} .
$$

Proposition 3.2. We have a decomposition

$$
W_{N}=\mathfrak{P}_{N} \oplus \bigoplus_{d \mid N} \bigoplus_{\chi \in \mathfrak{C}_{d} \backslash \chi_{0, d}} \mathbb{C} \omega_{\chi} .
$$

Proof. It is clear that $W_{N}$ is isomorphic to $\left\{v \in \mathbb{C}^{N} \mid \sum_{j=1}^{N} v_{j}=0\right\} \cong \mathbb{C}^{N-1}$. We can now formally write

$$
\mathbb{C}^{N-1}=\mathbb{C}^{\sigma_{0}(N)-1} \oplus \bigoplus_{d \mid N} \mathbb{C}^{\varphi(d)-1}
$$

Recall that characters are linearly independent. Each summand $\mathbb{C}^{\varphi(d)-1}$ corresponds to a subspace of $W_{N}$ given by the span of the $\omega_{\chi}$, where the $\chi$ are the non-principal characters modulo $d$. Therefore the quotient $\mathbb{C}^{\sigma_{0}(N)-1}$ is generated by the principal characters and since we have the vanishing condition of $W_{N}$ this is given by $\mathfrak{P}_{N}$, as required.

We use the same definition in the context of residue functions, i.e. we basically split them into non-principal characters and principal part elements. Let $\phi$ be a non-principal character or an element of the principal part $\phi=\sum_{d \mid N_{\phi}} c_{d} d_{N_{\phi}}^{d} \chi_{0, d}$, modulo $N_{\phi}$. We then define the corresponding character by

$$
\phi^{*}(n)= \begin{cases}\phi(n), & \text { if } \phi \text { is non-principal, } \\ \chi_{0, N_{\phi}}(n), & \text { if } \phi \text { is in the principal part. }\end{cases}
$$

Note that we have $\phi(M n)=\phi^{*}(M) \phi(n)$ for any $M$ coprime to $N_{\phi}$.

Definition 3.3. Let $N$ be a positive integer. For any positive integer $M$ which is coprime to $N$, we define the Atkin-Lehner operator $[M]: W_{N} \rightarrow W_{N}$ by

$$
(\omega)[M]=\sum_{j \in \mathbb{F}_{N}}\left(\kappa_{N} \beta_{\omega}\right)(M j) h_{j / N}
$$

Remark 3.4. Each $\omega_{\phi}$ is an eigenvector of $[M]$ with eigenvalue $\phi^{*}(M)$.
On $W_{\infty}$ we define an involution $\hat{.}: W_{\infty} \rightarrow W_{\infty}$ given by $\hat{\omega}(z):=\omega(-z)$. One easily checks that this map is well-defined and level preserving. In particular, it restricts to maps $\hat{.}: W_{N} \rightarrow W_{N}$. We define $W_{T}^{ \pm} \subset W_{T}$ for $T \in \mathbb{N} \cup\{\infty\}$ as the spaces of even and odd weak functions, respectively. This induces a canonical decomposition map $\iota_{T}: W_{T} \rightarrow W_{T}^{+} \oplus W_{T}^{-}$ given by $\omega \mapsto(\omega+\hat{\omega}) / 2+(\omega-\hat{\omega}) / 2$. Hence we obtain multiplicative decompositions

$$
\iota_{T_{1}} \otimes \iota_{T_{2}}: W_{T_{1}} \otimes W_{T_{2}} \longrightarrow\left(W_{T_{1}}^{+} \oplus W_{T_{2}}^{-}\right) \otimes\left(W_{T_{1}}^{+} \oplus W_{T_{2}}^{-}\right)
$$

and we define

$$
\left(W_{T_{1}} \otimes W_{T_{2}}\right)^{+}:=W_{T_{1}}^{+} \otimes W_{T_{2}}^{+} \oplus W_{T_{1}}^{-} \otimes W_{T_{2}}^{-}
$$

and

$$
\left(W_{T_{1}} \otimes W_{T_{2}}\right)^{-}:=W_{T_{1}}^{+} \otimes W_{T_{2}}^{-} \oplus W_{T_{1}}^{-} \otimes W_{T_{2}}^{+}
$$

Fix an integer $k$. Every pair $\omega \otimes \eta$ in $W_{M} \otimes W_{N}$ defines a holomorphic function on the union of the upper and lower half plane $\mathbb{H}:=\mathbb{H}^{+} \cup \mathbb{H}^{-}$by

$$
\begin{gathered}
\vartheta_{k}: W_{M} \otimes W_{N} \longrightarrow \mathcal{O}(\underline{\mathbb{H}}) \\
\vartheta_{k}(\omega \otimes \eta ; \tau):=-2 \pi i \sum_{x \in \mathbb{Q}^{\times}} \operatorname{res}_{z=x}\left(z^{k-1} \eta(z) \omega(z \tau)\right) .
\end{gathered}
$$

