RATIONAL FUNCTIONS, COTANGENT SUMS AND EICHLER INTEGRALS

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Abstract. With the help of so called pre-weak functions, we formulate a very general transformation law for some holomorphic functions on the upper half plane and motivate the term of a generalized Eisenstein series with real-exponent Fourier expansions. Using the transformation law in the case of negative integers k, we verify a close connection between finite cotangent sums of a specific type and generalized L-functions at integer arguments. Finally, we expand this idea to Eichler integrals and period polynomials for some types of modular forms.

Introduction

We recall that an elliptic modular form f of weight $k \in \mathbb{Z}$ for a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ with multiplier system $v : \Gamma \to \mathbb{C}^\times$ is a holomorphic function on the extended upper half plane $\{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\} \cup \mathbb{Q} \cup \{\infty\}$, which satisfies the transformation law

$$f|_k M(\tau) = v(M) f(\tau).$$

Here $f|_k M$ denotes the usual Petersson slash operator

$$f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau) = (ad-bc)^{\frac{k}{2}} (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right).$$

One can show that there are no non-constant modular forms for $k \leqslant 0$ and that the spaces $M_k(\Gamma, v)$ are finite-dimensional. A useful tool for computing the exact value of the dimensions is the Riemann-Roch formula, for more explicit details see for example [6]. Modular forms play an extraordinary important role in many fields of mathematics and physics such as number theory, geometry and string theory. Also many generalizations of the classical modular forms have been found, such as Siegel modular forms (see also [1] and [11]) for matrix valued arguments that transform under congruence subgroups of the symplectic group Sp_n; and Hilbert modular forms (for a great introduction, the reader may wish to consult [10]) that transform under congruence subgroups of $\mathrm{SL}_2(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers of a number field K.

Basically, two elementary ideas for constructing modular forms dominate in literature. One of them uses so called Poincaré series, which give in the simplest case Eisenstein series. The other one goes via Fourier analysis and quadratic forms. This leads to theta functions. In [9] a third elementary approach to modular forms was presented. It is based on a class of very simple functions which we will call *weak functions*. A weak function ω is a 1-periodic meromorphic function in the entire plane, which has the following properties:

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(i) All poles of ω are simple and lie in $\mathbb{Q}$.

(ii) The function ω tends to 0 rapidly as the absolute value of the imaginary part increases, so
$$\omega(x+iy) = O(|y|^{-M})$$
for all $M > 0$ as $|y| \to \infty$.

By Liouville's theorem one quickly sees that each weak ω is essentially just a rational function $R \in \mathbb{C}(X)$ with (only simple) poles only in roots of unity, such that $R(0) = R(\infty) = 0$. Here we put $\omega(z) := R(e(z))$, where $e(z) := e^{2\pi i z}$. One defines W_N to be the space of weak functions with the property that $\omega(z/N)$ only has poles in $\mathbb{Z}$. We associate to ω a periodic residue function $\beta_\omega(x) := -2\pi i \mathrm{res}_{z=x}\omega(z)$. Now one can show the following construction theorem for modular forms for the congruence subgroup
$$\Gamma(N_1N_2) \subset \Gamma_1(N_1, N_2) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N_1, N_2) \middle| a \equiv d \equiv 1 \pmod{N_1N_2} \right\}.$$

Theorem 0.1. *Let $k \geqslant 3$ and $N_1, N_2 > 1$ be integers. There is a homomorphism*
$$W_{N_1} \otimes W_{N_2} \longrightarrow M_k(\Gamma_1(N_1, N_2))$$
$$\omega \otimes \eta \longmapsto \vartheta_k(\omega \otimes \eta; \tau) := \sum_{x \in \mathbb{Q}^\times} x^{k-1}\beta_\eta(x)\omega(x\tau).$$
In the case that $k = 1$ and $k = 2$ the map stays well-defined under the restriction that the function $z \mapsto z^{k-1}\eta(z)\omega(z\tau)$ is removable in $z = 0$.

To any modular form of weight $k \geqslant 2$, that vanishes in the cusps in $\tau = 0$ and $\tau = i\infty$, we can associate an Eichler integral. It has the form
$$\mathcal{E}(f;\tau) := c_k \int_\tau^{i\infty} f(z)(z-\tau)^{k-2}\mathrm{d}z,$$
where c_k is some normalizing constant. This integral represents a holomorphic and periodic function on the upper half plane and is tied to the so called period polynomial $p(f;\tau)$ of f by the functional equation
$$\mathcal{E}(f; -1/\tau) - \tau^{2-k}\mathcal{E}(f;\tau) = p(f;\tau).$$
Explicitly, we have a connection to the critical values of the L-function associated to f by
$$p(f;\tau) = \sum_{n=0}^{k-2} \binom{k-2}{n} i^{1-n}\Lambda(f; n+1)\tau^{k-2-n}.$$
These period polynomials are very important objects in number theory. For example, they appear in the context of a conjecture by Delinge-Beilinson-Scholl which makes an assertion about the nature of values of derivatives of L-functions of Hecke cuspforms f, see also [15]. Also, an immediate implication of the Eichler-Shimura isomorphism, see [14], applied to the period polynomial is Manin's Periods Theorem [17], which provides important information about the arithmetic nature of critical L-values. For a detailed investigation of the values of Eichler integrals at algebraic points, also in the context of

Ramanujan identities for L-values at integer arguments, see [12]. Finally, a fairly good introduction to the so called Riemann hypothesis for period polynomials attached to derivatives of L-functions is given in [5].

In this paper, we will continue the study of the relation between weak functions and modular forms. We look at a more general class of weak functions, the pre-weak functions, that are allowed to have poles at real numbers and only have to be bounded as the imaginary part of the argument tends to $\pm\infty$. We collect these functions in the vector space W_{pre}. What we get is the following transformation law.

Theorem 0.2 (cf. Theorem 1.4). *Let $\omega \otimes \eta \in W^{\otimes}_{(k)}$, then we have for all $k \in \mathbb{Z}$ and $\tau \in \mathbb{H}$*

$$\vartheta_k(\omega \otimes \eta; -1/\tau) = \tau^k \vartheta_k(\eta \otimes -\widehat{\omega}; \tau) + 2\pi i \mathrm{res}_{z=0}(z^{k-1}\eta(z)\widehat{\omega}\left(\frac{z}{\tau}\right)).$$

Here $\hat{\omega}(z) = \omega(-z)$ and $W^{\otimes}_{(k)}$ denotes a sufficiently good subspace of $W_{\mathrm{pre}} \otimes W_{\mathrm{pre}}$ and is explained below.

This transformation law is just a straight generalization of a theorem in [9], where modular forms are constructed with the help of rational functions. A first example of application of this generalized transformation law is the definition and investigation of generalized Eisenstein series. These functions can be written as a holomorphic limit of modular forms and hold a Fourier series $\sum_{t \in \mathbb{R}_{>0}} a(t)q^t$ with real exponents.
In the case the pre-weak functions $\omega \otimes \eta$ only live on points of finite order, the rational function at the end of the transformation refers to a period polynomial if $k < 0$. It implies that critical values of L-functions attached to Eisenstein series are just residues of elementary functions.
We motivate this elementary approach to period polynomials by some applications cotangent sums and generalized L-functions that are defined by

$$L(\omega; s) := \sum_{x \in \mathbb{R}_{>0}} \beta_\omega(x) x^{-s}.$$

Here, β_ω is the 1-periodic coefficient function of the pre-weak function ω. Basically, we analyze the case $\omega \equiv 1$, hence $1 \otimes \eta$, in more detail. As a result we give a detailed description of cotangent sums in terms of L-functions. A (general) cotangent sum is a finite sum of the form

$$C(\beta; m) := \sum_{0<x<1} \beta(x) \cot^m(\pi x),$$

where of course $\beta(x)$ is zero everywhere except finitely many points. In case of a character χ we use the different but more convenient denotation

$$C(\chi; m) := \sum_{j=1}^{N-1} \chi(j) \cot^m\left(\frac{\pi j}{N}\right).$$

A famous example for a cotangent sum is given in [13] on p. 262:

$$\sum_{j=1}^{N-1} \cot^2\left(\frac{\pi j}{N}\right) = \frac{(N-1)(N-2)}{3}, \qquad N = 2, 3, ... \tag{0.1}$$

Note that the sum is always rational independent of the choice of N. Generally, it turns out that the arithmetic nature of such cotangent sums is strongly tied with the arithmetic nature of corresponding L-functions. The key idea is to construct (for fixed ω) a lower diagonal isomorphism $A_m \in \mathbb{Q}^{m\times m}$ between the spaces $\{a_1 C(\beta_\omega; 1) + \cdots + a_m C(\beta_\omega; m) | a_j \in \mathbb{Q}\}$ and $\{b_1 \widetilde{L}^*(\beta_\omega; 1)/\pi + \cdots + b_m \widetilde{L}^*(\beta_\omega; m)/\pi^m | b_j \in \mathbb{Q}\}$, where the $\widetilde{L}^*$ are essentially L-functions. A consequence of this construction is the following theorem.

Theorem 0.3 (cf. Theorem 2.14). *Let $\omega \in W_{\text{pre}}$ be a pre-weak function that is removable in $z = 0$. Let $K|\mathbb{Q}$ be a field extension (not necessarily finite) and $m \in \mathbb{N}$ be any positive integer. Assume that $C(\beta_\omega; 0) \in K$. Then we have*

$$\frac{\widetilde{L}(\omega; 1)}{\pi}, \frac{\widetilde{L}(\omega; 2)}{\pi^2}, \cdots, \frac{\widetilde{L}(\omega; m)}{\pi^m} \in K \iff C(\beta_\omega; 1), C(\beta_\omega; 2), \cdots, C(\beta_\omega; m) \in K.$$

For example, with $\zeta(2k) \in \mathbb{Q}\pi^{2k}$ an easy consequence of Theorem 0.3 is

$$C_N(m) := \sum_{j=1}^{N-1} \cot^m \left(\frac{j\pi}{N} \right) \in \mathbb{Q}, \qquad \forall m, N \in \mathbb{N}.$$

This is well-known and was verified by Berndt and Yeap (see [2], p. 6). We can use Theorem 0.3 to show some more interesting relations for cotangent sums.

Theorem 0.4 (cf. Corollary 2.20). *Let p be a prime and χ be the Legendre symbol modulo p. Then we have for all $m \in \mathbb{N}$*

$$\sqrt{p}C(\chi; m) \in \mathbb{Q}.$$

An example for $m = 13$ and the Legendre symbol modulo 7 is

$$\cot^{13}\left(\frac{\pi}{7}\right) + \cot^{13}\left(\frac{2\pi}{7}\right) - \cot^{13}\left(\frac{3\pi}{7}\right) + \cot^{13}\left(\frac{4\pi}{7}\right) - \cot^{13}\left(\frac{5\pi}{7}\right) - \cot^{13}\left(\frac{6\pi}{7}\right) = \frac{494370}{49\sqrt{7}}.$$

Furthermore, with our method it is possible to derive explicit formulas for the cotangent sums $C(\chi; m)$ where χ is an *arbitrary* primitive character. These will be stated in Corollary 2.19. Similarly, we can give (rather complicated) formulas for Dirichlet series with trigonometric coefficients at integer arguments, see Corollary 2.22 and Remark 2.23. Finally, using Fourier analysis and the generalized Clausen functions one can derive closed formulas for cotangent sums presented by Berndt and Yeap [2] involving sine and cosine functions. Here we use explicit terms (described in Theorem 2.15) of the rational isomorphisms briefly described above.

In the last section, we prove a duality result that concerns pre-weak functions. We prove that the $k-1$-fold integrals of ϑ_k can be expressed as linear combinations of expressions ϑ_j (with some negative j) and apply this to Eisenstein series. This is realized by an injective linear map

$$\mathcal{E}_{k,a}^{N,M} : W_{\text{weak},a}[\mathcal{T}_N] \otimes W_{\text{pre},1}^{i\infty}[\mathcal{T}_M] \longrightarrow W_{\text{pre},1}^{i\infty}[\mathcal{T}_M] \otimes \bigoplus_{j=0}^{a-1} z^j W_{\text{pre},1}^{i\infty}[\mathcal{T}_N]$$

that is given in terms of several Fourier transforms and will be specified below. Note that the $\mathcal{T}_*$ stand for (pre-)weak functions of level $*$. In particular, we prove the following theorem.

Theorem 0.5 (cf. Theorem 4.10). *The diagram*

$$\begin{tikzcd} W_{\mathrm{weak},a}[\mathcal{T}_N] \otimes W^{i\infty}_{\mathrm{pre},1}[\mathcal{T}_M] \arrow[r, "\mathcal{E}^{N,M}_{k,a}"'] \arrow[d, "\vartheta_k"'] & W^{i\infty}_{\mathrm{pre},1}[\mathcal{T}_M] \otimes \bigoplus_{j=0}^{a-1} z^j W^{i\infty}_{\mathrm{pre},1}[\mathcal{T}_N] \arrow[d, "(\tau \mapsto \frac{N}{M}\tau) \circ \vartheta_{2-k}"] \\ \mathbb{C}_0^+[[q^{1/M}]] \arrow[r, "\int_{k-1}"'] & \mathbb{C}_0^+[[q^{1/M}]] \end{tikzcd}$$

is commutative. Here, $\mathbb{C}_0^+[[q^{1/M}]]$ is the space of Fourier series of the form $\sum_{n=1}^{\infty} a(n)q^{n/M}$ that converge to holomorphic functions on the upper half plane.

As a consequence, for example, we can find the following interpretation of critical values of L-functions associated to Eisenstein series.

Theorem 0.6 (cf. Theorem 4.13). *Let $k \geqslant 2$ be an integer, χ and ψ be two primitive Dirichlet characters with $\chi(-1)\psi(-1) = (-1)^k$ and $f(\tau) = E_k(\chi, \psi; \tau)$. We then have the following identity between rational functions:*

$$\sum_{\ell=0}^{k-2} \binom{k-2}{\ell} i^{1-\ell}\Lambda(f; \ell+1)\tau^{-\ell} = -\frac{4\pi^2}{N_\psi^{k-1} N_\chi (k-1)} \mathrm{res}_{z=0}\left(z^{1-k}\omega_\psi(z)\omega_\chi\left(\frac{N_\psi z}{N_\chi \tau}\right)\right).$$

The paper is organized as follows. First we define the term pre-weak function and consider generalized Eisenstein series, that have q-series expansions $\sum_{t\geqslant 0} a(t)q^t$ with real exponents. We prove the functional equations of the associated generalized L-functions.
In the second section we apply the theory of pre-weak functions to cotangent sums and generalize some results by Berndt, Yeap and Zaharescu. In the third section we expand our main ideas to pre-weak functions of higher degree, which means that we allow poles of higher order. Finally, in the last section, we generalize the ideas of the second section to period polynomials and prove a duality result.

Notation. Throughout the paper k, $N \geqslant 1$ and $M \geqslant 1$ will denote integers. We sometimes use the notation $\mathrm{sgn}(f) = \pm 1$ to indicate that f is an even or odd function, respectively.

We define for any set L to be $L^{\mathbb{C}_0}$ the space of all functions $f : L \to \mathbb{C}$, that are zero everywhere except *finitely* many $x \in L$. The subspace $L_0^{\mathbb{C}_0} \subset L^{\mathbb{C}_0}$ is given by all f satisfying $\sum_{x\in L} f(x) = 0$. In the case $0 \in L$ we write $L^{\mathbb{C}_{0,0}}$ for the subspace of functions with $f(0) = 0$.

For positive integers N we abbreviate $\mathbb{F}_N := \mathbb{Z}/N\mathbb{Z}$. For the complex variable $z = x + iy$ we write $e(z) := e^{2\pi i z}$ and for the complex variable τ we define $q := e^{2\pi i\tau}$. We write

$\partial_\tau := \frac{1}{2\pi i}\frac{\partial}{\partial \tau}$ and $\partial_z := \frac{1}{2\pi i}\frac{\partial}{\partial z}$. If the variable is clear we only write ∂.

We will write $S(f) \subset U$ for the set of poles of a meromorphic function $f : U \to \overline{\mathbb{C}}$. Throughout, we write $W_{\mathrm{weak},a}$ and $W_{\mathrm{pre},a}$ for the space of weak and pre-weak functions of degree at most a (which means that $\omega \in W_{\mathrm{weak},a}$ has poles of order maximal a), respectively. The notation $W[\mathcal{T}]$ means that the contained (pre)-weak functions shall only have poles in $\mathcal{T} \subset \mathbb{R}/\mathbb{Z}$ modulo $\mathbb{Z}$. We write $\mathcal{T}_N := \{0, \frac{1}{N}, ..., \frac{N-1}{N}\}$.
We denote W_*^0 as the subspace of (pre-)weak functions that are removable in $z = 0$.

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1. Pre-weak functions, Eisenstein series and generalized periodic L-functions

We denote the vector space of all generalized weak functions of degree 1 (this means, that only poles os degree 1 are allowed) by W_{weak}. I.e., each function $\omega \in W_{\mathrm{weak}}$ has period 1, is meromorphic in $\mathbb{C}$ and of rapid decay as $|\mathrm{Im}(z)| \to \infty$ and only has poles of degree at most 1 at real values.

We now call a 1-periodic *pre-weak*, if it has all properties of a weak function except that it is just bounded as $y \to \pm\infty$ in the strip $\{0 \leqslant x < 1\}$. In other words, we have the exact sequence

$$0 \longrightarrow W_{\mathrm{weak}} \longrightarrow W_{\mathrm{pre}} \overset{f\mapsto(f(-i\infty),f(i\infty))}{\longrightarrow} \mathbb{C}^2 \longrightarrow 0.$$

The subspaces $W_{\mathrm{pre}}^{\pm i\infty} \subset W_{\mathrm{pre}}$ contain all pre-weak functions that additionally vanish in $z = \pm i\infty$. All introduced notations for weak functions will also apply to pre-weak functions, if appropriate. Note that each $\omega \in W_{\mathrm{pre}}$ also has a representation

$$\omega(z) = \omega(i\infty) + \sum_{x\in\mathbb{R}/\mathbb{Z}} \beta_\omega(x)h_x(z), \qquad h_x(z) := \frac{e(z)}{e(x) - e(z)},$$

where the sum is of course finite. Now consider the homomorphism

$$(\mathbb{R}/\mathbb{Z})^{\mathbb{C}_{0,0}} \longrightarrow \mathcal{O}(\{s \in \mathbb{C} \mid \sigma > 1\}) \tag{1.1}$$
$$\beta \longmapsto L(\beta; s) := \sum_{x\in\mathbb{R}_{>0}} \beta(x)x^{-s}.$$

The holomorphic functions on the right will be called *periodic L-functions* (since the input function lives on the 1-torus). We have the decomposition

$$L(\beta; s) = \sum_{x\in(0,1]} \beta(x)\zeta(s, x), \tag{1.2}$$

where

$$\zeta(s, x) := \sum_{n=0}^{\infty}(n + x)^{-s}, \qquad x > 0,$$

is the Hurwitz zeta function. By analytic continuation we may consider the subspace $\frac{1}{s-1}\mathcal{O}(\mathbb{C}) \subset \mathcal{O}(\{s \in \mathbb{C} \mid \sigma > 1\})$ for the image in (1.1). The residue map $\beta \longmapsto$

$\mathrm{res}_{s=1}L(\beta;s)$ has kernel $(\mathbb{R}/\mathbb{Z})_0^{\mathbb{C}_{0,0}}$. In the case that β has support on $\frac{1}{N}\mathbb{Z}\backslash\mathbb{Z}$ for some N, we obtain an ordinary Dirichlet series with an exponential factor.

$$L(\beta;s) = N^s \sum_{n=1}^{\infty} \beta\left(\frac{n}{N}\right) n^{-s}.$$

The aim of this section is to associate periodic L-functions with generalized Eisenstein series that satisfy certain transformation properties. These Eisenstein series $E_k(\omega\otimes\eta;\tau)$ arise from (generalized) weak functions $\omega\otimes\eta$ with real (but not necessarily rational) poles. Since we are not able to assign ω and η a meaningful finite integer level in the case they have irrational poles, the functions $E_k(\omega\otimes\eta;\tau)$ will not be modular forms (except of course they identically vanish).

