Borcherds Products for Unitary Groups

Building Bridges 2nd EU/US Workshop on Automorphic Forms and Related Topics, Bristol

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Overview

1. Setup: Hermitian spaces and unitary groups $\mathrm{U}(1, n)$.
2. The Borcherds lift for $\mathrm{U}(1, n)$
3. Example case: $\mathrm{U}(1, 1)$
4. Further results & outlook.

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A hermitian space

- $\mathbb{F} = \mathbb{Q}(\sqrt{d})$, $d \in \mathbb{Z}_{<0}$, d square-free.
- $D_{\mathbb{F}}$ the discriminant, $\delta = \sqrt{D_{\mathbb{F}}}$ the different of $\mathbb{F}$.
- $\mathcal{O}_{\mathbb{F}}$ ring of integers, $\mathcal{D}_{\mathbb{F}}^{-1}$ inverse different ideal of $\mathbb{F}$

$$\mathcal{O}_{\mathbb{F}} = \mathbb{Z} \oplus \zeta\mathbb{Z}, \quad \mathcal{D}_{\mathbb{F}}^{-1} = \delta^{-1}\mathcal{O}_{\mathbb{F}}$$

- $V_{\mathbb{F}}, \langle \cdot, \cdot \rangle$ a hermitian space over $\mathbb{F}$, of dimension $q+1$
- $\langle \cdot, \cdot \rangle$ a non-degenerate, indefinite hermitian form, of signature $(1, q)$.
- $L \subset V$ an even hermitian lattice, $L \otimes_{\mathcal{O}_{\mathbb{F}}} \mathbb{F} = V_{\mathbb{F}}$, with $\mathbb{Z}$-dual L'
  Fix two vectors u, u' with $u \in L$ primitive, $u' \in L'$ and

$$\langle u, u \rangle = \langle u', u' \rangle = 0, \quad \langle u, u' \rangle \neq 0$$

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The unitary group

Denote by $\mathrm{U}(V)$ be the unitary group of $V_{\mathbb{F}}$.

- $\mathrm{U}(L) = \mathsf{Stab}_{\mathrm{U}(V)}(L)$
- The discriminant kernel $\Gamma_L \subset \mathrm{U}(L)$.

The symmetric domain is isomorphic to the quotient

$$\mathrm{U}(V)(\mathbb{R})/\mathcal{C} \qquad (\mathcal{C} \text{ max. compact subgroup}).$$

The modular variety $X_\Gamma = \Gamma_L \backslash \mathrm{U}(V)(\mathbb{R})/\mathcal{C}$ is called a *ball-quotient*.
A projective model is given by the cone of positive lines in V,

$$\mathcal{K}_{\mathrm{U}} = \{[v]\,;\, \langle v, v\rangle > 0\} \subset \mathbb{P}^1\mathbb{C}.$$

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The Siegel domain model

An affine model can obtained as follows:
For each $[z] \in \mathcal{K}_{\mathrm{U}}$ fix a representative of the form $z = u' - \tau\delta\langle u', u\rangle u + \sigma$ (with $\sigma \in V \cap u^{\perp} \cap u'^{\perp}$ and $\tau \in \mathbb{C}$).

The *Siegel domain model* for the symmetric domain is defined as

$$\mathcal{H}_{\mathrm{U}} = \{(\tau, \sigma) \in \mathbb{C} \times \mathbb{C}^{q-1} \,;\, 2\Im\tau|\delta||\langle u, u'\rangle|^2 > -\langle\sigma, \sigma\rangle\} \simeq \mathcal{K}_{\mathrm{U}}.$$

Example $q = 1$:
Here, $\mathrm{SU}(V) \simeq \mathrm{SL}_2(\mathbb{R})$, and $\mathcal{H}_{\mathrm{U}}$ just the usual complex upper half-plane,

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Unitary modular forms

Definition

Let Γ be of finite index in Γ_L, and $k \in \mathbb{Z}$. Then, $f : \mathcal{H}_{\mathrm{U}} \to \mathbb{C}$ is a *unitary modular form* (for Γ with weight k), if

1. f is holomorphic on $\mathcal{H}_{\mathrm{U}}$.
2. For all $\gamma \in \Gamma$, $f(\gamma(\tau, \sigma)) = j(\gamma; \tau, \sigma)^k f(\tau, \sigma)$.
3. f is entire at the cusps of $\mathcal{H}_{\mathrm{U}}$.
   (For $q > 1$ this follows from the Köcher-principle.)