It can be checked that the series converges absolutely and uniformly on compact subsets of $\mathbb{H}$. So $\vartheta_{k}(\omega \otimes \eta ; \tau)$ is indeed holomorphic in this region. By simple symmetry arguments one sees $\left(W_{M} \otimes W_{N}\right)^{\mp} \subseteq \operatorname{ker}\left(\vartheta_{k}\right)$ if $(-1)^{k}= \pm 1$.
The next theorem is one of the central statements of the paper. It states that there is in some sense a modular duality induced by the isomorphism

$$
\begin{gathered}
W_{M} \otimes W_{N} \xrightarrow{\sim} W_{N} \otimes W_{M}, \\
\omega \otimes \eta \longmapsto \eta \otimes-\widehat{\omega} .
\end{gathered}
$$

Theorem 3.5 (Main transformation law). Let $\omega \otimes \eta$ be a pair in $W_{M} \otimes W_{N}$. Then we have

$$
\vartheta_{k}(\omega \otimes \eta ;-1 / \tau)=\tau^{k} \vartheta_{k}(\eta \otimes-\widehat{\omega} ; \tau)+2 \pi i \operatorname{res}_{z=0}\left(z^{k-1} \eta(z) \widehat{\omega}\left(\frac{z}{\tau}\right)\right)
$$

Remark 3.6. Note that the second summand on the right is a rational function of $\tau$, which is holomorphic in $\mathbb{C}^{\times}$.

Proof. Let $\tau \in \mathbb{H}$ be arbitrary and fixed. Define

$$
g_{\tau}(z):=-2 \pi i z^{k-1} \eta(z) \widehat{\omega}\left(\frac{z}{\tau}\right)
$$

Then $g_{\tau}$ is a meromorphic function in the plane whose poles are simple and $S\left(g_{\tau}\right) \subset$ $\frac{1}{M} \mathbb{Z} \cup \frac{1}{N} \mathbb{Z} \tau$. For fixed arbitrary small $\varepsilon>0$ define a sequence $R_{n}$ of radii with $R_{n} \rightarrow \infty$ as $n \rightarrow \infty$ and

$$
d\left(\left\{|z|=R_{n}\right\}, S\left(g_{\tau}\right)\right)>\varepsilon
$$

for all $n \in \mathbb{N}$, where $d(U, V)$ defines the infimum of the euclidean distance of points $u \in U$ and $v \in V$. Now consider the closed contour integrals

$$
I_{n}(\tau)=\frac{1}{2 \pi i} \oint_{|z|=R_{n}} g_{\tau}(z) \mathrm{d} z
$$

taken as usual counter clockwise. A tedious calculation (which is omitted here) using the growth properties of weak functions shows $\lim _{n \rightarrow \infty} I_{n}(\tau)=0$ and hence by the Residue theorem

$$
\sum_{\alpha \in \frac{1}{M} \mathbb{Z} \backslash\{0\}} \operatorname{res}_{z=\alpha}\left(g_{\tau}(z)\right)+\operatorname{res}_{z=0}\left(g_{\tau}(z)\right)+\sum_{\alpha \in \frac{1}{N} \mathbb{Z}\{0\}} \operatorname{res}_{z=\alpha \tau}\left(g_{\tau}(z)\right)=0 .
$$

Since $\tau \in \mathbb{H}$ and the poles of $\omega$ are a subset of $\mathbb{Q}$, the poles of $z \mapsto \eta(z) \widehat{\omega}\left(\frac{z}{\tau}\right)$ in $\mathbb{Q}^{\times}$are a subset of the poles of $\eta$, and hence the first sum clearly equals $\vartheta_{k}\left(\widehat{\omega} \otimes \eta ; \frac{1}{\tau}\right)=\vartheta_{k}\left(\omega \otimes \eta ;-\frac{1}{\tau}\right)$. Since

$$
\operatorname{res}_{z=\alpha \tau}\left(g_{\tau}(z)\right)=\tau^{k} \operatorname{res}_{z=\alpha}\left(g_{\tau}(\tau z)\right)
$$

we obtain for the second sum

$$
\sum_{\alpha \in \frac{1}{N} \mathbb{Z}\{\{0\}} \operatorname{res}_{z=\alpha \tau}\left(g_{\tau}(z)\right)=\tau^{k} \vartheta_{k}(\eta \otimes \widehat{\omega} ; \tau) .
$$

This proves the claim.