We will use the notation $W^{\pm}$ to indicate the sub-spaces spanned by odd and even functions. What we need is the following: for $k \in \mathbb{Z}$ we define

$$W_{(k)}^{\otimes} := \begin{cases} W_{\text{weak}} \otimes W_{\text{weak}}, & \text{if } k > 0, \\ \left\langle W_{\text{pre}} \otimes W_{\text{weak}}, W_{\text{weak}} \otimes W_{\text{pre}}, W_{\text{pre}}^{+} \otimes W_{\text{pre}}^{-}, W_{\text{pre}}^{-} \otimes W_{\text{pre}}^{+} \right\rangle, & \text{if } k = 0, \\ W_{\text{pre}} \otimes W_{\text{pre}}, & \text{if } k < 0. \end{cases}$$

Also we use the notation $W_{(k)}^{\otimes}[\mathcal{T}_1,\mathcal{T}_2]$ to indicate, that the first and the second space are associated to the subsets $\mathcal{T}_1,\mathcal{T}_2 \subset \mathbb{R}/\mathbb{Z}$, e.g. $W_{(1)}^{\otimes}[\mathcal{T}_N,\mathcal{T}_M] = W_{\text{weak}}[\mathcal{T}_N] \otimes W_{\text{weak}}[\mathcal{T}_M]$. Consider the following linear map between pairs of pre-weak functions and holomorphic functions

$$\vartheta_k : V_k \longrightarrow \mathcal{O}(\mathbb{H}^+ \cup \mathbb{H}^-),$$

$$\omega\otimes\eta \longmapsto -2\pi i \lim_{R\to\infty} \sum_{\substack{x\in\mathbb{R}^\times \\ |x|\leqslant R}} \mathrm{res}_{z=x}\left(z^{k-1}\eta(z)\omega(z\tau)\right) =: \vartheta_k(\omega\otimes\eta;\tau). \tag{1.3}$$

We explain V_k by $V_k := W_{\text{weak}} \otimes W_{\text{pre}}$ if $k > 0$ and $V_k := W_{(k)}^{\otimes}$, else. A proof that this is well-defined is given in Proposition 1.3.

Remark 1.1. *If one considers the decomposition $W = W^+ \oplus W^-$ into even and odd functions, respectively, one can easily show by symmetry that $(W^+\otimes W^+)\oplus(W^-\otimes W^-) \subset \ker(\vartheta_k)$ if $k \equiv 1 \pmod 2$, and $(W^+\otimes W^-)\oplus(W^-\otimes W^+) \subset \ker(\vartheta_k)$, else. We use for elements $\omega \in W^{\pm}\backslash\{0\}$ the notation $\mathrm{sgn}(\omega) = \pm 1$.*

Note that $W_{(0)}^{\otimes}$ is also spanned by the spaces $W_{\text{pre}}^{+} \otimes W_{\text{pre}}^{-}$ and $W_{\text{pre}}^{-} \otimes W_{\text{pre}}^{+}$ that entirely map to the constant zero function by Remark 1.1. But we will still use this notation for formal reasons.

Remark 1.2. *With the still valid functional equation*

$$h_x(-z) = -1 - h_{-x}(z)$$

one easily sees that

$$\omega \in W_{\text{weak}}^{\pm} = \left\{ \sum_{x\in\mathbb{R}/\mathbb{Z}} \beta_\omega(x) h_x(z) \;\middle|\; \beta_\omega(-x) = \mp\beta_\omega(x) \right\}.$$

Proposition 1.3. *The map ϑ_k is well-defined.*

Proof. Let $x \in \mathbb{R}^\times$ and $K \subset \mathbb{H}^+ \cup \mathbb{H}^-$ be a compact subset. Then we have the estimate

$$|\mathrm{res}_{z=x} z^{k-1}\eta(z)\omega(z\tau)| \leqslant \max_{\tau \in K} |\omega(\tau x)| \cdot |\mathrm{res}_{z=x}\eta(z)| \cdot |x|^{k-1}.$$

We distinguish three cases.

1. In the case $k > 0$ the claim now follows easily since then $\omega \in W_{\mathrm{weak}}$ and hence there is a $\delta > 0$ (depending on K and ω), such that

$$\max_{\tau \in K} |\omega(\tau x)| = O\left(e^{-\delta |x|}\right).$$

On the other hand, the term $|\mathrm{res}_{z=x}\eta(z)|$ is bounded since η is periodic.

2. If $k < 0$ it follows that

$$\left|\mathrm{res}_{z=x} z^{k-1}\eta(z)\omega(z\tau)\right| \leqslant C|x|^{k-1}$$

where the constant $C > 0$ may be chosen as

$$C = \max_{w \in \bigcup_{0 \neq t \in S(\eta)} tK} |\omega(w)| \cdot \max_{\lambda \in [0,1]} |\mathrm{res}_{z=\lambda}\eta(z)|.$$

Since the sum $\sum_{x \in S(\eta)\setminus\{0\}} |x|^{-1-|k|}$ converges the claim follows.

3. In the case $k = 0$ we note that the map is defined on the subspace $W_{\mathrm{weak}} \otimes W_{\mathrm{pre}}$ by the arguments of 1. It is clearly defined for $W_{\mathrm{pre}}^+ \otimes W_{\mathrm{pre}}^-$ and $W_{\mathrm{pre}}^- \otimes W_{\mathrm{pre}}^+$ since then all summands cancel each other. So we are left to show that we can define it on $W_{\mathrm{pre}} \otimes W_{\mathrm{weak}}$. Without loss of generality we assume that $\omega \otimes \eta \in W_{\mathrm{pre}}^\pm \otimes W_{\mathrm{weak}}^\pm$. First let both functions be even. Then $\omega = c + \omega_{\mathrm{w}}$ with some constant c and $\omega_{\mathrm{w}} \in W_{\mathrm{weak}}$. In conclusion, we only have to show that the sequence

$$S := -2\pi i c \lim_{R \to \infty} \sum_{\substack{x \in \mathbb{R}^\times \\ |x| \leqslant R}} \mathrm{res}_{z=x}\eta(z) z^{-1}$$

converges. Let $0 < x_1 < x_2 < x_3 < \cdots$ the sequence of all positive poles of η. With partial summation we obtain

$$\sum_{j=1}^N \beta_\eta(x_j) x_j^{-1} = \left(\sum_{j=1}^N \beta_\eta(x_j)\right) x_N^{-1} + \sum_{u=1}^{N-1} \left(\sum_{j=1}^u \beta_\eta(x_j)\right)(x_{u+1}^{-1} - x_u^{-1}).$$

Since η is weak, the term $\sum_{j=1}^N \beta_\eta(x_j)$ is bounded and hence the right hand side converges as N tends to infinity. The odd case works similarly, since then we have $\omega = c\cot(\pi z) + \omega_{\mathrm{w}}$ and hence

$$\mathrm{res}_{z=x} z^{-1}\eta(z)\omega(\tau z) = \pm i c \beta_\eta(x) x^{-1} + O(e^{-\delta x}), \qquad \delta > 0,$$

but since $(\pm x)^{-1} = \pm x^{-1}$ we are reduced to the even case. Finally, since in both cases we obtain homomorphisms that coincide on the common subspace $W_{\mathrm{weak}} \otimes W_{\mathrm{weak}}$ we may extend it to the resultant space $\langle W_{\mathrm{pre}} \otimes W_{\mathrm{weak}}, W_{\mathrm{weak}} \otimes W_{\mathrm{pre}} \rangle$. ☐

We now obtain the following very general transformation law.

Theorem 1.4. *Let $\omega \otimes \eta \in W_{(k)}^{\otimes}$, then we have for all $k \in \mathbb{Z}$ and $\tau \in \mathbb{H}$*

$$\vartheta_k\left(\omega \otimes \eta; -\frac{1}{\tau}\right) = \tau^k \vartheta_k\left(\eta \otimes -\widehat{\omega}; \tau\right) + 2\pi i \operatorname{res}_{z=0}\left(z^{k-1}\eta(z)\widehat{\omega}\left(\frac{z}{\tau}\right)\right). \tag{1.4}$$

Here $\widehat{\omega}(z) = \omega(-z)$.

Proof. Let $y > 0$ and $\tau = iy \in \mathbb{H}$. Define

$$g_y(z) := -2\pi i z^{k-1}\eta(z)\widehat{\omega}\left(\frac{z}{iy}\right).$$

Then g_y is a meromorphic function in the plane with simple poles at $S(g_y) = S(\eta) \cup S(\omega)iy\backslash\{0\}$ (all lying on the real and imaginary axes). Consider the closed contour integrals

$$I_n(y) = \frac{1}{2\pi i}\oint_{R_n(y)} g_y(z)\mathrm{d}z,$$

where $R_n(y)$ is a sequence of rectangles that cross the axes half between the respective poles x_n and x_{n+1}. We are left to show $I_n(y) \overset{n\to\infty}{\longrightarrow} 0$ since then the claim follows with the identity and residue theorem. Using periodicity of η, ω and the decay of g_y we find that this will certainly be the case for $k \neq 0$. So we are left to show it for $k = 0$.
We first consider the case $\omega\otimes\eta \in W_{\mathrm{pre}}^{\pm}\otimes W_{\mathrm{pre}}^{\mp}$. Then the functions $\vartheta_0(\omega\otimes\eta)$ and $\vartheta_0(\eta\otimes-\widehat{\omega})$ are constant zero. Since the product $\omega(z/iy)\eta(z)/z$ is an even function in this case, its residue at $z = 0$ will be 0. Hence the transformation law is trivially satisfied in this case. Now let $\omega \in W_{\mathrm{weak}}$. Then the integrals on the right and the left in the rectangle will go to zero because of the exponential decay of ω and the periodicity of η. So we can express $I_n(y)$ in the form

$$I_n(y) = \int_{\sigma_n+it_n}^{-\sigma_n+it_n} g_y(z)\mathrm{d}z + \int_{-\sigma_n-it_n}^{\sigma_n-it_n} g_y(z)\mathrm{d}z + o(1) \tag{1.5}$$

where $0 < \sigma_n \to \infty$ and $0 < t_n \to \infty$ are chosen in the sense of $R_n(y)$. Now we divide the integrals into three parts:

$$\int_{-\sigma_n-it_n}^{\sigma_n-it_n} g_y(z)\mathrm{d}z = \int_{-\sigma_n-it_n}^{-c\sqrt{n}-it_n} g_y(z)\mathrm{d}z + \int_{-c\sqrt{n}-it_n}^{c\sqrt{n}-it_n} g_y(z)\mathrm{d}z + \int_{c\sqrt{n}-it_n}^{\sigma_n-it_n} g_y(z)\mathrm{d}z.$$

Here, $c > 0$ is some fixed constant (note that $\sqrt{n} = o(\sigma_n)$). There is a constant $C > 0$ such that we have $|\eta(z)| \leqslant C$ for all $|\mathrm{Im}(z)| \geqslant 1$. Also on the segments $[-\sigma_n \pm it_n, \sigma_n \pm it_n]$ the function $\widehat{\omega}(z/yi)$ is uniformly bounded (with respect to $n = 1, 2, 3, ...$) by some $D > 0$ since it is periodic along the imaginary axes. Hence for sufficiently large n we obtain

$$\int_{-c\sqrt{n}-it_n}^{c\sqrt{n}-it_n} g_y(z)\mathrm{d}z \ll \frac{\sqrt{n}}{t_n} = o(1).$$

On the other hand, since $\widehat{\omega}(z/yi)$ is of rapid decay as $\mathrm{Re}(z) \to \pm\infty$ we have $|g_y(z)| = O(e^{-\delta|\mathrm{Re}(z)|})$ uniformly on $\{z \in \mathbb{C} \mid |\mathrm{Re}(z)| > 1, |\mathrm{Im}(z)| > 1\}$ for some $\delta > 0$. Hence the integrals

$$\int_{\pm 1 - it_n}^{\pm\infty - it_n} g_y(z)\mathrm{d}z$$

will certainly converge absolutely and also

$$\int_{\pm c\sqrt{n} - it_n}^{\pm\sigma_n - it_n} g_y(z)\mathrm{d}z = o(1).$$

The first integral in (1.5) tends to zero by the same argumentation. The case $\omega \otimes \eta \in W_{\mathrm{pre}} \otimes W_{\mathrm{weak}}$ works analogously. This proves the transformation formula. □

Definition 1.5. *Let β be any function in $(\mathbb{R}/\mathbb{Z})^{\mathbb{C}_0}$. Then we define its Fourier transform $\mathcal{F}(\beta) : \mathbb{R} \to \mathbb{C}$ by*

$$\mathcal{F}(\beta)(y) = \sum_{x \in \mathbb{R}/\mathbb{Z}} \beta(x) e^{-2\pi i x y}.$$

Definition 1.6. *Let $k \geqslant 3$ be an integer and β, γ be functions in $(\mathbb{R}/\mathbb{Z})_0^{\mathbb{C}_0}$, such that $\mathrm{sgn}(\beta)\mathrm{sgn}(\gamma) = (-1)^k$. We assign these data an Eisenstein series by*

$$E_k(\beta, \gamma; \tau) := \sum_{t \in \mathbb{R}_{>0}} a_k(\beta, \gamma; t) q^t$$

with the coefficients

$$a_k(\beta, \gamma; t) := \sum_{\substack{d_1 \in \mathbb{R}_{>0} \\ d_2 \in \mathbb{N} \\ d_1 d_2 = t}} d_1^{k-1} \beta(d_1) \mathcal{F}(\gamma)(d_2).$$

In the cases $k = 2$ and $k = 1$ we have the same definition under the restrictions $\beta(0)\gamma(0) = 0$ and $\beta(0) = \gamma(0) = 0$, respectively.

Note that the (non-trivial) exponents in the above Fourier series can be irrational numbers too.

Theorem 1.7. *Let all assumptions hold as above. The generalized Eisenstein series satisfies the modular identity*

$$E_k\left(\beta, \gamma; -\frac{1}{\tau}\right) = \tau^k E_k(\gamma, -\widehat{\beta}; \tau).$$

Proof. We find

$$\begin{aligned}\vartheta_k(\omega \otimes \eta; \tau) &= 2 \sum_{\alpha \in \mathbb{R}_{>0}} \alpha^{k-1} \beta(\alpha) \sum_{x \in \mathbb{R}/\mathbb{Z}} \gamma(x) \frac{e(\alpha\tau)}{e(x) - e(\alpha\tau)} \\ &= 2 \sum_{\alpha \in \mathbb{R}_{>0}} \sum_{\nu=1}^{\infty} \alpha^{k-1} \beta(\alpha) \left(\sum_{x \in \mathbb{R}/\mathbb{Z}} \gamma(x) e(-\nu x) \right) q^{\alpha\nu} = 2E_k(\beta, \gamma; \tau).\end{aligned}$$

The claim now follows by Theorem 1.4. Note that in the case $k = 2$ at least one and in the case $k = 1$ both of the functions ω_β and η_γ have a removable singularity in $z = 0$, such that in every case the rational part in (1.4) vanishes. $\square$

Analogous to ordinary Eisenstein series we can assign a generalized L-function to $E_k(\beta, \gamma; \tau)$. The result is a generalized Dirichlet series

$$\sum_{t \in D} a(t) t^{-s},$$

where $D \subset \mathbb{R}_{>0}$ is a discrete subset and $a : D \to \mathbb{C}$ a sequence of complex numbers. Like in the classical case one can show (for example by Mellin transform, using the transformation law of the Eisenstein series) that these L-functions have a meromorphic continuation to the entire plane and satisfy a functional equation of the standard type.

Proposition 1.8. *The generalized L-function associated to $E_k(\beta, \gamma; \tau)$ is given by*

$$L(E_k(\beta, \gamma); s) = L(\beta; s + 1 - k) \sum_{x \in \mathbb{R}/\mathbb{Z}} \gamma(x) \mathrm{Li}_s(e^{-2\pi i x}),$$

where $\mathrm{Li}_s(z)$ denotes the polylogarithm. It converges on $\{s \in \mathbb{C} | \mathrm{Re}(s) > k\}$ and has a meromorphic continuation to the entire plane.

Note that $L(\beta; s)$ represents a holomorphic function on $\{s \in \mathbb{C} | \mathrm{Re}(s) > 1\}$ by (1.2) (β is 1-periodic and zero at all but finitely many points) and has a holomorphic continuation to $\mathbb{C} \backslash \{1\}$ with a possible simple pole in $s = 1$.

Proof. Starting with Definition 1.6 we obtain

$$\sum_{t \in \mathbb{R}_{>0}} \left(\sum_{\substack{d_1 \in \mathbb{R}_{>0} \\ d_2 \in \mathbb{N} \\ d_1 d_2 = t}} d_1^{k-1} \beta(d_1) \mathcal{F}(\gamma)(d_2) \right) t^{-s} = \left(\sum_{t \in \mathbb{R}_{>0}} \beta(t) t^{-s+k-1} \right) \sum_{n=1}^{\infty} \mathcal{F}(\gamma)(n) n^{-s}.$$

The function γ is zero almost everywhere. Since by

$$|\mathcal{F}(\gamma)(n)| \leqslant \sum_{x \in \mathbb{R}/\mathbb{Z}} |\gamma(x)|$$

its Fourier transform $\mathcal{F}(\gamma)(n)$ is bounded and hence the corresponding Dirichlet series converges absolutely on $\{s \in \mathbb{C} | \mathrm{Re}(s) > 1\}$. We now have

$$\sum_{n=1}^{\infty} \mathcal{F}(\gamma)(n) n^{-s} = \sum_{n=1}^{\infty} \sum_{x \in \mathbb{R}/\mathbb{Z}} \gamma(x) e^{-2\pi i n x} n^{-s} = \sum_{x \in \mathbb{R}/\mathbb{Z}} \gamma(x) \mathrm{Li}_s(e^{-2\pi i x}).$$

The claim follows with the analytic properties of $s \mapsto \mathrm{Li}_s(e^{-2\pi i x})$ and $L(\beta; s)$. □

In the next theorem we prove a functional equation for the completed L-function associated to a generalized Eisenstein series.