Modular forms on $\mathcal{H}_{\mathrm{U}}$ can be developed as Fourier-Jacobi series

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Vector valued modular forms

As a quadratic module over $\mathbb{Z}$, the lattice L has signature $(2, 2q)$ (with the bilinear form $(\cdot,\cdot) := \mathrm{Tr}_{\mathbb{F}/\mathbb{Q}}\langle\cdot,\cdot\rangle$).

Now, the Weil-representation ρ_L is a unitary representation of $\mathrm{SL}_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[L'/L]$,

$$\rho_L(T)\mathfrak{e}_\gamma = e\left(\tfrac{1}{2}(\gamma,\gamma)\right)\mathfrak{e}_\gamma,\ \rho_L(S)\mathfrak{e}_\gamma = \frac{\sqrt{i}^{q-1}}{\sqrt{L'/L}} \sum_{\delta\in L'/L} e(-(\gamma,\delta))\mathfrak{e}_\delta.$$

Define the usual $|_{k,\rho}$ operation on functions $f : \mathbb{H} \to \mathbb{C}[L'/L]$:

$$(f\,|_{k,\rho}\,M)(\tau) = (c\tau+d)^{-k}\rho_L(M)^{-1}f(M\tau), \quad M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}).$$

Then $\mathcal{M}_k^!(\rho_L)$ is the set of holomorphic functions $f : \mathbb{H} \to \mathbb{C}[L'/L]$ with $f\,|_{k,\rho}\,M = f$, $\forall M \in \mathrm{SL}_2(\mathbb{Z})$, and meromorphic around the cusp $i\infty$.

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Heegner-divisors

Let m be a negative integer and $\gamma \in L'/L$.
The *Heegner-divisor* of index (m, γ) is defined a follows:

- For $\lambda \in L'$ with $\langle \lambda, \lambda \rangle = m < 0$, set

$$\mathbf{H}(\lambda) := \{(\tau, \sigma) \in \mathcal{H}_{\mathrm{U}};\, \langle z(\tau, \sigma), \lambda \rangle = 0\}\,.$$

  (Note: $\mathbf{H}(\lambda) = \mathbf{H}(\mu\lambda)$, for $\mu \in S^1 \cap \mathcal{O}_{\mathbb{F}}$)

- For the index (m, γ) we may now define

$$\mathbf{H}(m, \gamma) := \sum_{\substack{\lambda \in \gamma + L \\ \langle \lambda, \lambda \rangle = m}} \mathbf{H}(\lambda).$$

  (Γ_L invariant divisor on $\mathcal{H}_{\mathrm{U}}$.)

More generally, any (integer) linear combination of divisors of this type is called a Heegner divisor.

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Main theorem: The Borcherds lift

$$\Xi(z; f, W) = Ce\left(\frac{\langle z,\rho\rangle}{\langle u',u\rangle}\right) \prod_{\substack{\lambda\in K' \\ (\lambda,W)>0}} \prod_{\substack{\gamma\in L_0'/L \\ p(\gamma)=\lambda+K}} \left(1 - e\left(\frac{\langle z,\lambda\rangle}{\langle u',u\rangle}\right)\right)^{c(\langle\lambda,\lambda\rangle,\gamma)}$$

$$\uparrow$$

$$f(\tau) = \sum_{\gamma\in L'/L} \sum_{\substack{n\in\mathbb{Z}+q(\gamma) \\ n\gg-\infty}} c(n,\gamma)e(n\tau)\,\mathfrak{e}_\gamma \in \mathcal{M}^!_{1-q}(\rho_L)$$

with $\quad z = u' - \tau\delta\langle u',u\rangle u + \sigma$

1. Meromorphic modular form for Γ_L on $\mathcal{H}_{\mathrm{U}}$ of weight $c(0,0)/2$.
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2. $$\mathrm{div}(\Xi(f)) = \frac{1}{2}\sum_{\gamma\in L'/L}\sum_{\substack{n\in\mathbb{Z}+q(\gamma)\\ n<0}} c(n,\gamma)\mathbf{H}(n,\gamma)$$

Example case: $\mathrm{U}(1,1)$

Let $V_{\mathbb{F}} = \mathbb{F}^2$ and $L = \mathcal{O}_{\mathbb{F}} \oplus \mathcal{D}_{\mathbb{F}}^{-1}$.