It is clear that every $\vartheta_{k}(\omega \otimes \eta ; \tau)$ admits a Fourier expansion. Since we only focus on the non-trivial cases we assume $\omega \otimes \eta \in\left(W_{M} \otimes W_{N}\right)^{ \pm}$if $(-1)^{k}= \pm 1$. It is given by

$$
\begin{equation*}
\vartheta_{k}(\omega \otimes \eta ; \tau)=2 N^{1-k} \sum_{m=1}^{\infty} \sum_{d \mid m}\left(d^{k-1}\left(\kappa_{N} \beta_{\eta}\right)(d)\left(\mathcal{F}_{M} \kappa_{M} \beta_{\omega}\right)\left(\frac{m}{d}\right)\right) q^{m / N} \tag{3.1}
\end{equation*}
$$

According to (2.4) we conclude for non-principal characters

$$
\begin{equation*}
E_{k}(\chi, \psi ; \tau)=\frac{\psi(-1)(-2 \pi i)^{k}}{N(k-1)!} \vartheta_{k}\left(\omega_{\mathcal{F}_{M}^{-1}(\chi)} \otimes \omega_{\mathcal{F}_{N}(\psi)} ; \tau\right) \tag{3.2}
\end{equation*}
$$

In particular, if $\chi$ and $\psi$ are primitive and hence conjugate up to a constant under the Forurier transform, this simplifies to

$$
\begin{equation*}
E_{k}(\chi, \psi ; \tau)=\frac{\chi(-1)(-2 \pi i)^{k} \mathcal{G}(\psi)}{N(k-1)!\mathcal{G}(\bar{\chi})} \vartheta_{k}\left(\omega_{\bar{\chi}} \otimes \omega_{\bar{\psi}} ; \tau\right) . \tag{3.3}
\end{equation*}
$$

A detailed description of the space $\vartheta_{k}\left(W_{M} \otimes W_{N}\right)$ will be given in 8 .

## 4. Construction of modular forms

In this section we present an alternative proof that the $\vartheta_{k}(\omega \otimes \eta ; \tau)$ define modular forms. We use the properties of weak functions and contour integration methods. The proof underlines the naturalness of the construction and gives modularity for all values $k \in \mathbb{N}$ simultaneously, without using the Hecke trick. Our main tool is the transformation law 3.5 and Weil's converse theorem, which is presented in [13] on p. 128. With Lemma 4.3.9 on p. 123 we avoid the need of constructing $L$-functions.

Let $N_{1}, N_{2}$ be positive integers and $\chi$ be a Dirichlet character modulo $N=N_{1} N_{2}$. Then we have an isomorphism

$$
\begin{gather*}
M_{k}\left(\Gamma_{0}\left(N_{1}, N_{2}\right), \chi\right) \xrightarrow{\sim} M_{k}\left(\Gamma_{0}(N), \chi\right) \\
f(\tau) \longmapsto f\left(N_{2} \tau\right) \tag{4.1}
\end{gather*}
$$

where $\Gamma_{0}\left(N_{1}, N_{2}\right)$ was defined in $(2.2)$ and $\Gamma_{0}(N)$ is as usual defined by

$$
\Gamma_{0}(N):=\left\{\left.\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right) \in \mathrm{SL}_{2}(\mathbb{Z}) \right\rvert\, c \equiv 0 \quad(\bmod N)\right\}
$$

In the same way we obtain an isomorphism $M_{k}\left(\Gamma_{1}\left(N_{1}, N_{2}\right)\right) \xrightarrow{\sim} M_{k}\left(\Gamma_{1}(N)\right)$, where $\Gamma_{1}\left(N_{1}, N_{2}\right)$ is the congruence subgroup

$$
\Gamma_{1}\left(N_{1}, N_{2}\right):=\left\{\left.\left(\begin{array}{ll}
a & b  \tag{4.2}\\
c & d
\end{array}\right) \in \Gamma_{0}\left(N_{1}, N_{2}\right) \right\rvert\, a \equiv d \equiv 1 \quad\left(\bmod N_{1} N_{2}\right)\right\}
$$

and

$$
\Gamma_{1}(N):=\left\{\left.\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right) \in \mathrm{SL}_{2}(\mathbb{Z}) \right\rvert\, c \equiv 0, a \equiv d \equiv 1 \quad(\bmod N)\right\}
$$