Theorem 1.9. *The completed L-function*

$$\Lambda(\beta, \gamma; s) := (2\pi)^{-s} \Gamma(s) L(E_k(\beta, \gamma); s)$$

extends to an entire function and satisfies the functional equation

$$\Lambda(E_k(\beta, \gamma); k - s) = \Lambda\left(E_k\left(\gamma, -\widehat{\beta}\right); s\right).$$

Proof. By Mellin transformation we obtain

$$\Lambda(\beta, \gamma; s) = \int_0^{\infty} E_k(\beta, \gamma; iy) y^{s-1} \mathrm{d}y.$$

By splitting the integral in the intervals $[0, 1]$ and $[1, \infty)$ and making the substitution $y \mapsto y^{-1}$ in the first integral we obtain

$$\begin{aligned} \Lambda(\beta, \gamma; s) &= \int_1^{\infty} E_k\left(\beta, \gamma; \frac{i}{y}\right) y^{-s-1} \mathrm{d}y + \int_1^{\infty} E_k(\beta, \gamma; iy) y^{s-1} \mathrm{d}y \\ &= \int_1^{\infty} E_k(\gamma, -\widehat{\beta}; iy) y^{k-s-1} \mathrm{d}y + \int_1^{\infty} E_k(\beta, \gamma; iy) y^{s-1} \mathrm{d}y. \end{aligned}$$

From this one sees that $\Lambda(\beta, \gamma; s)$ is entire. The symmetry on the right hand side leads to the desired functional equation. □

2. Cotangent sums

Besides periodic L-functions we may associate other objects to a pre-weak function. For integers $m = 1, 2, 3, \ldots$ we define the corresponding *cotangent sum*

$$C(\omega; m) := \sum_{x \in \mathbb{R}/\mathbb{Z}} \beta_\omega(x) \cot^m(\pi x).$$

The primary goal of this section is to develop a principle which helps to write cotangent sums as rational combinations of L-functions, and vice versa. With this we may conclude several results about cotangent sums using well known results about L-functions, and of course vice versa again.

A famous example for a cotangent sum is given in [13] on p. 262:

$$\sum_{j=1}^{N-1} \cot^2\left(\frac{\pi j}{N}\right) = \frac{(N-1)(N-2)}{3}, \qquad N = 2, 3, \ldots \tag{2.1}$$

Note that the sum is always rational independent of the choice of N. This was generalized by Chu and Marini in [4] and Berndt and Yeap [2] on p. 6.

Theorem 2.1. *Let N and n be positive integers. Then*

$$\sum_{j=1}^{N-1} \cot^{2n}\left(\frac{\pi j}{N}\right) = (-1)^n N - (-1)^n 2^{2n} \sum_{j_0=0}^{n} \left(\sum_{\substack{j_1,\ldots,j_{2n}\geqslant 0\\ j_0+j_1+\cdots+j_{2n}=n}} \prod_{r=0}^{2n} \frac{B_{2j_r}}{(2j_r)!} \right) N^{2j_0}.$$

In particular, we have

$$\sum_{j=1}^{N-1} \cot^{2n}\left(\frac{\pi j}{N}\right) \in \mathbb{Q}.$$

Note that the B_n denote the *Bernoulli numbers* defined by generating series

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = \frac{x}{e^x - 1}.$$

The interesting identity in Theorem 2.1 can be proved by looking at

$$f(z) = \cot^{2n}(\pi z)\cot(\pi k z)$$

and using contour integration. Another more general result is presented in [2] on p. 17 (there is a mistake in the original paper) and looks as follows.

Theorem 2.2. *For positive integers $0 < a < k$ and n let*

$$s_n(k,a) := \sum_{j=1}^{k-1} \sin\left(\frac{2\pi a j}{k}\right) \cot^n\left(\frac{\pi j}{k}\right)$$

and

$$c_n(k,a) := \sum_{j=1}^{k-1} \cos\left(\frac{2\pi a j}{k}\right) \cot^n\left(\frac{\pi j}{k}\right).$$

Then we have for all positive integers m

$$s_{2m-1}(k,a) = (-1)^m 2^{2m-1} \sum_{\substack{j_1,\ldots,j_{2m-1},\mu,\nu\geqslant 0\\ 2j_1+\cdots+2j_{2m-1}+\mu+\nu=2m-1}} a^\mu k^\nu \frac{1}{\mu!}\frac{B_\nu}{\nu!} \prod_{r=1}^{2m-1} \frac{B_{2j_r}}{(2j_r)!} \tag{2.2}$$

and

$$c_{2m}(k,a) = (-1)^{m+1} 2^{2m} \sum_{\substack{j_1,\ldots,j_{2m},\mu,\nu\geqslant 0\\ 2j_1+\cdots+2j_{2m}+\nu+\mu=2m}} a^\mu k^\nu \frac{1}{\mu!}\frac{B_\nu}{\nu!} \prod_{r=1}^{2m} \frac{B_{2j_r}}{(2j_r)!}. \tag{2.3}$$

In particular, both s_m and c_m define sequences of elements in $\mathbb{Q}[k,a]$.

In other words, the theories of generalized periodic L-functions and cotangent sums are in some way equivalent. To understand this, we modify the definition (1.1) of a periodic L-function in the following way. In the entire section we denote W_{pre}^0 as the subspace of pre-weak functions that have a removable singularity in $z = 0$, which is equivalent to $\beta_\omega(0) = 0$. Consider now the homomorphism between the space of pre-weak functions and an infinite tuple of complete L-values at positive integers

$$W_{\text{pre}}^0 \longrightarrow \mathbb{C}^{\mathbb{N}}$$

$$\omega \longmapsto \left(\widetilde{L}(\omega; 1), \widetilde{L}(\omega; 2), ...\right), \qquad \widetilde{L}(\omega; k) := \sum_{x \in \mathbb{R}^\times} \beta_\omega(x) x^{-k}.$$

In the case $k = 1$, we interpret the sum as

$$L(\omega; 1) = \lim_{N \to \infty} \sum_{-N \leqslant x \leqslant N, x \neq 0} \beta_\omega(x) x^{-1} = \sum_{x > 0} (\beta_\omega(x) - \beta_\omega(-x)) x^{-1}. \tag{2.4}$$

Remark 2.3. *Note that by Remark 1.2* $\operatorname{sgn}(\omega) = (-1)^k$ *implies* $L(\omega; k) = 0$ *for* $k > 1$ *(an even pre-weak function is weak up to a constant and an odd up to a cotangent function). If* $k = 1$ *this relation still holds if we restrict to weak functions or odd* ω*.*

Before we move on, we define a sequence of numbers which is of great importance in combinatorics.

Definition 2.4. *Let* $n \in \mathbb{N}_0$ *and* $k \in \mathbb{Z}$*. We define the Stirling numbers of the second kind by*

$$\left\{ n \atop k \right\} := \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k - j)^n, \qquad 0 \leqslant k \leqslant n,$$

where $\left\{ 0 \atop 0 \right\} := 1$ *and* $\left\{ n \atop k \right\} := 0$ *whenever* $k > n$ *or* $k < 0$*.*

Put

$$\Delta(\ell, u) := \binom{\ell}{u} - \binom{\ell}{u - 1}$$

and

$$S^*(n, k) := k! \left\{ n \atop k \right\} = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k - j)^n, \qquad k \leqslant n.$$

To find the connection between (generalized) L-functions and cotangent sums we need the following lemma.

Lemma 2.5. *Define a sequence* $\delta : \mathbb{N}_0^2 \to \mathbb{C}$ *by*

$$\delta_0(0) = \delta_1(0) = \delta_0(1) := 0,$$

and for integers $\nu, u \geqslant 0$ *with* $\nu + u \geqslant 2$*:*

$$\delta_\nu(u) := \frac{i^{\nu+u}}{(\nu - 1)!} \sum_{\ell=u-1}^{\nu-1} (-1)^{\nu+\ell-u-1} 2^{\nu-1-\ell} S^*(\nu - 1, \ell) \Delta(\ell, u).$$

Let $a \in \mathbb{C}\backslash\mathbb{Z}$. Then we have in an arbitrary small neighborhood of $z = 0$

$$\cot(\pi(z-a)) = \sum_{\nu=0}^{\infty} P_\nu(\cot(\pi a))z^\nu = -\cot(\pi a) + (-\pi - \pi\cot^2(\pi a))z + \cdots ,$$

where

$$P_\nu(X) = \pi^\nu \sum_{u=0}^{\nu+1} \delta_{\nu+1}(u)X^u.$$

Remark 2.6.

(i) The first polynomials P_ν are given by

$$\begin{aligned} P_0(X) &= -X, \\ P_1(X) &= -\pi - \pi X^2, \\ P_2(X) &= -\pi^2 X - \pi^2 X^3, \\ P_3(X) &= -\frac{\pi^3}{3} - \frac{4\pi^3}{3}X^2 - \pi^3 X^4, \\ P_4(X) &= -\frac{2\pi^4}{3}X - \frac{5\pi^4}{3}X^3 - \pi^4 X^5. \end{aligned}$$

(ii) We have for all $\nu \geqslant 1$ the formulas

$$\delta_\nu(\nu) = -1 \tag{2.5}$$

and for all $\nu \geqslant 2$

$$\delta_\nu(0) = \frac{i^\nu}{(\nu-1)!}\sum_{\ell=0}^{\nu-1}(-1)^{\nu+\ell-1}2^{\nu-1-\ell}S^*(\nu-1,\ell),$$

since then $\Delta(\ell, 0) = 1$.

(iii) It is $\delta_\nu(u) = 0$ if $u > \nu$. Since the function $\cot(x)$ *is odd, we obtain $\delta_\nu(u) = 0$ if $\nu + u \equiv 1 \pmod 2$.*

Proof. It is clear that the function $f(z) = \cot(\pi(z-a))$ is holomorphic in a neighborhood of $z = 0$ in the case $a \in \mathbb{C}\backslash\mathbb{Z}$. For the constant term we find

$$\cot(\pi(-a)) = -\cot(\pi a) = \pi^0\left(\delta_1(0) + \delta_1(1)\cot(\pi a)\right),$$

and indeed this coefficient is

$$\delta_1(1) = i^2 \cdot (-1) \cdot 2^0 \cdot S^*(0,0) \cdot \left(\binom{0}{1} - \binom{0}{0}\right) = -1.$$

Using the formula in [19] on p. 2,

$$\cot^{(n)}(x) = (2i)^n(\cot(x) - i)\sum_{v=0}^{n}\frac{v!}{2^v}\left\{ {n \atop v} \right\}(i\cot(x) - 1)^v, \qquad n \geqslant 1,$$

(note that in the paper, the sum starts at $v = 1$ but we have $n \geqslant 1$, hence $\left\{ {n \atop 0} \right\} = 0$) and the binomial theorem, for $\nu \geqslant 1$, we end up with

$$f^{(\nu)}(0) = -(-2\pi i)^{\nu} \sum_{\ell=0}^{\nu} \sum_{u=0}^{\ell+1} (\alpha_{\nu,\ell}(u-1) - i\alpha_{\nu,\ell}(u)) \cot^{u}(\pi a),$$

where

$$\alpha_{\nu,\ell}(u) := \frac{S^*(\nu,\ell)}{2^{\ell}} \binom{\ell}{u} (-1)^{\ell-u} i^{u}.$$

Put

$$(2.6) b_{\nu}(\ell,u) := \alpha_{\nu,\ell}(u-1) - i\alpha_{\nu,\ell}(u) = \frac{S^*(\nu,\ell) i^{u-1} (-1)^{\ell-u}}{2^{\ell}} \left(\binom{\ell}{u} - \binom{\ell}{u-1} \right),$$

and note that this implies $b_{\nu}(-1,0) = 0$. With the additional summand $b_{\nu}(-1,0)$ we obtain

$$\sum_{\ell=0}^{\nu} \sum_{u=0}^{\ell+1} b_{\nu}(\ell,u) = \sum_{u=0}^{\nu+1} \sum_{\ell=u-1}^{\nu} b_{\nu}(\ell,u)$$

and conclude

$$\frac{f^{(\nu)}(0)}{\nu!} = -\frac{(-2\pi i)^{\nu}}{\nu!} \sum_{u=0}^{\nu+1} \sum_{\ell=u-1}^{\nu} b_{\nu}(\ell,u) \cot^{u}(\pi a).$$

Together with (2.6) this proves the formula for $\delta_{\nu}(u)$, after the index shift $\nu \mapsto \nu - 1$. $\square$

We can use Lemma 2.5 to determine the local Taylor expansion of $\omega(z)$ at $z = 0$. This will later help to explain the relationship between periodic L-functions and cotangent sums.

Lemma 2.7. *Let $\omega \in W_{\text{pre}}^{0}$. Then we have*

$$\omega(z) = \omega(i\infty) - \frac{1}{2} C(\omega;0) + \frac{i}{2} \sum_{\nu=0}^{\infty} \left(\sum_{u=0}^{\nu+1} \delta_{\nu+1}(u) C(\omega;u) \right) (z\pi)^{\nu}.$$

Proof. With the behavior of the function $\cot(\pi z)$ at $z = i\infty$ we obtain the following canonical representation of ω:

$$\omega(z) = \omega(i\infty) + \sum_{x \in \mathbb{R}/\mathbb{Z}} \beta_{\omega}(x) \left(\frac{i}{2} \cot\left(\pi\left(z - x\right)\right) - \frac{1}{2} \right),$$

and with Lemma 2.5 we obtain

$$\frac{i}{2} \sum_{x \in \mathbb{R}/\mathbb{Z}} \beta_{\omega}(x) \cot\left(\pi\left(z-x\right)\right) = \frac{i}{2} \sum_{\nu=0}^{\infty} \sum_{x \in \mathbb{R}/\mathbb{Z}} \beta_{\omega}(x) P_{\nu}\left(\cot \pi x\right) z^{\nu}$$

$$= \frac{i}{2} \sum_{\nu=0}^{\infty} \sum_{x \in \mathbb{R}/\mathbb{Z}} \beta_{\omega}(x) \pi^{\nu} \sum_{u=0}^{\nu+1} \delta_{\nu+1}(u) \cot^{u}\left(\pi x\right) z^{\nu} = \frac{i}{2} \sum_{\nu=0}^{\infty} \left(\sum_{u=0}^{\nu+1} \delta_{\nu+1}(u) C(\omega;u) \right) (z\pi)^{\nu}.$$

The claim now follows with some simple rearrangements. $\square$

At this point we stress the simple but important fact, that the coefficients $\delta_\nu(u)$ are independent of the choice of ω.

Lemma 2.8 (Generalized Abel's theorem)**.** *Let $f_n : \mathbb{E} \cup \{1\} \to \mathbb{C}$ be a sequence of continuous functions that are holomorphic in the unit disc $\mathbb{E}$, such that $f_n(z) \to f(z)$ as $n \to \infty$ for all $z \in \mathbb{E}$. We assume that f is bounded on $[0,1]$ and put $D := \sup_{0 \leqslant t \leqslant 1} |f(t)|$. Let $\sum_{n=1}^\infty a(n)$ be a converging series and $F(z) = \sum_{n=1}^\infty a(n) f_n(z)$ be holomorphic in $\mathbb{E}$. Assume that the f_n satisfy the Abelian condition: there is a constant $C > 0$ such that uniformly for all $n > 0$ and all $0 \leqslant t \leqslant 1$:*

$$|f_n(t) - f_{n+1}(t)| \leqslant C(1-t)t^n.$$

Then we have

$$\lim_{t \to 1^-} \sum_{n=1}^\infty a(n) f_n(t) = f(1) \sum_{n=1}^\infty a(n).$$

Note that the important case $f_n(z) = z^n$ is *Abel's theorem.*

Proof. We show that for each $\varepsilon > 0$ there is an N_0 such that for all $N \geqslant N_0$:

$$\sup_{0 \leqslant t \leqslant 1} \left| \sum_{n>N} a(n) f_n(t) \right| \leqslant \varepsilon. \tag{2.7}$$

Let $\varepsilon > 0$. Choose $\delta > 0$ such that $\max\{|f_1(1)|\delta, \delta(C+D)\} \leqslant \varepsilon$. We choose an integer N such that if $A_n = \sum_{k=N+1}^n a(k)$, we have

$$\sup_{n>N} |A_n| \leqslant \delta.$$

This is possible since the series $\sum_{n=1}^\infty a(n)$ converges. By partial summation we obtain with $f_n(z) \to f(z)$ and $0 \leqslant t < 1$:

$$\begin{aligned} \left| \sum_{n>N} a(n) f_n(t) \right| &= \left| A_\infty f(t) - \sum_{n>N} A_n (f_n(t) - f_{n+1}(t)) \right| \\ &\leqslant \delta D + \delta C(1-t) \sum_{n>N} t^n \leqslant \delta(C+D) \leqslant \varepsilon. \end{aligned}$$

On the other hand we have

$$\left| \sum_{n>N} a(n) f_n(1) \right| = |f(1)| \left| \sum_{n>N} a(n) \right| \leqslant |f(1)|\delta \leqslant \varepsilon.$$

From this follows (2.7) and we conclude the lemma. □

We consider the following special case.

Lemma 2.9. *Let g be holomorphic on $\mathbb{E}$ and a neighborhood U of $z = 1$. Then $f_n(z) := g(z^n)$ satisfies the assertions of Lemma 2.8.*

Proof. Let $0 < b < a < 1$. To see the lemma one uses the Cauchy integral formula

$$\frac{g(a)-g(b)}{a-b} = \frac{1}{2\pi i}\oint_\gamma \frac{g(z)}{(z-a)(z-b)}\mathrm{d}z,$$

where the closed and smooth integration path $\gamma \subset \mathbb{E} \cup U$ with length $\mathcal{L}(\gamma)$ surrounds the compact line $[0,1]$ once in positive direction. We find a minimum distance $\varepsilon > 0$ between γ and $[0,1]$. Hence

$$\left|\frac{1}{2\pi i}\oint_\gamma \frac{g(z)}{(z-a)(z-b)}\mathrm{d}z\right| \leqslant \frac{1}{2\pi}\max_{z\in\gamma}\left|\frac{g(z)}{(z-a)(z-b)}\right|\mathcal{L}(\gamma) \leqslant C\frac{\max_{z\in\gamma}|g(z)|}{\varepsilon^2},$$

where $C > 0$ is independent from a and b. Put $a = t^n$ and $b = t^{n+1}$ for $0 < t < 1$. Since $g(t^n)$ converges to $g(0)$ if $0 \leqslant t < 1$ and to $g(1)$ if $t = 1$, one has $D := \max\{|g(0)|, |g(1)|\}$. $\square$

We are now in the position to prove a result that ties values of L-functions with Taylor coefficients of pre-weak functions.