- $\Gamma_L \simeq \mathrm{SL}_2(\mathbb{Z})$:

$$\mathrm{SL}_2(\mathbb{Z}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \leftrightarrow \begin{pmatrix} a & b\epsilon \\ c\epsilon^{-1} & d \end{pmatrix} \in SU(L),$$
$$\text{with} \quad \epsilon = -\delta \langle u', u \rangle.$$

- The symmetric domain is just the usual upper half-plane, $\mathbb{H} = \{\tau \in \mathbb{C};\ \Im\tau > 0\}$.

The space of input functions is $\mathcal{M}_0^!(\mathrm{SL}_2(\mathbb{Z}))$; a basis is given by

$$1,\ j_1 := j - 744,\ j_2, \ldots,$$
$$\text{with} \quad j_n(\tau) = q^{-n} + \mathbf{O}(q) \quad (n \in \mathbb{Z}_{>0}).$$

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Heegner-divisors & Weyl-chambers

Let $n \in \mathbb{Z}$, $n > 0$.

- For $\lambda = \lambda_1 u + \lambda_2 u' \in L$, with $\langle \lambda, \lambda \rangle = -n < 0$,

$$\mathbf{H}(\lambda) = [\tau_\lambda], \quad \text{with} \quad \tau_\lambda = \frac{-\operatorname{Tr}(\lambda_1 \bar{\lambda}_2) + n\delta}{\mathrm{N}(\lambda_2)},$$

  a CM-point with CM-order $\mathbb{Z} + n'\mathcal{O}_{\mathbb{F}}$ (with $n' \mid n$).

$$\mathbf{H}(-n) = \sum_{\substack{\lambda \in L \\ \langle \lambda, \lambda \rangle = -n}} [\tau_\lambda] \quad \text{(Heegner-divisor of index } -n\text{)}$$

- Let $d = \sharp\{t \in \mathbb{Z}_{>0};\, t \mid n\}$. Let $t_1, \dots, t_d$ be the divisors of n, in ascending order ($t_i \leq t_{i+1}$). Set $t_0 = 0$, $t_{d+1} = \infty$.

$$W(t_i, t_{i+1}) = \left\{ |\delta| t_i^2 < 2n \Im\tau < |\delta| t_{i+1}^2 \right\}, \quad (i = 0, \dots, d).$$

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Heegner-divisors & Weyl-chambers

Let $n \in \mathbb{Z}$, $n > 0$.

- For $\lambda = \lambda_1 u + \lambda_2 u' \in L$, with $\langle \lambda, \lambda \rangle = -n < 0$,

$$\mathbf{H}(\lambda) = [\tau_\lambda], \quad \text{with} \quad \tau_\lambda = \frac{-\operatorname{Tr}(\lambda_1 \bar{\lambda_2}) + n\delta}{\mathrm{N}(\lambda_2)},$$

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Theorem

$\Xi(\tau; j_n)$ is a meromorphic modular form of weight 0 on $\mathbb{H}$ satisfying

- *It's divisor is given by $\mathrm{div}(\Xi) = \frac{1}{2}H(-n)$.*
- *The product expansion attached to the Weyl-chamber $W = W(t_i, t_{i+1})$ is given by:*

$$\Xi(\tau; j_n, W) = e(\rho_1\tau - \rho_2\bar{\zeta}) \prod_{\substack{k,l\in\mathbb{Z} \\ nl > kt_i^2}} \left(1 - e\left(k\tau - l\bar{\zeta}\right)\right)^{c(kl)},$$

$$\text{where} \quad \rho_1 = \sum_{\substack{t|n \\ t \geq t_{i+1}}} t, \quad \rho_2 = \sum_{\substack{t|n \\ 0<t\leq t_i}} t.$$

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Further Results & Outlook

- Values around cusps: If Ξ is regular and non-zero at $[u]$, the value $\lim_{\tau \to i\infty} \Xi(z; f)$ is a CM-value of an eta-quotient.
- Modularity of divisors (Borcherds-GZK): The series

$$A(\tau) = c_1(\mathcal{L}_{-1/2}) + \sum_{\beta \in L'/L} \sum_{\substack{n \in \mathbb{Z} + q(\beta) \\ n > 0}} \mathbf{H}(-n, \beta)\ q^n \mathfrak{e}_\beta,$$

is a modular form contained in $\mathcal{M}_{1+q,\rho_L^*} \otimes \left(\mathrm{CH}^1(X_\Gamma/\mathcal{B})\right)_{\mathbb{Q}}$.
- Work in preparation: Local Borcherds products.
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Thank you for your attention!