Furthermore we have a useful decomposition

$$
M_{k}\left(\Gamma_{1}(N)\right)=\bigoplus_{\chi} M_{k}\left(\Gamma_{0}(N), \chi\right)
$$

where the sum runs over all Dirichlet characters modulo $N$. Together with 4.1 this gives the decomposition

$$
\begin{equation*}
M_{k}\left(\Gamma_{1}\left(N_{1}, N_{2}\right)\right)=\bigoplus_{\chi} M_{k}\left(\Gamma_{0}\left(N_{1}, N_{2}\right), \chi\right) \tag{4.3}
\end{equation*}
$$

where the sum runs over all Dirichlet characters modulo $N$.
Let $M \geqslant 1$ be an integer and $f(\tau)=\sum_{n=0}^{\infty} a(n) q^{n / M}$ be a holomorphic function on the upper half plane. Let $\psi$ a Dirichlet character modulo $r$. Then we put

$$
f_{\psi}(\tau)=\sum_{n=0}^{\infty} \psi(n) a(n) q^{n / M}
$$

We say in this case that $f$ is twisted by the character $\psi$. In the following and for all $\lambda \in \mathbb{R}$ we put $T(\lambda)=\left(\begin{array}{cc}1 & \lambda \\ 0 & 1\end{array}\right)$. The following result is well known.

Proposition 4.1. Let $f(\tau)=\sum_{n=0}^{\infty} a(n) q^{n / M}$ be holomorphic on the upper half plane such that $a(n)=O\left(n^{L}\right)$ for some $L>0$. Let $\psi$ be a primitive Dirichlet character with conductor $m_{\psi}$. Then for any integer $k>0$ we have

$$
f_{\psi}(\tau)=\left.\mathcal{G}(\bar{\psi})^{-1} \sum_{u=1}^{m_{\psi}} \bar{\psi}(u) f\right|_{k} T\left(\frac{u M}{m_{\psi}}\right) .
$$

Now let $N>1$ and $M>1$ be coprime integers. We observe that two maps $\beta_{N}: \mathbb{F}_{N} \rightarrow \mathbb{C}$ and $\beta_{M}: \mathbb{F}_{M} \rightarrow \mathbb{C}$ induce a new map $\beta_{N} \times \beta_{M}: \mathbb{F}_{N M} \rightarrow \mathbb{C}$ when putting

$$
\left(\beta_{N} \times \beta_{M}\right)(m)=\beta_{N}(v) \beta_{M}(u)
$$

where $m=v M-u N$. We use this type of notation because it is more natural for later applications. According to the Chinese remainder theorem this is well defined. Note that

$$
\left(\beta_{N} \times \beta_{M}\right)(m)=\beta_{N}\left[M^{-1}\right](m) \beta_{M}\left[N^{-1}\right](-m)
$$

where $M^{-1}$ is the multiplicative inverse of $M$ modulo $N$ and $N^{-1}$ is the multiplicative inverse of $N$ modulo $M$.

Definition 4.2. Let $N$ and $M$ be coprime. Then we define a bilinear map

$$
\times: W_{N} \times W_{M} \rightarrow W_{N M}
$$

by putting

$$
(\omega \times \eta)(z):=\sum_{j \in \mathbb{F}_{N M}}\left(\beta_{\omega} \times \beta_{\eta}\right)(j) h_{j / M N}(z)
$$

Note that this is well defined since

$$
\sum_{j \in \mathbb{F}_{N M}}\left(\beta_{\omega} \times \beta_{\eta}\right)(j)=\sum_{u \in \mathbb{F}_{N}} \sum_{v \in \mathbb{F}_{M}} \beta_{\omega}(u) \beta_{\eta}(v)=0 .
$$

Lemma 4.3. Let $\psi$ be a primitive Dirichlet character modulo $N$ and $d$ a proper divisor of $N$. Then for all integers $u$ we have

$$
\sum_{j=0}^{\frac{M}{d}-1} \psi(d j+u)=0
$$

Proof. This is well-known and will be omitted.
The next lemma is a technical statement for rearranging sums over $\mathbb{F}_{M} \times \mathbb{F}_{N}$ over $\mathbb{F}_{M N}$ using the above cross product.

Lemma 4.4. Let $N$ and $M$ be two coprime integers. Let $\beta: \mathbb{F}_{N} \rightarrow \mathbb{C}$ and $f: \mathbb{F}_{N M} \rightarrow \mathbb{C}$ be functions. Let also $\alpha$ be an integer and $\psi$ a primitive Dirichlet character modulo $M$. Then we have the identity

$$
\sum_{u \in \mathbb{F}_{M}} \sum_{j \in \mathbb{F}_{N}} \psi(u) \beta(j) f(j M-\alpha u N)=\bar{\psi}(\alpha) \sum_{\ell \in \mathbb{F}_{N M}}(\psi \times \beta)(\ell) f(\ell) .
$$

Proof. We omit the simple proof.

The next theorem considers the twists of the functions $\vartheta_{k}$.