Proposition 2.10. *Let $k \geqslant 1$ be an integer and $\omega \otimes \eta \in W_{\mathrm{pre}} \otimes W_{\mathrm{pre}}$ if $k > 1$ and $\omega \otimes \eta \in \left\langle W_{\mathrm{pre}} \otimes W_{\mathrm{weak}}, W_{\mathrm{pre}}^{+} \otimes W_{\mathrm{pre}}^{-}, W_{\mathrm{pre}}^{-} \otimes W_{\mathrm{pre}}^{+}\right\rangle$ else, such that ω has a removable singularity in $z = 0$. We then have*

$$\lim_{y\to 0^+} \vartheta_{1-k}(\omega \otimes \eta; iy) = \omega(0)\widetilde{L}(\eta; k). \tag{2.8}$$

In particular, for $\omega \otimes \eta \in W_{\mathrm{pre}} \otimes W_{\mathrm{pre}}$ (and $\omega \otimes \eta \in \left\langle W_{\mathrm{pre}} \otimes W_{\mathrm{weak}}, W_{\mathrm{pre}}^{+} \otimes W_{\mathrm{pre}}^{-}\right\rangle$ if $k = 1$) we have the key identity

$$\widetilde{L}(\eta; k) = 2\pi i \ \mathrm{res}_{z=0}\left(z^{-k}\eta(z)\right). \tag{2.9}$$

Proof. First we note that in the case $k = 1$ (2.8) is trivial for elements $\omega\otimes\eta$ in $W_{\mathrm{pre}}^{\pm}\otimes W_{\mathrm{pre}}^{\mp}$, since then both the left hand side and the right hand side are zero (note that either $\omega(0) = 0$ or $\widetilde{L}(\eta; 1) = 0$). Also if $\omega \in \eta \in W_{\mathrm{pre}}^{+} \otimes W_{\mathrm{pre}}^{-}$ both sides vanish according to Remark 2.3 and since η is odd. So we can assume η to be weak in this case.
We have $\omega(z) = R(e(z))$ with a rational function R, which fulfills the conditions of Lemma 2.9 (note that ω has a removable singularity in $z = 0$). We obtain:

$$\vartheta_{1-k}(\omega \otimes \eta; iy) = \sum_{\alpha>0} \alpha^{-k}\beta_\eta(\alpha)\omega(\alpha iy) + \sum_{\alpha>0}(-1)^k\alpha^{-k}\beta_\eta(-\alpha)\omega(-\alpha iy).$$

Since η is weak for $k = 1$ both series will converge for $y = 0$ separately. Hence with Lemma 2.8 we conclude

$$\lim_{y\to 0^+} \vartheta_{1-k}(\omega \otimes \eta; iy) = \omega(0)\widetilde{L}(\eta; k).$$

Note that we have a homeomorphism between the segments $[0, i\infty]$ and $[0,1]$ given by $z \mapsto e^{2\pi i z}$. On the other hand, with Theorem 1.4 we obtain

$$\begin{aligned}\lim_{\tau\to 0} \vartheta_{1-k}(\omega\otimes\eta;\tau) &= \lim_{\tau\to 0}\left[(-\tau)^{k-1}\vartheta_{1-k}\left(\eta\otimes -\widehat{\omega}; -\frac{1}{\tau}\right) + 2\pi i \mathrm{res}_{z=0}\left(z^{-k}\eta(z)\omega(z\tau)\right)\right]\\ &= 2\pi i\omega(0)\mathrm{res}_{z=0}\left(z^{-k}\eta(z)\right).\end{aligned}$$

In the case of $k = 1$, the first term on the right side vanishes because η is weak. The choice $\omega = 1$ finally proves (2.9). $\square$

Throughout our analysis of cotangent sums we assume the first component of the $W_{\mathrm{pre}} \otimes W_{\mathrm{pre}}$ to be the function which is constant 1. It is trivial but crucial that this function is even. Since we want to consider all values of completed L-functions simultaneously, we only look at elements $1 \otimes \omega \in \langle W^+_{\mathrm{pre}} \otimes W^0_{\mathrm{weak}}, W^+_{\mathrm{pre}} \otimes W^{0,-}_{\mathrm{pre}}\rangle$. In other words, throughout, ω it is an odd pre-weak function or weak function - both have a removable singularity in $z = 0$. Together with Lemma 2.7 we can now suggest closed formulas for cotangent sums in terms of corresponding L-functions at integer arguments.

Proposition 2.11. *Let $k \geqslant 1$ and $\omega \in \langle W^0_{\mathrm{weak}}, W^{0,-}_{\mathrm{pre}}\rangle$. We have the formula*

$$\widetilde{L}(\omega;k) = \sum_{\alpha\in\mathbb{R}^\times}\beta_\omega(\alpha)\alpha^{-k} = -\pi^k\sum_{n=0}^{k}\delta_k(n)C(\omega;n),$$

which is equivalent to

$$\widetilde{L}^*(\omega;k) := -\frac{\widetilde{L}(\omega;k)}{\pi^k} - \delta_k(0)C(\omega;0) = \sum_{n=1}^{k}\delta_k(n)C(\omega;n). \tag{2.10}$$

Proof. First note that $\delta_1(0) = 0$. In the case ω is odd it is trivial that $L(\omega;1) = 0 = C(\omega;1)$, which proves the formula in this case. So let ω be weak if $k = 1$. With Lemma 2.7 we see that the residue of $z^{-k}\omega(z)$ in $z = 0$ is given by

$$\mathrm{res}_{z=0}\left(z^{-k}\omega(z)\right) = \frac{i}{2}\pi^{k-1}\sum_{u=0}^{k}\delta_k(u)C(\omega;u).$$

Multiplying by $2\pi i$ proves the claim when using (2.9). $\square$

Definition 2.12. *For Dirichlet characters χ modulo N we put*

$$C(\chi;m) := \sum_{j=1}^{N-1}\chi(j)\cot^m\left(\frac{j\pi}{N}\right).$$

Remark 2.13. *Let $k > 0$ be an integer. In* [3] *a relation between the class number h_K of the field $K = \mathbb{Q}(\sqrt{-k})$ and cotangent sums is proved. If χ is an odd (real) character for K, we have*

$$C(\chi;1) = 2\sqrt{k}h_K.$$

The present method now gives a further viewpoint to this equation since by Proposition (2.10) *we have*

$$\widetilde{L}(\omega_\chi;1) = -\pi\delta_1(1)C(\beta_{\omega_\chi};1)$$

and by the class number formula $L(\chi;1)$ *is directly tied to* h_K. *Here we have put*

$$\omega_\chi(z) := \sum_{j=1}^{N-1} \chi(j) h_{\frac{j}{N}}(z),$$

where χ *is a character modulo* N.

Let Δ_∞ be the linear operator

$$\Delta_\infty : \prod_{n\in\mathbb{N}} \mathbb{R} \longrightarrow \prod_{n\in\mathbb{N}} \mathbb{R}$$

$$(a_1, a_2, a_3, ...)^T \longmapsto \left(\sum_{j=1}^{m} \delta_m(j) a_j\right)_{m\in\mathbb{N}}.$$

We can write this formally as an infinite lower triangular matrix:

$$\Delta_\infty := \begin{pmatrix} \delta_1(1) & 0 & 0 & 0 & 0 & \cdots \\ \delta_2(1) & \delta_2(2) & 0 & 0 & 0 & \cdots \\ \delta_3(1) & \delta_3(2) & \delta_3(3) & 0 & 0 & \cdots \\ \delta_4(1) & \delta_4(2) & \delta_4(3) & \delta_4(4) & 0 & \cdots \\ \delta_5(1) & \delta_5(2) & \delta_5(3) & \delta_5(4) & \delta_5(5) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.11}$$

Proposition 2.11 provides us a linear system with countable many unknowns. In other words, we can find values for the cotangent sums recursively. We obtain:

$$\begin{pmatrix} \widetilde{L}^*(\omega;1) \\ \widetilde{L}^*(\omega;2) \\ \widetilde{L}^*(\omega;3) \\ \widetilde{L}^*(\omega;4) \\ \widetilde{L}^*(\omega;5) \\ \vdots \end{pmatrix} = \Delta_\infty \begin{pmatrix} C(\omega;1) \\ C(\omega;2) \\ C(\omega;3) \\ C(\omega;4) \\ C(\omega;5) \\ \vdots \end{pmatrix}. \tag{2.12}$$

Note that in the case that ω is weak we have $\widetilde{L}^*(\omega;k) = -\pi^{-k}\widetilde{L}(\omega;k)$. With $\delta_\nu(\nu) = -1$ (see (2.5)) we see that the system (2.12) is invertible, since we have a lower diagonal operator. In other words, for all positive integers m we have

$$\Delta_m^{-1}\boldsymbol{L}_m(\omega) = \boldsymbol{C}_m(\omega), \tag{2.13}$$

where $\boldsymbol{L}_m(\omega)$ and $\boldsymbol{C}_m(\omega)$ denote the first m rows vectors of (2.12) and Δ_m the regular major $m\times m$ block of the operator. Note that since $\Delta_m \in \mathbb{Q}^{m\times m}$ we have $\Delta_m^{-1} \in \mathbb{Q}^{m\times m}$. Therefore we obtain the following theorem.

Theorem 2.14. *Let $\omega \in \left\langle W^0_{\text{weak}}, W^{0,-}_{\text{pre}} \right\rangle$ be a pre-weak function. Let $K|\mathbb{Q}$ be a field extension (not necessarily finite) and $m \in \mathbb{N}$ be any positive integer. Assume that $C(\omega; 0) \in K$. Then we have*

$$\frac{\widetilde{L}(\omega;1)}{\pi}, \frac{\widetilde{L}(\omega;2)}{\pi^2}, \cdots, \frac{\widetilde{L}(\omega;m)}{\pi^m} \in K \iff C(\omega;1), C(\omega;2), \cdots, C(\omega;m) \in K.$$

Proof. As (2.12) proves, we can express the terms $\widetilde{L}(\omega;k)\pi^{-k} + C(\omega;0)\delta_k(0)$ as rational combinations of $C(\omega;m)$, $1 \leqslant m \leqslant k$ and vice versa the terms $C(\omega;k)$ as rational combinations of $\widetilde{L}(\omega;m)\pi^{-m} + C(\omega;0)\delta_m(0)$. Since $\delta_m(0) \in \mathbb{Q}$ for all $m \geqslant 0$, the claim follows with $C(\omega;0) \in K$. □

We see that it turns out that there is an arithmetic connection between cotangent sums and generalized L-functions. Together with Theorems 2.1 and 2.2 we are able to find explicit formulas. Here, the key ingredient is the fact that expressions like

$$\sum_{j=1}^{N-1} \cot^m\left(\frac{j\pi}{N}\right)$$

are polynomials $P_m(N)$ for fixed m. Compare Theorem 2.1. For the next theorem we need the Euler numbers E_n that are defined by the generating series

$$\frac{2}{e^z + e^{-z}} = \sum_{n=0}^{\infty} \frac{\mathrm{E}_n}{n!} z^n.$$

Theorem 2.15. *Let $k \geqslant 1$ and $\omega \in \left\langle W^0_{\text{weak}}, W^{0,-}_{\text{pre}} \right\rangle$.*

(i) There are rational numbers $\delta_k(\ell)$ (given in 2.5) and $\delta_k^*(\ell)$, independent from the choice of ω, such that

$$-\frac{\tilde{L}(\omega;k)}{\pi^k} - \delta_k(0)C(\omega;0) = \sum_{\ell=1}^{k} \delta_k(\ell)C(\omega;\ell) \tag{2.14}$$

and

$$C(\omega;k) = \sum_{\ell=1}^{k} \delta_k^*(\ell)\left(-\frac{\tilde{L}(\omega;\ell)}{\pi^\ell} - \delta_\ell(0)C(\omega;0)\right). \tag{2.15}$$

(ii) Explicitly, we obtain $\delta_\nu^*(u) = 0$ if $\nu + u \equiv 1 \pmod 2$ and for $0 < \ell \leqslant k$

$$\delta_{2k}^*(2\ell) = (-1)^{k+\ell+1} 2^{2k-2\ell} \sum_{\substack{j_1,\ldots,j_{2k} \geqslant 0 \\ \ell + j_1 + \cdots + j_{2k} = k}} \prod_{r=1}^{2k} \frac{B_{2j_r}}{(2j_r)!} \tag{2.16}$$

and

$$\delta_{2k-1}^*(2\ell-1) = (-1)^{k+\ell+1} 2^{2k-2\ell} \sum_{\substack{j_1,\ldots,j_{2k-1} \geqslant 0 \\ 2\ell-1+2j_1+\cdots+2j_{2k-1} = 2k-1}} \prod_{r=1}^{2k-1} \frac{B_{2j_r}}{(2j_r)!}. \tag{2.17}$$

(iii) (Supplementary laws) We have for all positive integers k

$$\text{(1)} \qquad \sum_{\ell=1}^{k} \delta_{2k}^*(2\ell)\delta_{2\ell}(0) = (-1)^{k-1},$$

$$\text{(2)} \qquad \sum_{\ell=1}^{k} \delta_{2k}^*(2\ell)\zeta(2\ell)\pi^{-2\ell} = \frac{(-1)^k}{2}\left(1 - 2^{2k} \sum_{\substack{j_1,\dots,j_{2k}\geqslant 0 \\ j_1+\cdots+j_{2k}=k}} \prod_{r=1}^{2k} \frac{B_{2j_r}}{(2j_r)!}\right).$$

Remark 2.16. *Supplementary law (1) reduces* (2.15) *to the formula*

$$C(\omega;k) + i^k\frac{1+(-1)^k}{2}C(\omega;0) = -\sum_{\ell=1}^{k}\delta_k^*(\ell)\tilde{L}(\omega;\ell)\pi^{-\ell}. \tag{2.18}$$

Proof.

(i) The formula (2.14) follows from Proposition 2.11. Let $k \leqslant m$ be arbitrarily chosen. Formula (2.15) follows with (2.13) and the fact that $\Delta_m^{-1} \in \mathbb{Q}^{m\times m}$ is again a lower triangular matrix, when denoting its coefficients by $\delta_\nu^*(u)$ (analogously as it was done in (2.11)). It is clear that all values $\delta_\nu^*(u)$ are independent of m and ω.

(ii) We first show by induction that for $\nu, u \geqslant 1$ the $\delta_\nu^*(u)$ vanish if $\nu + u \equiv 1 \pmod 2$. This is clear for $\nu < u$, so we assume that $u \leqslant \nu$. Obviously, with the vanishing of the above triangle in mind, the statement is equivalent to the vanishing of all "odd" lower diagonals

$$\begin{aligned} D_1 &:= (\delta_\nu^*(\nu-1))_{\nu=2,3,\dots} \\ D_3 &:= (\delta_\nu^*(\nu-3))_{\nu=4,5,\dots} \\ &\vdots \\ D_{2k-1} &:= (\delta_\nu^*(\nu-2k+1))_{\nu=2k,2k+1,\dots} \\ &\vdots \end{aligned}$$

We formally write $\Delta_\infty^{-1}\Delta_\infty = I_\infty$. First we show the vanishing of D_1. Let $\nu \geqslant 2$. Then we obtain, multiplying the ν-th row of the operator Δ_∞^{-1} with the $\nu-1$-th column of Δ_∞:

$$\sum_{u=1}^{\infty}\delta_\nu^*(u)\delta_u(\nu-1) = \sum_{u=\nu-1}^{\nu}\delta_\nu^*(u)\delta_u(\nu-1) = \delta_\nu^*(\nu-1)\delta_{\nu-1}(\nu-1) = 0.$$

Hence $\delta_\nu^*(\nu-1) = 0$, since $\delta_{\nu-1}(\nu-1) = -1$ (note that $\delta_\nu(\nu-1) = 0$ – remember that $\delta_\nu(u) = 0$ if $\nu + u \equiv 1 \pmod 2$ by Remark 2.6 (iii)). Note that the sum could be reduced to two summands in the first step since we have multiplied two lower diagonal operators. For the induction step, we assume that we have proved vanishing for $D_1, D_3, \dots, D_{2k-1}$. We show that under these circumstances we obtain the vanishing of D_{2k+1}. Let $\nu \geqslant 2k+2$, and multiply the ν-th row of Δ_∞^{-1} with the $\nu - 2k - 1$-th column of Δ_∞.

$$\sum_{u=1}^{\infty} \delta_\nu^*(u)\delta_u(\nu - 2k - 1) = \sum_{u=\nu-2k-1}^{\nu} \delta_\nu^*(u)\delta_u(\nu - 2k - 1) = 0. \tag{2.19}$$

If $\nu - 2k \leqslant u \leqslant \nu$ is of the form $u = \nu - 2\ell$ for an integer ℓ, we have $\delta_\nu^*(u)\delta_u(\nu - 2k - 1) = 0$ since $\delta_{\nu-2\ell}(\nu - 2k - 1) = 0$. Otherwise, if $u = \nu - 2\ell + 1$, we also have $\delta_\nu^*(u)\delta_u(\nu - 2k - 1) = 0$ since then $\delta_\nu^*(\nu - 2\ell + 1) = 0$ by assumption since $\ell \leqslant k$. Hence, (2.19) reduces to

$$\delta_\nu^*(\nu - 2k - 1)\delta_{\nu-2k-1}(\nu - 2k - 1) = 0.$$

Since $\delta_{\nu-2k-1}(\nu - 2k - 1) = -1$, we obtain $\delta_\nu^*(\nu - 2k - 1) = 0$.

To obtain the coefficients δ^* explicitly, we could of course simply use invert the operator Δ_∞, which would not be too bad, since all of its finite "blocks" are lower diagonal with determinant ± 1. However, there is even a quicker trick that uses a small subset of cotangent sums that are polynomials in the "period" variable N.

To prove the formula (2.16) for $\delta_{2k}^*(2\ell)$ with $1 \leqslant \ell \leqslant k$ choose

$$\omega_N(z) := \sum_{j=1}^{N-1} h_{\frac{j}{N}}(z) = \frac{i}{2}\left(N\cot(N\pi z) - \cot(\pi z)\right),$$

where $N > 1$ is a positive integer. A brief calculation shows $\omega_N \in W_{\mathrm{pre}}^{0,-}$. We have for integers $k > 0$

$$\widetilde{L}(\omega_N; k) = \sum_{r \not\equiv 0 \pmod{N}} \left(\frac{r}{N}\right)^{-k} = \begin{cases} 2\zeta(k)(N^k - 1), & \text{if } k \equiv 0 \pmod 2, \\ 0, & \text{else}, \end{cases}$$

and for $k = 1$ the right sum is understood as in (2.4). Since ω_N is not weak, we have to include the terms $C(\beta_{\omega_N}; 0) = N - 1$. From (2.12) and 2.1 we conclude for all even positive integers $2k$

$$\begin{aligned} &\sum_{\ell=1}^{k} \delta_{2k}^*(2\ell)\pi^{-2\ell}\left(-2\zeta(2\ell)(N^{2\ell} - 1) - \pi^{2\ell}\delta_{2\ell}(0)(N - 1)\right) \\ &\quad = (-1)^k N - (-1)^k 2^{2k} \sum_{j_0=0}^{k} \left(\sum_{\substack{j_1,\ldots,j_{2k} \geqslant 0 \\ j_0 + j_1 + \cdots + j_{2k} = k}} \prod_{r=0}^{2k} \frac{B_{2j_r}}{(2j_r)!} \right) N^{2j_0}. \end{aligned} \tag{2.20}$$

Both sides are a polynomial in N and since this identity is valid for all $N > 1$, we obtain

$$-2\delta_{2k}^*(2\ell)\zeta(2\ell)\pi^{-2\ell} = -(-1)^k 2^{2k} \frac{B_{2\ell}}{(2\ell)!} \sum_{\substack{j_1,\ldots,j_{2k} \geqslant 0 \\ \ell + j_1 + \cdots + j_{2k} = k}} \prod_{r=1}^{2k} \frac{B_{2j_r}}{(2j_r)!}$$

by comparing coefficients. Note that by the classical result

$$\zeta(2\ell) = (-1)^{\ell-1} \frac{(2\pi)^{2\ell} B_{2\ell}}{2(2\ell)!}, \qquad \ell = 1, 2, 3, \ldots,$$

this is equivalent to

$$\delta^*_{2k}(2\ell)2^{2\ell}(-1)^\ell\frac{B_{2\ell}}{(2\ell)!} = (-1)^{k+1}2^{2k}\frac{B_{2\ell}}{(2\ell)!}\sum_{\substack{j_1,\dots,j_{2k}\geqslant 0\\ \ell+j_1+\cdots+j_{2k}=k}}\prod_{r=1}^{2k}\frac{B_{2j_r}}{(2j_r)!}$$

and with $B_{2\ell}\neq 0$ formula (2.16) follows easily.

The proof of formula (2.17) works similar. Take a positive integer $N\equiv 0 \pmod 4$, set $a=\frac{N}{4}$ and

$$\eta_N(z) := \sum_{j=1}^{N-1}\sin\left(\frac{\pi j}{2}\right)h_{\frac{j}{N}}(z) = \sum_{j=1}^{N-1}\chi_4(j)h_{\frac{j}{N}}(z),$$

where χ_4 is the non-principal character modulo 4. Clearly η_N is weak with level N. Together with (2.12) and (2.2) we obtain for positive integers $2k-1$

$$-2\sum_{\ell=1}^{k}\delta^*_{2k-1}(2\ell-1)\pi^{1-2\ell}L(\chi_4;2\ell-1)N^{2\ell-1} = \sum_{j=1}^{N-1}\sin\left(\frac{\pi j}{2}\right)\cot^{2k-1}\left(\frac{\pi j}{N}\right)$$

$$= (-1)^k2^{2k-1}\sum_{\substack{j_1,\dots,j_{2k-1},\mu,\nu\geqslant 0\\ 2j_1+\cdots+2j_{2k-1}+\mu+\nu=2k-1}}\left(\frac{N}{4}\right)^\mu N^\nu\frac{1}{\mu!}\frac{B_\nu}{\nu!}\prod_{r=1}^{2k-1}\frac{B_{2j_r}}{(2j_r)!}$$

$$= (-1)^k2^{2k-1}\sum_{\substack{j_0,j_1,\dots,j_{2k-1}\geqslant 0\\ j_0+2j_1+\cdots+2j_{2k-1}=2k-1}}\sum_{a=0}^{j_0}\frac{B_a4^{a-j_0}}{(j_0-a)!a!}\prod_{r=1}^{2k-1}\frac{B_{2j_r}}{(2j_r)!}N^{j_0}.$$

Using the classical formula

$$L(\chi_4;2\ell-1) = (-1)^{\ell-1}\frac{\mathrm{E}_{2\ell-2}\pi^{2\ell-1}}{4^\ell(2\ell-2)!}, \qquad \ell=1,2,3,...,$$

we obtain by comparing coefficients:

$$\frac{(-1)^\ell\delta^*_{2k-1}(2\ell-1)\mathrm{E}_{2\ell-2}}{2^{2\ell-1}(2\ell-2)!} = (-1)^k2^{2k-1}2^{2-4\ell}S_{2\ell-1}\sum_{\substack{j_1,\dots,j_{2k-1}\geqslant 0\\ 2\ell-1+2j_1+\cdots+2j_{2k-1}=2k-1}}\prod_{r=1}^{2k-1}\frac{B_{2j_r}}{(2j_r)!}$$

with

$$S_{2\ell-1} := \sum_{a=0}^{2\ell-1}\frac{B_a4^a}{(2\ell-1-a)!a!}.$$

The identity

$$S_{2\ell-1} = -\frac{\mathrm{E}_{2\ell-2}}{(2\ell-2)!}$$

follows with the fact that

$$\left(\sum_{m=0}^{\infty}\frac{B_m4^m}{m!}x^m\right)\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right) + x\sum_{p=0}^{\infty}\frac{\mathrm{E}_{2p}}{(2p)!}x^{2p} = \frac{4xe^x}{e^{4x}-1}+\frac{2xe^x}{e^{2x}+1} = \frac{2x}{e^x-e^{-x}}$$

is an even function. The formula now follows after a simple rearrangement.

(iii) Looking again at (2.20) we obtain by comparing the coefficients belonging to N:

$$-\sum_{\ell=1}^{k} \delta_{2k}^{*}(2\ell)\delta_{2\ell}(0) = (-1)^k = -i^{2k}.$$

This proves supplementary law (1). On the other hand, making this comparison for the constant terms we find

$$2\sum_{\ell=1}^{k} \delta_{2k}^{*}(2\ell)\zeta(2\ell)\pi^{-2\ell} + \sum_{\ell=1}^{k} \delta_{2k}^{*}(2\ell)\delta_{2\ell}(0) = -(-1)^k 2^{2k} \sum_{\substack{j_1,\ldots,j_{2k}\geqslant 0 \\ j_1+\cdots+j_{2k}=k}} \prod_{r=1}^{2k} \frac{B_{2j_r}}{(2j_r)!}$$

and using supplementary law (1) we immediately see (2).

This completes the proof. □

It is clear by the vanishing of δ^* and δ for arguments $\nu + u \equiv 1 \pmod 2$ that

$$\sum_{\ell=1}^{2k-1} \delta_{2k-1}^{*}(\ell)\delta_{\ell}(0) = 0.$$

Hence

$$\sum_{\ell=1}^{k} \delta_{k}^{*}(\ell)\delta_{\ell}(0) = i^k \frac{1+(-1)^k}{2}$$

and with (2.15) we obtain (2.18).

We want to apply these theorems to make statements about cotangent sums using L-functions. What we need is the following classical result due to Leopoldt.

Theorem 2.17. *Let χ be a primitive character modulo N and k be a positive integer. Put $\delta := \frac{1-\chi(-1)}{2}$. If $k \equiv \delta \pmod 2$, then*

$$L(\chi;k) = (-1)^{1+\frac{k-\delta}{2}} \frac{\mathcal{G}(\chi)}{2i^{\delta}} \frac{B_{k,\overline{\chi}}}{k!} \left(\frac{2\pi}{N}\right)^k.$$

Here the numbers $B_{k,\overline{\chi}}$ are the generalized Bernoulli numbers defined by the identity

$$\sum_{a=1}^{N} \frac{\chi(a) z e^{az}}{e^{Nz}-1} = \sum_{n=0}^{\infty} \frac{B_{n,\chi}}{n!} z^n.$$

Remark 2.18. *Let χ be a character modulo N. Note that we can express $B_{n,\chi}$ in terms of the standard Bernoulli numbers by the formula*

$$B_{n,\chi} = \sum_{j=1}^{N-1} \chi(j) \sum_{u=0}^{n} \binom{n}{u} B_u j^{n-u} N^{u-1}. \tag{2.21}$$

It follows that if χ is real we have $B_{n,\chi} \in \mathbb{Q}$.

We can use this to determine a closed formula for the character cotangent sums

$$C(\chi; m) := \sum_{j=1}^{N-1} \chi(j) \cot^m\left(\frac{\pi j}{N}\right).$$

Corollary 2.19. *Let χ^+ be an even and χ^- be an odd primitive character modulo $N > 1$ and $m \geqslant 1$ be an integer. We have the explicit formulas*

$$C(\chi^+; 2m) = \mathcal{G}(\chi^+) \sum_{\ell=1}^{m} (-1)^\ell 2^{2\ell} \delta^*_{2m}(2\ell) \frac{B_{2\ell,\overline{\chi^+}}}{(2\ell)!}. \tag{2.22}$$

and

$$C(\chi^-; 2m-1) = i\mathcal{G}(\chi^-) \sum_{\ell=1}^{m} (-1)^\ell 2^{2\ell-1} \delta^*_{2m-1}(2\ell-1) \frac{B_{2\ell-1,\overline{\chi^-}}}{(2\ell-1)!}. \tag{2.23}$$

In particular, independently of m, one has

$$\mathcal{G}(\chi^+)^{-1} C(\chi^+; 2m) \in \mathbb{Q}(\chi^+(g_1), ..., \chi^+(g_t)) \subset \mathbb{Q}(e^{\frac{2\pi i}{\varphi(N)}}) \tag{2.24}$$

and

$$i\mathcal{G}(\chi^-)^{-1} C(\chi^-; 2m-1) \in \mathbb{Q}(\chi^-(g_1), ..., \chi^-(g_t)) \subset \mathbb{Q}(e^{\frac{2\pi i}{\varphi(N)}}) \tag{2.25}$$

respectively, where the integers $g_1, ..., g_t$ modulo N are generators of $\mathbb{F}_N^\times$ and $\varphi(N)$ is Euler's totient function.

Proof. Define

$$\omega_{\chi^\pm}(z) := \sum_{j=1}^{N-1} \chi^\pm(j) h_{\frac{j}{N}}(z).$$

Then $\omega_{\chi^\pm}(z)$ is weak and hence $C(\beta_{\omega_{\chi^\pm}}; 0) = 0$. By Theorem 2.17 one obtains

$$\widetilde{L}(\omega_{\chi^+}; 2\ell) = (-1)^{\ell+1} \mathcal{G}(\chi) \frac{B_{2\ell,\overline{\chi^+}}}{(2\ell)!} (2\pi)^{2\ell}$$

and similarly

$$\widetilde{L}(\omega_{\chi^-}; 2\ell-1) = (-1)^{\ell+1} i\mathcal{G}(\chi) \frac{B_{2\ell-1,\overline{\chi^-}}}{(2\ell-1)!} (2\pi)^{2\ell-1}.$$

Note that we obtain an additionally factor 2 (by symmetry) and $N^{2\ell}$ and $N^{2\ell-1}$ (by the residues), respectively, in this calculation. The formulas (2.22) and (2.23) now follow with Theorem 2.15.
To see (2.24) and (2.25) we first note that the right inclusions follow from $g_j^{\varphi(N)} \equiv 1 \pmod N$. By (2.21) we see $B_{n,\overline{\chi}} \in \mathbb{Q}(\chi(g_1), ..., \chi(g_t))$ and with (2.22) and (2.23) we are done. □

Corollary 2.20. *Let p be a prime and χ be the Legendre symbol modulo p. Then we have for all $m \in \mathbb{N}$*

$$\sqrt{p} C(\chi; m) \in \mathbb{Q}.$$

Proof. For the Legendre symbol χ we have the identity

$$\mathcal{G}(\chi) = \begin{cases} \sqrt{p}, & \text{if } p \equiv 1 \pmod 4 \\ i\sqrt{p}, & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Since χ is real, it is rational, and the claim follows with Corollary 2.19. $\square$

There has been lots of effort finding closed values for Gauss sums. The reader may wish to consult for example [16] for an elementary overview.

Example 2.21. *With Mathematica we obtain the identities*

$$\cot^2\left(\frac{\pi}{5}\right) - \cot^2\left(\frac{2\pi}{5}\right) - \cot^2\left(\frac{3\pi}{5}\right) + \cot^2\left(\frac{4\pi}{5}\right) = \frac{8}{\sqrt{5}},$$

$$\cot^6\left(\frac{\pi}{13}\right) - \cot^6\left(\frac{2\pi}{13}\right) + \cot^6\left(\frac{3\pi}{13}\right) + \cot^6\left(\frac{4\pi}{13}\right) - \cot^6\left(\frac{5\pi}{13}\right) + \cot^6\left(\frac{6\pi}{13}\right) - \cot^6\left(\frac{7\pi}{13}\right)$$
$$- \cot^6\left(\frac{8\pi}{13}\right) + \cot^6\left(\frac{9\pi}{13}\right) + \cot^6\left(\frac{10\pi}{13}\right) - \cot^6\left(\frac{11\pi}{13}\right) + \cot^6\left(\frac{12\pi}{13}\right) = \frac{31832}{\sqrt{13}},$$

and

$$\cot^{13}\left(\frac{\pi}{7}\right) + \cot^{13}\left(\frac{2\pi}{7}\right) - \cot^{13}\left(\frac{3\pi}{7}\right) + \cot^{13}\left(\frac{4\pi}{7}\right) - \cot^{13}\left(\frac{5\pi}{7}\right) - \cot^{13}\left(\frac{6\pi}{7}\right) = \frac{494370}{49\sqrt{7}}.$$

Also we can use the results about cotangent sums to derive properties about L-functions having trigonometric coefficients.

Corollary 2.22. *Let* $\widetilde{\cot} = \cot$ *except* $\widetilde{\cot}(\pi n) := 0$ *for all* $n \in \mathbb{Z}$. *Let* $N > a \geqslant 1$ *and* $n_1, n_2, n_3 \geqslant 0$ *be integers such that* $n_1 n_2 = 0$. *We then have for* $k \geqslant 1$ *with* $n_1 + n_3 \equiv k \pmod 2$*:*

$$\sum_{n=1}^{\infty} \frac{\sin^{n_1}\left(\frac{2\pi a n}{N}\right) \cos^{n_2}\left(\frac{2\pi a n}{N}\right) \widetilde{\cot}^{n_3}\left(\frac{\pi n}{N}\right)}{n^k} \in \mathbb{Q}\pi^k.$$

Proof. The condition $n_1 + n_3 \equiv r \pmod 2$ implies that the coefficients (when extended to $\mathbb{Z}$) define an even/odd function if and only if r is even/odd. The result now follows with the well-known expressions for $\sin^n$ and $\cos^n$ in terms of linear combinations of multiple arguments sin and cos functions and Theorems 2.2 and 2.14. $\square$

Remark 2.23. *Again, using Theorems 2.2 and 2.15, one can find rather complicated explicit formulas for the above Dirichlet series in terms of the values* $\delta_\nu(u)$.

We can use this formalism to give a purely Fourier analytic proof for Theorem 2.2. Remember the modified *Clausen function*

$$\mathrm{Sl}_{2k-1}(\theta) := \sum_{n=1}^{\infty} \frac{\sin(2\pi\theta n)}{n^{2k-1}}$$

and

$$\mathrm{Sl}_{2k}(\theta) := \sum_{n=1}^{\infty} \frac{\cos(2\pi\theta n)}{n^{2k}}.$$

Using standard Fourier analysis one obtains for $0 \leqslant \theta < 1$:

$$\mathrm{Sl}_{2k-1}(\theta) = \frac{(-1)^k (2\pi)^{2k-1}}{2(2k-1)!} \sum_{j=0}^{2k-1} \binom{2k-1}{j} B_j \theta^{2k-1-j} \tag{2.26}$$

and

$$\mathrm{Sl}_{2k}(\theta) = \frac{(-1)^{k-1} (2\pi)^{2k}}{2(2k)!} \sum_{j=0}^{2k} \binom{2k}{j} B_j \theta^{2k-j}. \tag{2.27}$$

We can now use Theorem 2.15 to find the closed formulas provided in Theorem 2.2. To see this, put $\theta = \frac{a}{k}$ for $0 < a < k$. Consider the function

$$\omega(z) = \sum_{j=1}^{k-1} \cos\left(\frac{2\pi a j}{k}\right) h_{\frac{j}{k}}(z)$$

which lies in $W_{\mathrm{pre}}^{0,-}$. Then, we have for even values $2\ell > 0$

$$\widetilde{L}(\omega; 2\ell) = \sum_{u \not\equiv 0 \pmod k} \cos\left(\frac{2\pi a u}{k}\right) \left(\frac{u}{k}\right)^{-2\ell} = 2\sum_{u=1}^{\infty} \cos\left(\frac{2\pi a u}{k}\right) \left(\frac{u}{k}\right)^{-2\ell} - 2\sum_{u=1}^{\infty} u^{-2\ell}$$

and by (2.27) this equals to

$$-2\zeta(2\ell) + \frac{(-1)^{\ell-1}(2\pi k)^{2\ell}}{(2\ell)!} \sum_{j=0}^{2\ell} \binom{2\ell}{j} B_j \left(\frac{a}{k}\right)^{2\ell-j}.$$

For odd values $2\ell - 1 > 0$ we find $\widetilde{L}(\omega; 2\ell-1) = 0$. Note also that the sum $C(\omega; 0) = -1$ for obvious reasons. Hence, with Theorem 2.15 we find

$$C(\omega; 2m) + (-1)^m = -\sum_{\ell=1}^{m} \delta_{2m}^{*}(2\ell) \widetilde{L}(\omega; 2\ell) \pi^{-2\ell}. \tag{2.28}$$

By supplementary law (2) we have

$$2\sum_{\ell=1}^{m} \delta_{2m}^{*}(2\ell)\zeta(2\ell)\pi^{-2\ell} = (-1)^m + (-1)^{m+1} 2^{2m} \sum_{\substack{j_1,\dots,j_{2m} \geqslant 0 \\ 2j_1+\cdots+2j_{2m}=2m}} \prod_{r=1}^{2m} \frac{B_{2j_r}}{(2j_r)!}. \tag{2.29}$$

On the other hand, a straightforward calculation shows

$$\begin{aligned} &- \delta_{2m}^{*}(2\ell) \frac{(-1)^{\ell-1}(2\pi k)^{2\ell}}{(2\ell)!} \sum_{j=0}^{2\ell} \binom{2\ell}{j} B_j \left(\frac{a}{k}\right)^{2\ell-j} \\ &= (-1)^{m+1} 2^{2m} \sum_{\mu=0}^{2\ell} \sum_{\substack{j_1,\dots,j_{2m} \geqslant 0 \\ 2\ell+2j_1+\cdots+2j_{2m}=2m}} \frac{a^{2\ell-\mu} k^{\mu} B_{\mu}}{(2\ell-\mu)!\mu!} \prod_{r=1}^{2m} \frac{B_{2j_r}}{(2j_r)!}. \end{aligned}$$

The cosine formula of Theorem 2.2 follows now by summing this over $\ell = 1, \dots, m$, making the substitution $2\ell = \nu + \mu$ and adding everything together. Note that the $(-1)^m$ in

(2.28) will cancel with that of (2.29) and that the formula (2.29) due to supplementary law (2) is just the case $2\ell = \mu + \nu = 0$, completing the sum in (2.3). Similarly, we can show the sine formula (2.2) in full generality.

Remark 2.24. *Note that we only have used the polynomials P_m and Q_m defined by*

$$P_m(N) = \sum_{j=1}^{N-1} \cot^m\left(\frac{\pi j}{N}\right)$$
$$Q_m(N) = \sum_{j=1}^{4N-1} \chi_4(j)\cot^m\left(\frac{\pi j}{4N}\right)$$

in the proof of Theorem 2.15.

3. The space $W_{\mathrm{pre},\infty}$ and applications

The proof of Theorem 1.4 did not use the order of the poles that occurred, only their locations. This motivates us to generalize the concept of pre-weak functions in the sense, that we allow them to have poles of arbitrary order. In this section we investigate analogous transformation laws for this kind of situation and will apply this to specific types of q-series, see also Theorem 3.11.