Theorem 4.5. Let $N_{1}, N_{2}$ and $M$ be integers such that $\operatorname{gcd}\left(N_{1}, M\right)=\operatorname{gcd}\left(N_{2}, M\right)=1$ and $\psi$ a primitive Dirichlet character modulo $M$. Then for any $\omega \in W_{N_{1}}$ and $\eta \in W_{N_{2}}$ we have

$$
\left(\vartheta_{k}\right)_{\psi}(\omega \otimes \eta ; \tau)=\mathcal{G}(\bar{\psi})^{-1} M^{k-1} \psi\left(N_{2}\right) \vartheta_{k}\left(\left(\omega_{\bar{\psi}} \times \omega\right) \otimes\left(\omega_{\psi} \times \eta[M]\right) ; M \tau\right) .
$$

Proof. With 4.1 we obtain

$$
\begin{aligned}
& \left(\vartheta_{k}\right)_{\psi}(\omega \otimes \eta ; \tau) \\
& =2 N_{2}^{1-k} \mathcal{G}(\bar{\psi})^{-1} \sum_{u \in \mathbb{F}_{M}} \bar{\psi}(u) \sum_{\alpha \in \mathbb{Z}\{00\}} \alpha^{k-1} \beta_{\eta}(\alpha) \omega\left(\frac{\alpha \tau}{N_{2}}+\frac{\alpha u}{M}\right) \\
& =2 N_{2}^{1-k} \mathcal{G}(\bar{\psi})^{-1} \sum_{\alpha \in \mathbb{Z} \backslash\{0\}} \alpha^{k-1} \beta_{\eta}(\alpha) \sum_{u \in \mathbb{F}_{M}} \sum_{j \in \mathbb{F}_{N_{1}}} \bar{\psi}(u) \beta_{\omega}(j) \frac{e\left(\frac{\alpha \tau}{N_{2}}-\frac{j M-\alpha u N_{1}}{M N_{1}}\right)}{1-e\left(\frac{\alpha \tau}{N_{2}}-\frac{j M-\alpha u N_{1}}{M N_{1}}\right)}
\end{aligned}
$$

Note that $f(x)=e\left(\frac{\alpha \tau}{N_{2}}-\frac{x}{M N_{1}}\right)\left(1-e\left(\frac{\alpha \tau}{N_{2}}-\frac{x}{M N_{1}}\right)\right)^{-1}$ is a function of $\mathbb{F}_{N_{1} M}$. Now with 4.4 this is

$$
\begin{aligned}
& =2 N_{2}^{1-k} \mathcal{G}(\bar{\psi})^{-1} \sum_{\alpha \in \mathbb{Z}\{\{0\}} \alpha^{k-1} \beta_{\eta}(\alpha) \psi(\alpha) \sum_{\ell \in \mathbb{F}_{N_{1} M}}\left(\bar{\psi} \times \beta_{\omega}\right)(\ell) \frac{e\left(\frac{\alpha \tau}{N_{2}}-\frac{\ell}{M N_{1}}\right)}{1-e\left(\frac{\alpha \tau}{N_{2}}-\frac{\ell}{M N_{1}}\right)} \\
& =2 N_{2}^{1-k} \psi\left(N_{2}\right) \mathcal{G}(\bar{\psi})^{-1} \sum_{\alpha \in \mathbb{Z}\{0\}} \alpha^{k-1} \beta_{\eta}\left[M^{-1}\right](M \alpha) \psi\left[N_{2}^{-1}\right](\alpha) \\
& \times \sum_{\ell \in \mathbb{F}_{N_{1} M}}\left(\bar{\psi} \times \beta_{\omega}\right)(\ell) \frac{e\left(\frac{\alpha \tau}{N_{2}}-\frac{\ell}{M N_{1}}\right)}{1-e\left(\frac{\alpha \tau}{N_{2}}-\frac{\ell}{M N_{1}}\right)} \\
& =2 N_{2}^{1-k} \psi\left(N_{2}\right) \mathcal{G}(\bar{\psi})^{-1} \sum_{\alpha \in \mathbb{Z}\{\{0\}} \alpha^{k-1}\left(\psi \times \beta_{\eta}[M]\right)(\alpha) \sum_{\ell \in \mathbb{F}_{N_{1} M}}\left(\bar{\psi} \times \beta_{\omega}\right)(\ell) \frac{e\left(\frac{\alpha \tau}{N_{2}}-\frac{\ell}{M N_{1}}\right)}{1-e\left(\frac{\alpha \tau}{N_{2}}-\frac{\ell}{M N_{1}}\right)} \\
& =2 M^{k-1} \psi\left(N_{2}\right) \mathcal{G}(\bar{\psi})^{-1}\left(M N_{2}\right)^{1-k} \sum_{\alpha \in \mathbb{Z} \backslash\{0\}} \alpha^{k-1}\left(\psi \times \beta_{\eta}[M]\right)(\alpha)\left(\omega_{\bar{\psi}} \times \omega\right)\left(\frac{\alpha M \tau}{N_{2} M}\right) .
\end{aligned}
$$

Theorem 4.6. Let $\chi$ and $\phi$ be two non-principal Dirichlet characters or principal elements modulo $N_{\chi}>1$ and $N_{\phi}>1$, respectively, and $k \geqslant 1$ an integer. Then if $f(\tau)=\vartheta_{k}\left(\omega_{\chi} \otimes\right.$ $\left.\omega_{\phi}, N_{\phi} \tau\right)$ we have $f \in M_{k}\left(\Gamma_{0}\left(N_{\chi} N_{\phi}\right), \overline{\chi^{*}} \phi^{*}\right)$.

Proof. We check the conditions of Weil's converse theorem. Here we use the equivalent version, which gets along without $L$-functions and uses the transformation properties of the twists of the Fourier series. For this we frequently use 3.5. Put $f(\tau)=\vartheta_{k}\left(\omega_{\chi} \otimes\right.$ $\left.\omega_{\phi} ; N_{\phi} \tau\right)$. It is clear by (3.1) that if we put $f(z)=\sum_{n=0}^{\infty} a(n) q^{n}$ we obtain $a(n)=O\left(n^{L}\right)$ for some $L>0$. Now we set

$$
\begin{aligned}
g(\tau) & =\left(\sqrt{N_{\phi} N_{\chi}} \tau\right)^{-k} f\left(-\frac{1}{N_{\phi} N_{\chi} \tau}\right) \\
& =\left(\sqrt{N_{\phi} N_{\chi}} \tau\right)^{-k} \vartheta_{k}\left(\omega_{\chi} \otimes \omega_{\phi} ;-\frac{1}{N_{\chi} \tau}\right) \\
& =-\left(\left(\sqrt{N_{\phi} N_{\chi}} \tau\right)^{-k} N_{\chi}^{k} \tau^{k} \vartheta_{k}\left(\omega_{\phi} \otimes \widehat{\omega_{\chi}} ; N_{\chi} \tau\right)\right. \\
& =-\left(\frac{N_{\chi}}{N_{\phi}}\right)^{\frac{k}{2}} \chi^{*}(-1) \vartheta_{k}\left(\omega_{\phi} \otimes \omega_{\chi} ; N_{\chi} \tau\right) .
\end{aligned}
$$

From this it is clear that $g(\tau)=\sum_{n=0}^{\infty} b(n) q^{n}$ for some sequence $b(n)$ with $b(n)=O\left(n^{L}\right)$ for some $L>0$. Let $\psi$ be a primitive Dirichlet character with conductor $M_{\psi}$ such that $\left(N_{\chi}, M_{\psi}\right)=\left(N_{\phi}, M_{\psi}\right)=1$. We denote

$$
C_{\psi}=\overline{\chi^{*}}\left(M_{\psi}\right) \phi^{*}\left(M_{\psi}\right) \psi\left(-N_{\chi} N_{\phi}\right) \mathcal{G}(\psi) \mathcal{G}(\bar{\psi})^{-1} .
$$

The theorem follows if we can show that

$$
\left.f_{\psi}\right|_{k} w\left(N_{\phi} N_{\chi} M_{\psi}^{2}\right)=C_{\psi} g_{\psi} .
$$

The left hand side we find

$$
\begin{aligned}
& \left.f_{\psi}\right|_{k} w\left(N_{\phi} N_{\chi} M_{\psi}^{2}\right) \\
& =\left(\sqrt{N_{\phi} N_{\chi} M_{\psi}^{2}} \tau\right)^{-k}\left(\vartheta_{k}\right)_{\psi}\left(\omega_{\chi} \otimes \omega_{\phi} ;-\frac{1}{N_{\chi} M_{\psi}^{2} \tau}\right)
\end{aligned}
$$