Definition 3.1. *We call a meromorphic function ω pre-weak of degree d, if all conditions for pre-weak functions are satisfied except that ω has a pole of order d (and all other poles have order at most d). We denote the vector space of pre-weak functions with degree at most a with $W_{\mathrm{pre},a}$. We collect all pre-weak functions of arbitrary degree in the space*

$$W_{\mathrm{pre},\infty} = \bigcup_{a=1}^{\infty} W_{\mathrm{pre},a}.$$

Even in the higher degree situation, we will still use the notation

$$\vartheta_k(\omega\otimes\eta;\tau) = -2\pi i \sum_{x\in\mathbb{R}^\times} \mathrm{res}_{z=x}\left(z^{k-1}\eta(z)\omega(z\tau)\right).$$

Like in the special case $a = 1$ it is quite easy to classify all pre-weak functions of degree at most a using elementary complex analytic ideas. For this purpose we abbreviate

$$h_{x,\ell}(z) = \frac{e(z)}{(e(x)-e(z))^\ell}. \tag{3.1}$$

We now find that there are uniquely determined functions $\beta_j : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$, $1 \leqslant j \leqslant a$, such that

$$\omega(z) = \omega(i\infty) + \sum_{j=1}^{a}\sum_{x\in\mathbb{R}/\mathbb{Z}} \beta_j(x)h_{x,j}(z).$$

In other words, there is an isomorphism

$$W_{\mathrm{pre},\infty} \cong \mathbb{C} \oplus \bigoplus_{\ell\geqslant 1} (\mathbb{R}/\mathbb{Z})^{\mathbb{C}_0}.$$

As we will see later, it is natural to study transformations of rational functions when applying the differential $\partial = \frac{1}{2\pi i}\frac{\partial}{\partial z}$. Note that $h_{x,\ell}(z)$ satisfies the differential equation

$$\partial h_{x,\ell}(z) = (1-\ell)h_{x,\ell}(z) + \ell e(x) h_{x,\ell+1}(z). \tag{3.2}$$

We define the projection $\pi_1 : W^{i\infty}_{\mathrm{pre},\infty} \to W^{i\infty}_{\mathrm{pre},1}$ by

$$\pi_1\left(\sum_{\ell=1}^{N_\omega}\sum_{x\in\mathbb{R}/\mathbb{Z}} \beta_{\omega,\ell}(x)h_{x,\ell}(z)\right) = \sum_{x\in\mathbb{R}/\mathbb{Z}} \beta_{\omega,1}(x)h_{x,1}(z).$$

This implies

Proposition 3.2. *We have the exact sequence*

$$0 \longrightarrow W^{i\infty}_{\mathrm{pre},\infty} \xrightarrow{\partial} W^{i\infty}_{\mathrm{pre},\infty} \xrightarrow{\pi_1} W^{i\infty}_{\mathrm{pre},1} \longrightarrow 0.$$

Proof. It is clear that π_1 is onto and that the extended homomorphism $W_{\mathrm{pre},\infty} \xrightarrow{\partial} W_{\mathrm{pre},\infty}$ has kernel $\mathbb{C}$. Since $W^{i\infty}_{\mathrm{pre},\infty} \cap \mathbb{C} = 0$, it follows that ∂ is injective.
To see $\mathrm{im}(\partial) \subset \ker(\pi_1)$ we observe that

$$\partial \sum_{\ell=1}^{r}\sum_{x\in\mathbb{R}/\mathbb{Z}} \beta_\ell(x)h_{x,\ell}(z) = \sum_{\ell=1}^{r}\sum_{x\in\mathbb{R}/\mathbb{Z}} \beta_\ell(x)((1-\ell)h_{x,\ell}(z) + \ell e(x)h_{x,\ell+1}(z))$$

has no non-vanishing term $h_{x,1}(z)$. On the other hand, if $f \in \ker(\pi_1)$, it is of the form

$$f(z) = \sum_{\ell=2}^{r}\sum_{x\in\mathbb{R}/\mathbb{Z}} \gamma_\ell(x)h_{x,\ell}(z).$$

Again with (3.2) we inductively see that there is a $g \in W^{i\infty}_{\mathrm{pre},\infty}$ such that $\partial(g) = f$. $\square$

Together with $W_{\mathrm{pre},1} \cong \mathbb{C} \oplus W^{i\infty}_{\mathrm{pre},1}$ we obtain the following.

Corollary 3.3. *We have a canonical isomorphism*

$$W_{\mathrm{pre},\infty} \cong \mathbb{C} \oplus \bigoplus_{n\geqslant 0} \partial^n W^{i\infty}_{\mathrm{pre},1}.$$

Together with the isomorphism

$$W^{i\infty}_{\mathrm{pre},1} \cong W_{\mathrm{weak},1} \oplus \mathbb{C}h_{0,1}$$

we quickly obtain

$$\partial W^{i\infty}_{\mathrm{pre},1} \cong \partial W_{\mathrm{weak},1} \oplus \mathbb{C}h_{0,2}.$$

Putting everything together we obtain the following decompositions.

Corollary 3.4. *Let a be an integer. Then we have the decompositions*

$$W_{\mathrm{weak},a} \cong W_{\mathrm{weak},1} \oplus \bigoplus_{n=1}^{a-1} \partial^n W^{i\infty}_{\mathrm{pre},1} \cong W_{\mathrm{weak},1} \oplus \bigoplus_{n=0}^{a-2} \partial^n \left(\partial W_{\mathrm{weak},1} \oplus \mathbb{C}h_{0,2}\right).$$

At some stage it will be crucial to change from $W_{\mathrm{weak},\infty}$ to $W_{\mathrm{pre},\infty}$ in the sense of decompositions into derivatives. This is done in the obvious way.

Proposition 3.5. *Let $\omega \in W_{\mathrm{weak},a}$. Then we have the following identity between decompositions provided by Corollary 3.4:*

$$\omega = \lambda_0 + \sum_{j=1}^{a-1} \partial^j \lambda_j = \lambda_0 + \sum_{j=1}^{a-1} \partial^{j-1}(\partial\omega_j + c_j h_{0,2}),$$

where $\lambda_0, \omega_j \in W_{\mathrm{weak},1}$, $\lambda_j \in W^{i\infty}_{\mathrm{pre},1}$ for $1 \leqslant j \leqslant a-1$. As a result, we get for all $0 \leqslant j \leqslant a-1$ the corresponding coefficients

$$\beta_{\lambda_j}(y) = \begin{cases} \beta_{\omega_j}(y), & y \neq 0, \\ \beta_{\omega_j}(0) + c_j, & y = 0, \end{cases}$$

where $c_0 := 0$.

The next lemma provides some useful differential identities.

Lemma 3.6. *Let be $k \in \mathbb{Z}$ and $\omega \otimes \eta \in W^{\otimes}_{(k)}$.*

(i) We have $\vartheta_k(\partial_z\omega \otimes \eta; \tau) = \partial_\tau \vartheta_{k-1}(\omega \otimes \eta; \tau)$.

(ii) We have $\vartheta_k(\omega \otimes \partial_z\eta; \tau) = \frac{1}{2\pi i}(1 - k - \tau\frac{\partial}{\partial\tau})\vartheta_{k-1}(\omega \otimes \eta; \tau)$.

Proof. Since interchanging residue and differential operator is legitimated we easily see

$$\partial_\tau \sum_{\alpha\in\mathbb{R}/\mathbb{Z}} \mathrm{res}_{z=\alpha}\left(z^{k-2}\eta(z)\omega(\tau z)\right) = \sum_{\alpha\in\mathbb{R}/\mathbb{Z}} \mathrm{res}_{z=\alpha}\left(z^{k-1}\eta(z)\frac{1}{2\pi i}\omega'(\tau z)\right).$$

This proves (i).
For (ii) let $f(z) = z^{k-1}\omega(\tau z)$. Then we note

$$0 = \mathrm{res}_{z=z_0}((f(z)\eta(z))') = \mathrm{res}_{z=z_0} f(z)\eta'(z) + \mathrm{res}_{z=z_0} f'(z)\eta(z)$$

and hence

$$\begin{aligned} \vartheta_k(\omega \otimes \eta'; \tau) &= 2\pi i \sum_{\alpha\in\mathbb{R}/\mathbb{Z}} \mathrm{res}_{z=\alpha}\left((k-1)z^{k-2}\omega(\tau z)\eta(z) + z^{k-1}\tau\omega'(\tau z)\eta(z)\right) \\ &= (1-k)\vartheta_{k-1}(\omega \otimes \eta; \tau) - \tau\vartheta_k(\omega' \otimes \eta; \tau) \\ &= \left((1-k) - \tau\frac{\partial}{\partial\tau}\right)\vartheta_{k-1}(\omega \otimes \eta; \tau), \end{aligned}$$

according to (i). □

As an application of the more general formalism we want to give a description of a special case of the main transformation law in the language of series of rational functions. To make things more explicit, we are going to use differentials of the form

$$w_0 + w_1\tau\frac{\partial}{\partial\tau} + w_2\tau^2\frac{\partial^2}{\partial\tau^2} + \cdots + w_n\tau^n\frac{\partial^n}{\partial\tau^n}, \qquad w_i \in \mathbb{C},$$

and apply the results of Lemma 3.6. Since the lemma tells us

$$\vartheta_k(\omega \otimes \partial_z\eta; \tau) = \frac{1}{2\pi i}\left(1 - k - \tau\frac{\partial}{\partial\tau}\right)\vartheta_{k-1}(\omega \otimes \eta; \tau),$$

it seems reasonable to look at differentials

$$\begin{aligned} D_{k,n} &= (2\pi i)^{-n}(1-k-\tau\frac{\partial}{\partial\tau})(2-k-\tau\frac{\partial}{\partial\tau})\cdots(n-k-\tau\frac{\partial}{\partial\tau}) \\ &= (2\pi i)^{-n}\sum_{\ell=0}^{n}\left(\sum_{j=0}^{n}(-1)^n \left\{ {j \atop \ell} \right\} \kappa_{1-k,n-k}(j)\right)\tau^\ell \frac{\partial^\ell}{\partial\tau^\ell} \end{aligned}$$

to find that

$$\vartheta_k(\omega\otimes\partial_z^n\eta;\tau) = D_{k,n}\vartheta_{k-n}(\omega\otimes\eta;\tau).$$

Here $\left\{ {j \atop \ell} \right\}$ denote the Stirling numbers of the second kind and for integers $b \geqslant a-1$ the numbers $\kappa_{a,b}(j)$ are defined by

$$(X-a)(X-a-1)\cdots(X-b) = \sum_{j=0}^{b-a+1}\kappa_{a,b}(j)X^j.$$

We abbreviate $s(n,\ell) := (2\pi i)^{\ell-n-1}\sum_{j=0}^{n}(-1)^{n+1}\left\{ {j \atop \ell} \right\}\kappa_{1-k,n-k}(j)$.

It is remarkable that we still obtain a simple modular relationship between $\vartheta_k(\omega\otimes\eta;\tau)$ and $\vartheta_k(\eta\otimes\widehat{\omega};\tau)$, as it was the case in Theorem 1.4.

Theorem 3.7. *Let $\omega\otimes\eta \in W_{\text{weak},\infty}\otimes W_{\text{weak},\infty}$, where ω and η have the Laurent expansions*

$$\omega(z) = \sum_{n=-U}^{\infty} a_n z^n,$$

$$\eta(z) = \sum_{n=-V}^{\infty} b_n z^n.$$

We then have the identity

$$\vartheta_k\left(\omega\otimes\eta;-\frac{1}{\tau}\right) = (-1)^{k-1}\tau^k\vartheta_k(\eta\otimes\widehat{\omega};\tau) + 2\pi i\sum_{c=0}^{U+V-k} b_{c-V}a_{V-k-c}(-1)^{V-c}\tau^{V-c}.$$

Proof. The proof is essentially the same as the one of Theorem 1.4. We may choose $\tau = iy$ with $y > 0$ and use the rapid decay of the functions ω and η for increasing imaginary parts to show

$$\begin{aligned} \frac{1}{2\pi i}\oint_{|z|=N+\varepsilon} z^{k-1}\eta(z)\omega(z\tau)\mathrm{d}z &= \sum_{\substack{-N\leqslant x\leqslant N \\ x\neq 0}}\left(\mathrm{res}_{z=x} + \mathrm{res}_{z=-\frac{x}{\tau}}\right)\left(z^{k-1}\eta(z)\omega(z\tau)\right) \\ &\quad + \mathrm{res}_{z=0}\left(z^{k-1}\eta(z)\omega(z\tau)\right) = o(1), \end{aligned}$$

where $\varepsilon > 0$ is fixed and sufficiently small (note that ω and η are periodic and only have real poles).

Put $g_\tau(z) := z^{k-1}\eta(z)\widehat{\omega}(z\tau)$ and $h_\tau(z) := z^{k-1}\omega(z)\eta(z\tau)$. For each $\tau \in \mathbb{H}$ we obtain the functional equation

$$g_\tau\left(-\frac{z}{\tau}\right) = (-\tau)^{1-k}z^{k-1}\eta\left(-\frac{z}{\tau}\right)\widehat{\omega}(-z) = (-\tau)^{1-k}h_{-\frac{1}{\tau}}(z).$$

Hence

$$\operatorname{res}_{z=-\frac{x}{\tau}}\left(g_\tau(z)\right) = -\frac{1}{\tau}\operatorname{res}_{z=x}\left(g_\tau\left(-\frac{z}{\tau}\right)\right) = (-\tau)^{-k}\operatorname{res}_{z=x}\left(h_{-\frac{1}{\tau}}(z)\right) \tag{3.3}$$

by the linearity of the residue. For the residue in $z=0$ we obtain

$$\begin{aligned}\operatorname{res}_{z=0}\left(g_\tau(z)\right) &= \operatorname{res}_{z=0}\left(z^{k-U-V-1}\left(\sum_{\ell=0}^{\infty} b_{\ell-V}z^\ell\right)\left(\sum_{\ell=0}^{\infty}(-1)^{\ell-U}a_{\ell-U}\tau^{\ell-U}z^\ell\right)\right)\\ &= \operatorname{res}_{z=0}\left(z^{k-U-V-1}\left(\sum_{\ell=0}^{\infty}\left(\sum_{c=0}^{\ell} b_{c-V}a_{\ell-c-U}(-1)^{\ell-c-U}\tau^{\ell-c-U}\right)z^\ell\right)\right)\\ &= \sum_{c=0}^{U+V-k} b_{c-V}a_{V-k-c}(-1)^{V-k-c}\tau^{V-k-c}\end{aligned}$$

and hence

$$-2\pi i\sum_{x\in\mathbb{R}^\times}\operatorname{res}_{z=x}\left(g_\tau(z)\right) - 2\pi i\sum_{x\in\mathbb{R}^\times}\operatorname{res}_{z=-\frac{x}{\tau}}\left(g_\tau(z)\right) - 2\pi i\ \operatorname{res}_{z=0}\left(g_\tau(z)\right) = 0,$$

and by (3.3) this implies

$$\vartheta_k(\eta\otimes\widehat{\omega};\tau) - 2\pi i\sum_{c=0}^{U+V-k} b_{c-V}a_{V-k-c}(-1)^{V-k-c}\tau^{V-k-c} = -(-\tau)^{-k}\vartheta_k\left(\omega\otimes\eta;-\frac{1}{\tau}\right).$$

Multiplying this by $(-1)^{k-1}\tau^k$ proves the claim. □

This framework can be used to derive transformation laws of "higher" functions $\vartheta_k(\omega\otimes\eta;\tau)$, where $\omega(z)$ and $\eta(z)$ are allowed to have poles of higher degree. The outcomes are functions of the form

$$f(\tau) = g_0(\tau) + \tau g_1(\tau) + \cdots + \tau^n g_n(\tau),$$

where the $g_j(\tau)$ are Fourier series on the upper half plane, such that the $f(\tau)$ possess non-trivial transformation properties. We will omit the details of this extremely technical setup but will give examples in order to convince the reader of its usefulness. We will not use Theorem 3.7 in full generality and show examples with rational poles and lower degrees.

Example 3.8. *Let $k \geqslant 6$ be an even integer. Put*

$$\omega(z) := \csc(2\pi z)$$

and

$$\eta(z) := i\cot(2\pi z)\csc(2\pi z).$$

Then we have

$$\partial_z(\omega(z)) = \eta(z)$$

and hence obtain

$$\vartheta_k(\partial_z\omega\otimes\partial_z\omega;\tau) = \partial_\tau\vartheta_{k-1}(\omega\otimes\partial_z\omega;\tau) = \frac{1}{2\pi i}\partial_\tau\left(2-k-\tau\frac{\partial}{\partial\tau}\right)\vartheta_{k-2}(\omega\otimes\omega;\tau).$$

This equals to

$$-\frac{1}{4\pi^2}\left((1-k)\frac{\partial}{\partial\tau}\vartheta_{k-2}(\omega\otimes\omega;\tau)-\tau\frac{\partial^2}{\partial\tau^2}\vartheta_{k-2}(\omega\otimes\omega;\tau)\right).$$

One could now use the transformation properties of $\vartheta_{k-2}(\omega\otimes\omega;\tau)$ given in Theorem 1.4 to make final conclusions. But we will use Theorem 3.7 to investigate $\vartheta_k(\eta\otimes\eta;\tau)$. Let n be a non-zero integer. We obtain with the series expansions

$$z^{k-1}\eta(z\tau)=A_0+A_1\left(z-\frac{n}{2}\right)+O\left(\left(z-\frac{n}{2}\right)^2\right)$$

with

$$A_0=i\left(\frac{n}{2}\right)^{k-1}\cot(n\pi\tau)\csc(n\pi\tau)$$

$$A_1=i(k-1)\left(\frac{n}{2}\right)^{k-2}\cot(n\pi\tau)\csc(n\pi\tau)-2i\left(\frac{n}{2}\right)^{k-1}\left(\pi\tau\csc(n\pi\tau)+2\pi\tau\cot^2(n\pi\tau)\csc(n\pi\tau)\right)$$

and

$$\eta(z)=\frac{i(-1)^n}{4\pi^2(z-\frac{n}{2})^2}+O(1)$$

for $k\geqslant 6$:

$$\begin{aligned}4\pi^2\mathrm{res}_{x=\frac{n}{2}}\left(z^{k-1}\eta(z)\eta(z\tau)\right)&=(-1)^{n+1}(k-1)\left(\frac{n}{2}\right)^{k-2}\cot(n\pi\tau)\csc(n\pi\tau)\\&\quad+2(-1)^n\left(\frac{n}{2}\right)^{k-1}\left(\pi\tau\csc(n\pi\tau)+2\pi\tau\cot^2(n\pi\tau)\csc(n\pi\tau)\right).\end{aligned}$$

Since we have

$$\begin{aligned}\cot(\pi n\tau)\csc(\pi n\tau)&=-\frac{2(q^n+1)q^{\frac{n}{2}}}{(q^n-1)^2},\\\csc(n\pi\tau)&=\frac{2iq^{\frac{n}{2}}}{q^n-1},\\\cot^2(n\pi\tau)\csc(n\pi\tau)&=-\frac{2i(q^n+1)^2q^{\frac{n}{2}}}{(q^n-1)^3},\end{aligned}$$

we obtain by symmetry and Theorem 3.7 (note that $\mathrm{sgn}(\eta)=1$), that $f_k(\tau)$ with series representation

$$(k-1)\sum_{n=1}^{\infty}(-1)^nn^{k-2}\frac{(1+q^n)q^{\frac{n}{2}}}{(1-q^n)^2}+\pi i\tau\sum_{n=1}^{\infty}(-1)^nn^{k-1}\frac{(1+6q^n+q^{2n})q^{\frac{n}{2}}}{(1-q^n)^3}$$

satisfies

$$f_k\left(-\frac{1}{\tau}\right)=-\tau^kf_k(\tau).$$

Example 3.9. *This example is very similar to Example 3.8, we choose*

$$\omega(z)=\eta(z):=\csc^2(2\pi z)$$

this time. The main difference is that $\omega(z)$ has no integral function that is weak, since the integral is given by $-\frac{1}{2\pi}\cot(2\pi z)+C$, compare also the result of Corollary 3.4. Let $k \geqslant 6$ be even. Very similar to Example 3.8 we find for

$$f_k(\tau) := (k-1)\sum_{n=1}^{\infty} n^{k-2}\frac{q^n}{(1-q^n)^2} + 2\pi i\tau \sum_{n=1}^{\infty} n^{k-1}\frac{(1+q^n)q^n}{(1-q^n)^3},$$

the transformation law

$$f_k\left(-\frac{1}{\tau}\right) = -\tau^k f_k(\tau).$$

Definition 3.10. *We say that a holomorphic q-series*

$$f(\tau) := \sum_{n=0}^{\infty} a(n) q^{\frac{n}{N}}$$

on the upper half plane has rational type (M,N), if there is a N-periodic arithmetic function $\psi(n)$, a polynomial P and a rational function R with poles only in $\{z=\zeta_M^j, 0 \leqslant j < M\}$ and $R(\infty)=R(0)=0$, such that

$$f(\tau) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \psi(n)P(n)R\left(q^{\frac{n}{N}}\right).$$

Theorem 3.11 (Transformation law for rational type q-series)**.** *Let f be a (M,N)-rational type q-series with periodic function satisfying $\sum_{j=1}^{N}\psi(j)=0$, and with rational R which has poles of degree at most a. Put $\delta(\psi)=0$ if $\psi(0)=0$ and $\delta(\psi)=1$, else. Then there is a polynomials $Q_{-1}(X)$ of degree at most $-\mathrm{ord}_{X=1}(R)-\mathrm{ord}_{X=0}(P)-1$, and complex numbers A_0 and A_1, such that*

$$f\left(-\frac{1}{\tau}\right) = Q_{-1}\left(-\frac{1}{\tau}\right) + A_0 + \delta(\psi)A_1\tau + \tau^{\deg(P)+1}\sum_{j=0}^{a-1}\tau^j s_j(\tau),$$

where each $s_j(\tau)$ is a finite sum of q-series of rational type (N,M).