Since $\psi$ is primitive we can apply 4.5 and obtain

$$
\begin{aligned}
& =\left(\sqrt{N_{\phi} N_{\chi} M_{\psi}^{2}} \tau\right)^{-k} \mathcal{G}(\bar{\psi})^{-1} M_{\psi}^{k-1} \psi\left(N_{\phi}\right) \vartheta_{k}\left(\left(\omega_{\bar{\psi}} \times \omega_{\chi}\right) \otimes\left(\omega_{\psi} \times \omega_{\phi}\left[M_{\psi}\right]\right) ;-\frac{1}{N_{\chi} M_{\psi} \tau}\right) \\
& =-\left(\frac{N_{\chi}}{N_{\phi}}\right)^{\frac{k}{2}} \mathcal{G}(\bar{\psi})^{-1} M_{\psi}^{k-1} \psi\left(N_{\phi}\right) \phi^{*}\left(M_{\psi}\right) \vartheta_{k}\left(\left(\omega_{\psi} \times \omega_{\phi}\right) \otimes\left(\widehat{\omega_{\bar{\psi}} \times \omega_{\chi}}\right) ; N_{\chi} M_{\psi} \tau\right) \\
& =-\psi(-1) \chi^{*}(-1)\left(\frac{N_{\chi}}{N_{\phi}}\right)^{\frac{k}{2}} \mathcal{G}(\bar{\psi})^{-1} M_{\psi}^{k-1} \psi\left(N_{\phi}\right) \phi^{*}\left(M_{\psi}\right) \\
& \times \vartheta_{k}\left(\left(\omega_{\psi} \times \omega_{\phi}\right) \otimes\left(\omega_{\bar{\psi}} \times \omega_{\chi}\right) ; N_{\chi} M_{\psi} \tau\right) .
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
g_{\bar{\psi}}(\tau) & =-\chi^{*}(-1)\left(\frac{N_{\chi}}{N_{\phi}}\right)^{\frac{k}{2}}\left(\vartheta_{k}\right)_{\bar{\psi}}\left(\omega_{\phi} \otimes \omega_{\chi} ; N_{\chi} \tau\right) \\
& =-\chi^{*}(-1) \mathcal{G}(\psi)^{-1}\left(\frac{N_{\chi}}{N_{\phi}}\right)^{\frac{k}{2}} M_{\psi}^{k-1} \bar{\psi}\left(N_{\chi}\right) \vartheta_{k}\left(\left(\omega_{\psi} \times \omega_{\chi}\right) \otimes\left(\omega_{\bar{\psi}} \times \omega_{\chi}\left[M_{\psi}\right]\right) ; N_{\chi} M_{\psi} \tau\right) \\
& =-\chi^{*}(-1) \mathcal{G}(\psi)^{-1}\left(\frac{N_{\chi}}{N_{\phi}}\right)^{\frac{k}{2}} M_{\psi}^{k-1} \bar{\psi}\left(N_{\chi}\right) \chi^{*}\left(M_{\psi}\right) \\
& \times \vartheta_{k}\left(\left(\omega_{\psi} \times \omega_{\chi}\right) \otimes\left(\omega_{\bar{\psi}} \times \omega_{\chi}\right) ; N_{\chi} M_{\psi} \tau\right) .
\end{aligned}
$$

Multiplying this by $C_{\psi}$ clearly gives us $\left.f_{\psi}\right|_{k} w\left(N_{\phi} N_{\chi} M_{\psi}^{2}\right)$. This proves the theorem.
Let $N_{1}$ and $N_{2}$ be positive integers with $\omega \otimes \eta \in W_{N_{1}} \otimes W_{N_{2}}$. This composes into elements $c_{i j} \omega_{i} \otimes \eta_{j}$, where both $\omega_{i}$ and $\eta_{j}$ are either the principal part or correspond to non-principal characters modulo $d_{1}$ and $d_{2}$ respectively, where $d_{i} \mid N_{i}$. Here $c_{i j}$ are proper constants. Hence $\vartheta_{k}(\omega \otimes \eta ; \tau)$ decomposes into $c_{i j} \vartheta_{k}\left(\omega_{i} \otimes \eta_{j} ; \tau\right)$, which belong to $M_{k}\left(\Gamma_{0}\left(d_{1}, d_{2}\right), \chi_{1,2}\right)$ according (4.1) and 4.6 with suitable characters $\chi_{1,2}$. But we have a canonical embedding $M_{k}\left(\Gamma_{0}\left(d_{1}, d_{2}\right), \chi_{1,2}\right) \rightarrow M_{k}\left(\Gamma_{0}\left(N_{1}, N_{2}\right), \chi_{1,2} \chi_{0, N_{1} N_{2}}\right)$. Together with (4.3) this proves the following theorem.

Theorem 4.7. Let $k \geqslant 3$ and $N_{1}, N_{2}>1$ be integers. There is a homomorphism

$$
\begin{gathered}
\left(\mathbb{F}_{\frac{1}{N_{1}}}\right)_{0}^{\mathbb{C}_{0}} \otimes\left(\mathbb{F}_{\frac{1}{N_{2}}}\right)_{0}^{\mathbb{C}_{0}} \longrightarrow M_{k}\left(\Gamma_{1}\left(N_{1}, N_{2}\right)\right) \\
\beta \otimes \gamma \longmapsto \sum_{x \in \mathbb{Q}^{\times}} x^{k-1} \gamma(x) \omega_{\beta}(x \tau) .
\end{gathered}
$$

In the case that $k=1$ and $k=2$ the map stays well-defined under the restriction that the function $z \mapsto z^{k-1} \omega_{\gamma}(z) \omega_{\beta}(z \tau)$ is removable in $z=0$.

Note that in this theorem we have used

$$
-2 \pi i \operatorname{res}_{z=x}\left(z^{k-1} \eta_{\gamma}(z) \omega(z \tau)\right)=x^{k-1} \gamma(x) \omega(x \tau)
$$

Remark 4.8. All these modular forms vanish in the cusps $\tau \in\{0, i \infty\}$. So if there were non-trivial weak functions with level 1, they would be odd (since there is a simple pole in $z=0$ ) and one could generate non-trivial cusp forms for any even weight $k \geqslant 2$ for $\mathrm{SL}_{2}(\mathbb{Z})$, which is impossible.

In the case $N_{1}=N_{2}=N$ and $k \in 2 \mathbb{N}$ we can even say a bit more. Let $\Gamma_{S}(N)$ be defined by the commodity of $\Gamma_{1}(N, N)$ and $S$. Then we can define an abelian character $\chi_{N}$ on $\Gamma_{S}(N)$ given by

$$
\chi_{N}(M)=\left\{\begin{array}{llr}
1, & M \equiv \pm E & \bmod N \\
-1, & M \equiv \pm S & \bmod N
\end{array}\right.
$$

Corollary 4.9. Let $\omega^{ \pm} \otimes \omega^{ \pm} \in W_{N}^{ \pm} \otimes W_{N}^{ \pm}$and $k \geqslant 2$ an even integer. Then we have $\vartheta_{k}\left(\omega^{+} \otimes \omega^{+}\right) \in M_{k}\left(\Gamma_{S}(N), \chi_{N}\right)$ and $\vartheta_{k}\left(\omega^{-} \otimes \omega^{-}\right) \in M_{k}\left(\Gamma_{S}(N)\right)$.

Proof. With 4.7 we obtain $\vartheta_{k}\left(\omega^{ \pm} \otimes \omega^{ \pm}\right) \in M_{k}\left(\Gamma_{1}(N, N)\right)$. Using 3.5 we additionally follow

$$
\left.\vartheta_{k}\left(\omega^{ \pm} \otimes \omega^{ \pm} ; \tau\right)\right|_{k} S=\mp \vartheta_{k}\left(\omega^{ \pm} \otimes \omega^{ \pm} ; \tau\right)
$$

Since $\Gamma_{S}(N)$ is generated by $\Gamma_{1}(N, N)$ and $S$, this proves the corollary.

We give an example of quick construction. The theta group $\Gamma_{\theta}$ is a congruence subgroup generated by the elements $T^{2}=\left(\begin{array}{ll}1 & 2 \\ 0 & 1\end{array}\right)$ and $S$.

Example 4.10. Let $v_{2}(n)$ be the exponent of 2 in the prime decomposition of $n$. For any even $k \geqslant 4$ we then have that

$$
f(\tau)=\sum_{n=1}^{\infty}(-1)^{n-1}\left(2^{v_{2}(n)}\right)^{k-1} \sigma_{k-1}\left(\frac{n}{2^{v_{2}(n)}}\right) q^{n / 2}
$$

is an entire modular form of weight $k$ for $\Gamma_{\theta}$.

Proof. The space $W_{2}^{-} \otimes W_{2}^{-}$has one dimension and is generated by $\omega_{2} \otimes \omega_{2}$, where

$$
\omega_{2}(z)=\frac{e(z)}{e\left(\frac{1}{2}\right)-e(z)}-\frac{e(z)}{1-e(z)}=-\frac{i}{\sin (2 \pi z)} .
$$

Hence due to 4.9 we obtain a modular form $f \in M_{k}\left(\Gamma_{\theta}\right)$ with

$$
f(\tau)=\sum_{n=1}^{\infty}(-1)^{n-1} n^{k-1} \frac{q^{n / 2}}{1-q^{n}}
$$

Rearranging the Lambert sum shows

$$
\begin{aligned}
f(\tau) & =\sum_{m=1}^{\infty} \sum_{\substack{n, r \\
n(2 r+1)=m}}(-1)^{m /(2 r+1)-1}\left(\frac{m}{2 r+1}\right)^{k-1} q^{m / 2} \\
& =\sum_{m=1}^{\infty} \sum_{\substack{u \mid m \\
u \text { odd }}}(-1)^{m / u-1}\left(\frac{m}{u}\right)^{k-1} q^{m / 2} .
\end{aligned}
$$

With

$$
\sum_{\substack{u \mid m \\ u \text { odd }}}(-1)^{m / u-1}\left(\frac{m}{u}\right)^{k-1}=(-1)^{m-1}\left(2^{v_{2}(m)}\right)^{k-1} \sigma_{k-1}\left(\frac{m}{2^{v_{2}(m)}}\right)
$$

the claim follows.

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