Proof. We are able to present a constructive proof, but we will only sketch the ideas of construction. Without loss of generality, we assume $P(n)=n^{k-1}$ with an arbitrary integer $k>0$. Hence $\mathrm{ord}_{X=0}(P)=k-1$. For each (M,N)-rational type series with the additional assumption $\sum_{j=1}^{N}\psi(j)=0$ we find weak functions

$$\eta(z) := N^{k-1}\sum_{j=1}^{N}\psi(j)h_{\frac{j}{N}}(z)$$

and

$$\omega(z) := R(e(z))$$

such that

$$f(\tau) = \vartheta_k(\omega\otimes\eta;\tau).$$

With Theorem 3.7 we find polynomials $Q_{-1}(X)$ and $Q_1(X)$ such that

$$f\left(-\frac{1}{\tau}\right) = Q_{-1}\left(-\frac{1}{\tau}\right) + Q_1(\tau) + (-1)^{k-1}\tau^k\vartheta_k(\eta\otimes\widehat{\omega};\tau).$$

But since $V \leqslant 1$ and $V \leqslant 0$ if and only if $\delta(\psi) = 0$, we see that Q_1 has degree at most 1 and 1 only if $\delta(\psi) = 1$. On the other hand, also by Theorem 3.7, $Q_{-1}(X)$ has degree at most $U - k = -\mathrm{ord}_{X=1}(R) - \mathrm{ord}_{X=0}(P) - 1$ (note that $\mathrm{ord}_{X=0}(P)$ is the correct measure at this point, since if P had more higher degree terms then the degree of Q_{-1} would be smaller for these terms). For any fixed $x = \frac{j}{M} \neq 0$, consider the expansions

$$\widehat{\omega}(z) = \sum_{\nu=-V}^{\infty} b_\nu\left(\frac{j}{M}\right)\left(z - \frac{j}{M}\right)^{\nu},$$
$$\eta(z\tau) = \sum_{u=0}^{\infty} \frac{\tau^u \eta^{(u)}\left(\frac{j\tau}{M}\right)}{u!}\left(z - \frac{j}{M}\right)^{u}.$$

With this we obtain, that $\mathrm{res}_{z=\frac{j}{M}}\left(z^{k-1}\widehat{\omega}(z)\eta(z\tau)\right)$ equals

$$\mathrm{res}_{z=\frac{j}{M}}\left(\sum_{\mu,\nu+V,u\geqslant 0}\binom{k-1}{\mu}\left(\frac{j}{M}\right)^{k-1-\mu} b_\nu\left(\frac{j}{M}\right)\frac{\tau^u\eta^{(u)}\left(\frac{j\tau}{M}\right)}{u!}\left(z-\frac{j}{M}\right)^{\mu+\nu+u}\right).$$

For any triple $\mu + \nu + u = -1$ with $0 \leqslant \mu \leqslant k-1$ this is essentially of the form

$$j^{k-1-\mu} b_\nu\left(\frac{j}{M}\right)\tau^u\eta^{(u)}\left(\frac{j\tau}{M}\right),$$

and since $\eta^{(u)}$ is weak again, hence of the form $W(q^{\frac{j}{M}})$ (note that $W(0) = W(\infty) = 0$ and W may only have poles in roots of unity ζ_N^j) and $\beta_\nu(x)$ is 1-periodic, we may sum this over all $j' \equiv j \pmod{M}$ to obtain a (N, M)-rational type series

$$\tau^u \sum_{j'\in\mathbb{Z}\setminus\{0\}} j'^{k-1-\mu}\widetilde{b_\nu}(j') W\left(q^{\frac{j'}{M}}\right),$$

where the M-periodic $\widetilde{b_\nu}(j')$ takes the value $b_\nu\left(\frac{j'}{M}\right)$ if $j' \equiv j \pmod{M}$ and 0 else. Summing up all the terms shows the claim, since $0 \leqslant u \leqslant a-1$ is not negative. $\square$

Finally, we give one more example.

Example 3.12. *Consider the weak functions* $\omega(z) := \csc^3(2\pi z)$ *and* $\eta(z) := \csc(2\pi z)$*. Put* $P(n) := n^{k-1}$ *with some even integer* $k \geqslant 6$*. This implies* $a = 3$*,* $\mathrm{ord}_{X=0}(P) = \deg(P) = k-1$*,* $V = 1$ *and* $U = 3$*. Following Theorem 3.11, the q-series*

$$f_k(\tau) := -2\pi i\sum_{n=1}^{\infty}\frac{1}{\pi}(-1)^n\left(\frac{n}{2}\right)^{k-1}\frac{(2i)^3 q^{\frac{3n}{2}}}{(q^n-1)^3} = 16\sum_{n=1}^{\infty}(-1)^n\left(\frac{n}{2}\right)^{k-1}\frac{q^{\frac{3n}{2}}}{(1-q^n)^3},$$

which is essentially $\vartheta_k(\omega\otimes\eta;\tau)$*, satisfies the transformation law*

$$f_k\left(-\frac{1}{\tau}\right) = -\tau^k(g_1(\tau)+g_4(\tau)) - \tau^{k+1}g_2(\tau) - \tau^{k+2}g_3(\tau),$$

where

$$g_1(\tau) := 2(-2\pi i)\sum_{n=1}^{\infty}\frac{(-1)^n}{4\pi}(-\csc(\pi n\tau))\left(\frac{n}{2}\right)^{k-1} = 2\sum_{n=1}^{\infty}(-1)^n\left(\frac{n}{2}\right)^{k-1}\frac{q^{\frac{n}{2}}}{1-q^n},$$

$$g_2(\tau) := 2(-2\pi i)\sum_{n=1}^{\infty}\frac{(-1)^n}{8\pi^3}2\pi\cot(\pi n\tau)\csc(\pi n\tau)(k-1)\left(\frac{n}{2}\right)^{k-2}$$
$$= \frac{2i(k-1)}{\pi}\sum_{n=1}^{\infty}(-1)^n\left(\frac{n}{2}\right)^{k-2}\frac{(1+q^n)q^{\frac{n}{2}}}{(1-q^n)^2},$$

$$g_3(\tau) := 2(-2\pi i)\sum_{n=1}^{\infty}\frac{(-1)^n}{8\pi^3}(-2)\left(\pi^2\csc(\pi n\tau)+2\pi^2\cot^2(\pi n\tau)\csc(\pi n\tau)\right)\left(\frac{n}{2}\right)^{k-1}$$
$$= 2\sum_{n=1}^{\infty}(-1)^n\left(\frac{n}{2}\right)^{k-1}\left(\frac{q^{\frac{n}{2}}}{1-q^n}-\frac{2(1+q^n)^2q^{\frac{n}{2}}}{(1-q^n)^3}\right),$$

$$g_4(\tau) := 2(-2\pi i)\sum_{n=1}^{\infty}\frac{(-1)^n}{8\pi^3}(-\csc(\pi n\tau))\frac{(k-1)(k-2)}{2}\left(\frac{n}{2}\right)^{k-3}$$
$$= \frac{(k-1)(k-2)}{2\pi^2}\sum_{n=1}^{\infty}(-1)^n\left(\frac{n}{2}\right)^{k-3}\frac{q^{\frac{n}{2}}}{1-q^n}.$$

Note that we were able to start summation at $n=1$ by symmetry.

4. Eichler duality

Let $k \geqslant 2$ be an integer. In this last section we develop an explicit formula for the $k-1$-fold integral of $\vartheta_k(\omega\otimes\eta;\tau)$ in the case $\omega\otimes\eta \in W_{\mathrm{weak},a}[\mathcal{T}_N]\otimes W^{i\infty}_{\mathrm{pre},1}[\mathcal{T}_M]$. On the rational function side it is given by a duality using Fourier transforms. In the following we give the definition of an m-fold integral as we will use it.

Definition 4.1. *Let $m \geqslant 0$ be an integer. Then we define the m-fold integral map $\int_m$ by*

$$\int_m : \mathbb{C}_0^+[[q^{1/M}]] \longrightarrow \mathbb{C}_0^+[[q^{1/M}]]$$
$$f(\tau) = \sum_{n=1}^{\infty} a_f(n)q^{n/M} \longmapsto M^m\sum_{n=1}^{\infty}a_f(n)n^{-m}q^{n/M}.$$

Note that this is the inverse function of ∂_τ^m defined on $\mathbb{C}_0^+[[q]]$.

Before we start we shortly introduce the Fourier transform of a pre-weak function with rational poles. Let N be an integer. Then we define

$$\mathcal{F}_N : W^{i\infty}_{\mathrm{pre},1}[\mathcal{T}_N] \xrightarrow{\sim} W^{i\infty}_{\mathrm{pre},1}[\mathcal{T}_N]$$
$$\sum_{j=1}^{N}\beta(j/N)h_{j/N} \longmapsto \sum_{j=1}^{N}\mathcal{F}_N(\beta)(j)h_{j/N}.$$

A simple calculation verifies, that the inverse of this isomorphism is given by

$$\mathcal{F}_N^{-1} : W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_N] \xrightarrow{\sim} W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_N]$$

$$\sum_{j=1}^{N} \beta(j/N) h_{j/N} \longmapsto \sum_{j=1}^{N} \mathcal{F}_N^{-1}(\beta)(j) h_{j/N},$$

where

$$\mathcal{F}_N^{-1}(\beta)(j) := \frac{1}{N} \sum_{k=1}^{N} \beta(k/N) e^{2\pi i jk/N}.$$

In order to prove Eichler duality we will introduce the following bracket notation which will simplify a lot.

Definition 4.2. *Let γ be a function in $(\mathbb{R}/\mathbb{Z})^{\mathbb{C}_0}$ and $\beta : \mathbb{R} \to \mathbb{C}$ be bounded. We put*

$$[\beta \otimes \gamma]_{k,\ell}(\tau) = 2 \sum_{t \in \mathbb{R}_{>0}} \left(\sum_{\substack{d_1 \in \mathbb{R}_{>0} \\ d_2 \in \mathbb{N} \\ d_1 d_2 = t}} d_1^{k-1} \gamma(d_1) d_2^{\ell} \beta(d_2) \right) q^t.$$

Note that $[\beta \otimes \gamma]_{k,\ell}$ always represents a holomorphic function on the upper half plane with a zero in $\tau = i\infty$.

In the following we want to find the Fourier expansion of $\vartheta_k(\omega, \eta; \tau)$ in the case that η has degree 1. This case is the most important one for most of our applications such as Eichler integrals. One of our main tools is a certain differential equation satisfied by the above introduced Fourier series.

Remark 4.3. *From now on, if not defined differently, we assume that if some $\omega \otimes \eta \in W_{\mathrm{pre},\infty} \otimes W_{\mathrm{pre},\infty}$ is used together with some integer k we have $\mathrm{sgn}(\omega)\mathrm{sgn}(\eta) = (-1)^k$.*

Lemma 4.4. *We have*

$$\partial_\tau [\beta \otimes \gamma]_{k,\ell} = [\beta \otimes \gamma]_{k+1,\ell+1}.$$

Proof. Since we can differentiate termwise we obtain

$$\begin{aligned}
&\partial_\tau 2M^{1-k} \sum_{m=1}^{\infty} \left(\sum_{d|m} d^{k-1} \gamma(d) \left(\frac{m}{d}\right)^{\ell} \beta\left(\frac{m}{d}\right) \right) e^{2\pi i \tau \frac{m}{M}} \\
&= 2M^{1-k} M^{-1} \sum_{m=1}^{\infty} \left(\sum_{d|m} d^{k-1} d \gamma(d) \left(\frac{m}{d}\right)^{\ell} \frac{m}{d} \beta\left(\frac{m}{d}\right) \right) e^{2\pi i \tau \frac{m}{M}} \\
&= 2M^{-k} \sum_{m=1}^{\infty} \left(\sum_{d|m} d^{k} \gamma(d) \left(\frac{m}{d}\right)^{\ell+1} \beta\left(\frac{m}{d}\right) \right) e^{2\pi i \tau \frac{m}{M}} \\
&= [\beta \otimes \gamma]_{k+1,\ell+1}(\tau).
\end{aligned}$$

□

Proposition 4.5. *Let* $k \equiv \frac{1 \mp 1}{2} \mod 2$ *be an integer and* $\omega \otimes \eta \in W_{(k)}^{\otimes,\pm}[\mathcal{T}_N, \mathcal{T}_M]$. *Then we have for all* τ *on the upper half plane*

(i)

$$\vartheta_k(\omega \otimes \eta; \tau) = A + [\mathcal{F}_N(\beta_\omega) \otimes \beta_\eta]_{k,0}(\tau)$$

where

$$A = \begin{cases} 2\omega^{\pm}(i\infty)L(\eta; 1-k), & k < 0, \\ 0, & k \geqslant 0. \end{cases}$$

(ii)

$$\vartheta_k\left(h_{0,2} \otimes \eta; \tau\right) = [1 \otimes \beta_\eta]_{k,1}(\tau).$$

Here $1(x) = 1$ *for all* $x \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}$.

Note that we use the convention of Remark 4.3 for all such assertions.

Proof. We first observe that for all $\alpha \in \mathbb{Z}\backslash\{0\}$

$$\mathrm{res}_{z=\frac{\alpha}{M}}\left(z^{k-1}\eta(z)\omega(z\tau)\right) = \frac{i}{2\pi}M^{1-k}\alpha^{k-1}\beta_\eta(\alpha)\omega\left(\frac{\alpha\tau}{M}\right).$$

Suppose that k is even. Let $\omega = \omega(i\infty) + \omega_0$ with $\omega_0 \in W_N$ and note that $\beta_\omega = \beta_{\omega_0}$. Now we obtain

$$\begin{aligned} \vartheta_k(\omega \otimes \eta; \tau) &= 2M^{1-k}\sum_{\alpha=1}^{\infty}\alpha^{k-1}\beta_\eta(\alpha)\left(\omega(i\infty) + \omega_0\left(\frac{\alpha\tau}{M}\right)\right) \\ &= A + 2M^{1-k}\sum_{\alpha=1}^{\infty}\alpha^{k-1}\beta_\eta(\alpha)\sum_{j\in\mathbb{F}_N}\beta_\omega(j)\frac{e\left(\frac{\alpha\tau}{M} - \frac{j}{N}\right)}{1 - e\left(\frac{\alpha\tau}{M} - \frac{j}{N}\right)} \\ &= A + 2M^{1-k}\sum_{\alpha=1}^{\infty}\sum_{\nu=1}^{\infty}\alpha^{k-1}\beta_\eta(\alpha)\sum_{j\in\mathbb{F}_N}\beta_\omega(j)e\left(-\frac{j\nu}{N}\right)q^{\alpha\nu/M} \\ &= A + 2M^{1-k}\sum_{m=1}^{\infty}\sum_{d|m}\left(d^{k-1}\beta_\eta(d)\sum_{j\in\mathbb{F}_N}\beta_\omega(j)e^{-2\pi i(m/d)j/N}\right)q^{m/M}. \end{aligned}$$

In the case $\omega(z) = h_{0,2}(z)$ we find

$$\begin{aligned} \vartheta_{k_e}\left(h_{0,2} \otimes \eta; \tau\right) &= 2M^{1-k}\sum_{\alpha=1}^{\infty}\alpha^{k-1}\beta_\eta(\alpha)\frac{e\left(\frac{\alpha\tau}{M}\right)}{\left(1 - e\left(\frac{\alpha\tau}{M}\right)\right)^2} \\ &= 2M^{1-k}\sum_{\alpha=1}^{\infty}\sum_{\nu=1}^{\infty}\alpha^{k-1}\beta_\eta(\alpha)\nu q^{\alpha\nu/M} \\ &= 2M^{1-k}\sum_{m=1}^{\infty}\sum_{d|m}\left(d^{k-1}\beta_\eta(d)\left(\frac{m}{d}\right)\right)q^{m/M}. \end{aligned}$$

The odd case works analogously. ☐

Note that the inverse Fourier transform of $1(x)$ is given by

$$\mathcal{F}_N^{-1}(1)(x) = \delta_0(x),$$

where $\delta_0(x) = 1$ if $x = 0 \mod \mathbb{Z}$ and $\delta_0(x) = 0$ for all other values $x \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}$. So we can also write

$$\vartheta_k(h_{0,2} \otimes \eta; \tau) = [\mathcal{F}_N(\delta_0) \otimes \beta_\eta]_{k,1}(\tau).$$

The work we have done so far now provides

Theorem 4.6. *Let $k \geqslant 0$ be an integer and $\eta \in W_{\mathrm{pre},1}[\mathcal{T}_M]$. Let $\omega \in W_{\mathrm{weak},a}[\mathcal{T}_N]$ with decomposition*

$$\omega = \lambda_0 + \sum_{j=1}^{a-1} \partial^j \lambda_j$$

such that $\lambda_0 \in W_N$ and $\lambda_j \in W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_N]$. Then the following identity is valid on the upper half plane:

$$\vartheta_k(\omega \otimes \eta; \tau) = \sum_{j=0}^{a-1} [\mathcal{F}_N(\beta_{\lambda_j}) \otimes \beta_\eta]_{k,j}(\tau).$$

Proof. Starting with an expression $\omega = \omega_0 + \sum_{j=1}^{a-1} \partial^{j-1}(\partial\omega_j + c_j h_{0,2})$ with $\omega_j \in W_N$, we obtain

$$\vartheta_k(\omega \otimes \eta; \tau) = \vartheta_k(\omega_0 \otimes \eta; \tau) + \sum_{j=1}^{a-1} \partial_\tau^{j-1} \vartheta_{k-j}((\partial_z \omega_j + c_j h_{0,2}) \otimes \eta; \tau)$$

and with Lemma 4.4 and Proposition 4.5 this simplifies to

$$\begin{aligned} &= [\beta_{\omega_0} \otimes \beta_\eta]_{k,0} + \sum_{j=1}^{a-1} \partial_\tau^j [\mathcal{F}_N(\beta_{\omega_j}) \otimes \beta_\eta]_{k-j,0}(\tau) + \partial_\tau^{j-1}[c_j \mathcal{F}_N(\delta_0) \otimes \beta_\eta]_{k-j+1,1}(\tau) \\ &= [\beta_{\omega_0} \otimes \beta_\eta]_{k,0} + \sum_{j=1}^{a-1} [\mathcal{F}_N(\beta_{\omega_j} + c_j \delta_0) \otimes \beta_\eta]_{k,j}(\tau) \end{aligned}$$

where $\delta_0(x) = 1$ if $x \in \mathbb{Z}$ and 0 else, and finally with Proposition 3.5

$$= \sum_{j=0}^{a-1} [\mathcal{F}_N(\beta_{\lambda_j}) \otimes \beta_\eta]_{k,j}(\tau).$$

□

The next lemma imitates a classical result by Bol.

Lemma 4.7 (Weak Bol's identity)**.** *Let $k \geqslant 1$ and $\beta : \mathbb{F}_N \to \mathbb{C}$, $\gamma : \mathbb{F}_M \to \mathbb{C}$. Then we have*

$$\int_{k-1} ([\beta \otimes \gamma]_{k,\ell})(\tau) = N^{1+\ell-k} [\gamma \otimes \beta]_{2-k+\ell,0}\left(\frac{N\tau}{M}\right).$$

Note that the choice of $k-1$ is crucial for this kind of formula.

Proof. This can be followed by direct calculation and for the convenience of the reader we provide the details.

$$\begin{aligned}\int_{k-1}\left([\beta\otimes\gamma]_{k,\ell}\right)(\tau) =&2\int_{k-1} M^{1-k}\sum_{m=1}^{\infty}\left(\sum_{d|m} d^{k-1}\gamma(d)\left(\frac{m}{d}\right)^{\ell}\beta\left(\frac{m}{d}\right)\right)q^{\frac{m}{M}}\\ =&\ 2\sum_{m=1}^{\infty}\left(\sum_{d|m}\gamma(d)\left(\frac{m}{d}\right)^{\ell-k+1}\beta\left(\frac{m}{d}\right)\right)q^{\frac{m}{M}}\\ =&\ 2N^{1+\ell-k}N^{k-\ell-1}\sum_{m=1}^{\infty}\left(\sum_{d|m}\gamma\left(\frac{m}{d}\right)d^{(2-k+\ell)-1}\beta\left(d\right)\right)\left(q^{\frac{N}{M}}\right)^{\frac{m}{N}}\\ =&\ N^{1+\ell-k}[\gamma\otimes\beta]_{2-k+\ell,0}\left(\frac{N\tau}{M}\right).\end{aligned}$$

□

Theorem 4.8. *Let $k \geqslant 1$ and $\omega\otimes\eta \in W_{\mathrm{weak},\infty}[\mathcal{T}_N]\otimes W^{i\infty}_{\mathrm{pre},1}[\mathcal{T}_M]$, where $\omega = \sum_{j=0}^{u}\partial_z^j\lambda_j$ with $\lambda_j \in W^{i\infty}_{\mathrm{pre},1}[\mathcal{T}_N]$ as in Theorem 4.6. Then we have*

$$\int_{k-1}\vartheta_k(\omega\otimes\eta;\tau) = \sum_{j=0}^{u} N^{1+j-k}\vartheta_{2-k+j}\left(\mathcal{F}_M^{-1}\eta\otimes\mathcal{F}_N\lambda_j;\frac{N\tau}{M}\right).$$

Proof. First of all, Theorem 4.6 gives us

$$\vartheta_k(\omega\otimes\eta;\tau) = \sum_{j=0}^{u}[\mathcal{F}_N(\beta_{\lambda_j})\otimes\beta_\eta]_{k,j}(\tau).$$

Now with Lemma 4.7 we conclude

$$\begin{aligned}\int_{k-1}\vartheta_k(\omega\otimes\eta;\tau) &= \sum_{j=0}^{u}\int_{k-1}[\mathcal{F}_N(\beta_{\lambda_j})\otimes\beta_\eta]_{k,j}(\tau)\\ &= \sum_{j=0}^{u} N^{1+j-k}[\beta_\eta\otimes\mathcal{F}_N(\beta_{\lambda_j})]_{2-k+j,0}\left(\frac{N\tau}{M}\right)\end{aligned}$$

and finally with Proposition 4.5

$$= \sum_{j=0}^{u} N^{1+j-k}\vartheta_{2-k+j}\left(\mathcal{F}_M^{-1}\eta\otimes\mathcal{F}_N\lambda_j;\frac{N\tau}{M}\right).$$

□

In order to study Eichler duality we extend the $\mathbb{C}$ vector space $W_{\mathrm{pre},\infty}[\mathcal{T}_N]$ to a $\mathbb{C}[z,z^{-1}]$ module by putting

$$\mathfrak{M}_N = W_{\mathrm{pre},\infty}[\mathcal{T}_N]\otimes\mathbb{C}[z,z^{-1}].$$

In particular, we obtain a graded algebra

$$\mathfrak{M}_N = \bigoplus_{j=-\infty}^{\infty} z^j W_{\mathrm{pre},\infty}[\mathcal{T}_N],$$

whose elements naturally stand with the function ϑ_k in the sense that

$$\vartheta_k(z^\ell \cdot \omega \otimes \eta; \tau) = \vartheta_k(\omega \otimes z^\ell \cdot \eta; \tau) = \vartheta_{k+\ell}(\omega \otimes \eta; \tau).$$

Definition 4.9. *Let $k \geqslant 3$ be an integer. Then for each $1 \leqslant a \leqslant k-2$ we define the Eichler homomorphism*

$$\mathcal{E}_{k,a}^{N,M} : \bigoplus_{j=0}^{a-1} \partial_z^j W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_N] \otimes W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_M] \longrightarrow W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_M] \otimes \bigoplus_{j=0}^{a-1} z^j W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_N]$$

by

$$\omega \otimes \eta = \sum_{j=0}^{a-1} \partial_z^j \lambda_j \otimes \eta \longmapsto N^{1-k} \mathcal{F}_M^{-1} \eta \otimes \sum_{j=0}^{a-1} (zN)^j \mathcal{F}_N \lambda_j.$$

Theorem 4.10. *We have the following assertions:*

(i) The map $\mathcal{E}_{k,a}^{N,M}$ is an isomorphism for each $1 \leqslant a \leqslant k-2$.

(ii) Consider the subspace $W_{\mathrm{weak},a}[\mathcal{T}_N] \otimes W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_M] \subset \bigoplus_{j=0}^{a-1} \partial_z^j W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_N] \otimes W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_M]$. The diagram

$$\begin{tikzcd}
W_{\mathrm{weak},a}[\mathcal{T}_N] \otimes W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_M] \arrow[r, "\mathcal{E}_{k,a}^{N,M}"'] \arrow[d, "\vartheta_k"'] & W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_M] \otimes \bigoplus_{j=0}^{a-1} z^j W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_N] \arrow[d, "(\tau \mapsto \frac{N}{M}\tau) \circ \vartheta_{2-k}"] \\
\mathbb{C}_0^+[[q^{1/M}]] \arrow[r, "\int_{k-1}"'] & \mathbb{C}_0^+[[q^{1/M}]]
\end{tikzcd}$$

is commutative.

Proof. (i) We show that the map

$$\mathcal{I}_{k,a}^{M,N} : W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_M] \otimes \bigoplus_{j=0}^{a-1} z^j W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_N] \longrightarrow \bigoplus_{j=0}^{a-1} \partial_z^j W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_N] \otimes W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_M]$$

with

$$x \otimes y = x \otimes \sum_{\ell=0}^{a-1} z^\ell y_\ell \longmapsto N^{k-1} \sum_{\ell=0}^{a-1} N^{-\ell} \partial_z^\ell \mathcal{F}_N^{-1} y_\ell \otimes \mathcal{F}_M x$$

is an inverse of $\mathcal{E}_{k,a}^{N,M}$. We find for $\omega \otimes \eta \in \bigoplus_{j=0}^{a-1} \partial_z^j W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_N] \otimes W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_M]$:

$$\begin{aligned}
\mathcal{I}_{k,a}^{M,N}\left(\mathcal{E}_{k,a}^{N,M}(\omega\otimes\eta)\right) &= \mathcal{I}_{k,a}^{M,N}\left(N^{1-k}\mathcal{F}_M^{-1}\eta \otimes \sum_{j=0}^{a-1}(zN)^j\mathcal{F}_N\lambda_j\right)\\
&= \sum_{j=0}^{a-1}\sum_{\ell=0}^{a-1} N^{j-\ell}(\delta_{j,\ell}\partial_z^\ell\mathcal{F}_N^{-1}\mathcal{F}_N\lambda_j\otimes\mathcal{F}_M\mathcal{F}_M^{-1}\eta)\\
&= \sum_{j=0}^{a-1}(\partial_z^j\lambda_j\otimes\eta)\\
&= \omega\otimes\eta.
\end{aligned}$$

The other way round we see for $a \otimes b \in W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_M] \otimes \bigoplus_{j=0}^{a-1} z^j W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_N]$:

$$\begin{aligned}
\mathcal{E}_{k,a}^{N,M}\left(\mathcal{I}_{k,a}^{M,N}(a\otimes b)\right) &= \mathcal{E}_{k,a}^{N,M}\left(N^{k-1}\sum_{\ell=0}^{a-1}N^{-\ell}\partial_z^\ell\mathcal{F}_N^{-1}b_\ell\otimes\mathcal{F}_M a\right)\\
&= N^{k-1}N^{1-k}\left(\mathcal{F}_M^{-1}\mathcal{F}_M x\otimes\sum_{j=0}^{a-1}N^{-j}(zN)^j\mathcal{F}_N\mathcal{F}_N^{-1}b_j\right)\\
&= a\otimes b.
\end{aligned}$$

(ii) We prove that the compositions give the same output. Let $\omega \otimes \eta \in W_{\mathrm{weak},a}[\mathcal{T}_N] \otimes W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_M]$ and ω be given by

$$\omega = \sum_{j=0}^{a-1}\partial^j\lambda_j, \qquad \lambda_j \in W_{\mathrm{pre},1}^{i\infty}[\mathcal{T}_N].$$

Then according to Theorem 4.8 we have that

$$\int_{k-1}\vartheta_k(\omega\otimes\eta;\tau) = \sum_{j=0}^{u}N^{1+j-k}\vartheta_{2-k+j}\left(\mathcal{F}_M^{-1}\eta\otimes\mathcal{F}_N\lambda_j;\frac{N\tau}{M}\right). \tag{4.1}$$

On the other hand we find

$$\begin{aligned}
\vartheta_{2-k}\left(\mathcal{E}_{k,a}^{N,M}(\omega\otimes\eta);\tau\right) &= N^{1-k}\vartheta_{2-k}\left(\mathcal{F}_M^{-1}\eta\otimes\sum_{j=0}^{a-1}(Nz)^j\mathcal{F}_N\lambda_j;\tau\right)\\
&= \sum_{j=0}^{a-1}N^{1-k+j}\vartheta_{2-k+j}\left(\mathcal{F}_M^{-1}\eta\otimes\mathcal{F}_N\lambda_j;\tau\right).
\end{aligned}$$

□

Before we apply this to several situations we first recall basic facts about Eichler integrals.

Proposition 4.11. *Let $f : \mathbb{H} \to \mathbb{C}$ be a holomorphic function with the following properties:*

(i) *f is periodic and has a Fourier expansion of the form* $f(\tau) = \sum_{n=1}^{\infty} a_f(n) q^{n/\lambda}$ *with some* $\lambda > 0$.

(ii) *There is an integer* $k \geqslant 2$ *and a dual function* f^* *with a Fourier expansion* $f^*(\tau) = \sum_{n=1}^{\infty} a_{f^*}(n) q^{n/\lambda^*}$ *with* $\lambda^* > 0$, *such that*

$$f(-1/\tau) = \tau^k f^*(\tau).$$

(iii) *The coefficients* $a_f(n)$ *and* $a_{f^*}(n)$ *are polynomially bounded, such that the corresponding L-functions* $L(f; s) = \sum_{n=1}^{\infty} a_f(n) n^{-s}$ *and* $L_{f^*}(s)$ *converge on some right half plane.*

Then the functions L_f *and* L_{f^*} *have meromorphic continuations to the entire plane. If we further put*

$$F(\tau) = \frac{(k-2)!}{(-2\pi i)^{k-1}} \lambda^{k-1} \sum_{n=1}^{\infty} a_f(n) \left(\frac{\lambda}{n}\right)^{k-1} q^n,$$

and

$$F^*(\tau) = \frac{(k-2)!}{(-2\pi i)^{k-1}} (\lambda^*)^{k-1} \sum_{n=1}^{\infty} a_{f^*}(n) \left(\frac{\lambda^*}{n}\right)^{k-1} q^n,$$

we obtain

$$F(\tau) - (-\tau)^{k-2} F^*(-1/\tau) = P_{f,f^*}(\tau) = (-1)^k \sum_{\ell=0}^{k-2} \binom{k-2}{\ell} i^{1-\ell} \Lambda_f(\ell+1) \tau^{k-2-\ell},$$

where

$$\Lambda(f; s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s) L(f; s).$$

A simple but very important observation was made in [8], we have the following formula.

Proposition 4.12. *For primitive non-principal Dirichlet characters* χ *and* ψ *we have*

$$E_k(\chi, \psi; \tau) = \frac{\chi(-1)(-2\pi i)^{k+1} \mathcal{G}(\psi)}{N_\psi (k-1)! \mathcal{G}(\overline{\chi})} \vartheta_k(\omega_{\overline{\chi}} \otimes \omega_{\overline{\psi}}; \tau). \tag{4.2}$$

It is well-known that if χ and ψ are primitive Dirichlet characters and $f = E_k(\chi, \psi; \tau)$ the corresponding L-function is

$$L(E_k(\chi, \psi; \tau); s) = \frac{2(-2\pi i)^k \mathcal{G}(\psi)}{N_\psi^k (k-1)!} L(\chi; s) L(\overline{\psi}; s-k+1).$$

We assume that both χ and ψ are non-principal. From identity (4.2) and Theorem 4.10 we obtain

$$\begin{aligned}\int_{k-1} E_k(\chi,\psi;\tau) &= \frac{\chi(-1)(-2\pi i)^{k+1}\mathcal{G}(\psi)}{N_\psi(k-1)!\mathcal{G}(\overline{\chi})}\int_{k-1}\vartheta_k(\omega_{\overline{\chi}}\otimes\omega_{\overline{\psi}};\tau)\\ &= \frac{\chi(-1)(-2\pi i)^{k+1}\mathcal{G}(\psi)}{N_\psi(k-1)!\mathcal{G}(\overline{\chi})}\times(-1)^k N_\chi^{1-k}N_\psi^{-1}\mathcal{G}(\overline{\psi})\mathcal{G}(\overline{\chi})\vartheta_{2-k}\left(\omega_\psi\otimes\omega_\chi;\frac{N_\chi\tau}{N_\psi}\right)\\ &= \frac{(-2\pi i)^{k+1}}{N_\chi^{k-1}N_\psi(k-1)!}\vartheta_{2-k}\left(\omega_\psi\otimes\omega_\chi;\frac{N_\chi\tau}{N_\psi}\right).\end{aligned}$$

Since

$$\int_{k-1} E_k(\chi,\psi;\tau) = \frac{(-2\pi i)^{k-1}}{(k-2)!}F(\tau),$$

where

$$F(\tau) = \int_\tau^{i\infty} E_k(\chi,\psi;z)(z-\tau)^{k-2}\mathrm{d}z,$$

this combines to

$$F(\tau) = -\frac{4\pi^2}{N_\chi^{k-1}N_\psi(k-1)}\vartheta_{2-k}\left(\omega_\psi\otimes\omega_\chi;\frac{N_\chi\tau}{N_\psi}\right).$$

In the sense of Proposition 4.11 we have $E_k^*(\chi,\psi;\tau) = \chi(-1)E_k(\psi,\chi;\tau)$ and this provides

$$F^*(\tau) = -\frac{4\pi^2}{N_\psi^{k-1}N_\chi(k-1)}\chi(-1)\vartheta_{2-k}\left(\omega_\chi\otimes\omega_\psi;\frac{N_\psi\tau}{N_\chi}\right).$$

Now according to Proposition 4.11 we have the functional equation

$$F(\tau)-(-\tau)^{k-2}F^*(-1/\tau) = (-1)^k\sum_{\ell=0}^{k-2}\binom{k-2}{\ell}i^{1-\ell}\Lambda_f(\ell+1)\tau^{k-2-\ell}.$$

On the other hand, from Theorem 1.4, we obtain

$$\begin{aligned}&(-\tau)^{k-2}F^*(-1/\tau)\\ =&-\tau^{k-2}(-1)^{k-2}\chi(-1)\frac{4\pi^2}{N_\psi^{k-1}N_\chi(k-1)}\vartheta_{2-k}\left(\omega_\chi\otimes\omega_\psi;-\frac{N_\psi}{N_\chi\tau}\right)\\ =&\frac{4\pi^2\tau^{k-2}(-1)^{k-2}}{N_\psi^{k-1}N_\chi(k-1)}\left(\mathrm{res}_{z=0}\left(z^{1-k}\omega_\psi(z)\omega_\chi\left(\frac{N_\psi z}{N_\chi\tau}\right)\right)+\left(\frac{N_\psi}{N_\chi\tau}\right)^{k-2}\vartheta_{2-k}\left(\omega_\psi\otimes\omega_\chi;\frac{N_\chi\tau}{N_\psi}\right)\right).\end{aligned}$$

And this concludes the following theorem.

Theorem 4.13. *Let $k\geqslant 2$ be an integer, χ and ψ be two primitive Dirichlet characters with $\chi(-1)\psi(-1)=(-1)^k$ and $f(\tau)=E_k(\chi,\psi;\tau)$. We then have the following identity between rational functions:*

$$\sum_{\ell=0}^{k-2}\binom{k-2}{\ell}i^{1-\ell}\Lambda_f(\ell+1)\tau^{-\ell} = -\frac{4\pi^2}{N_\psi^{k-1}N_\chi(k-1)}\mathrm{res}_{z=0}\left(z^{1-k}\omega_\psi(z)\omega_\chi\left(\frac{N_\psi z}{N_\chi\tau}\right)\right).$$

We can now use this to give detailed expressions for the L-functions in the critical strip.

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